Indian Geotechnical Journal, 41(2), 2011, 54-63

Distribution of Subgrade Modulus beneath Beams on Reinforced Elastic Foundations

Arindam Dey[1], Prabir Kumar Basudhar[2] and Sarvesh Chandra[3]

Key words

Subgrade modulus, confining stress, reinforced granular foundation bed, iterative finite difference method

Abstract: The paper deals with the estimation of contact pressure beneath a footing resting on a compacted granular bed, overlying a deposit of poor granular soil, with geosynthetic reinforcement placed at the interface of the soil strata. The footing and reinforcement are idealized as elastic beams resting on a Winkler medium. The variation in the modulus of subgrade reaction due to the variation in the confining pressure beneath the beam is considered in this study. Governing differential equations for predicting the flexural response of reinforced foundation bed under different loading conditions were developed and solved by using finite difference method. The study shows that the type of loading significantly affects the contact pressure and subgrade modulus profiles along the length of the beam. The study indicates that the modulus of subgrade reaction is not uniform along the length of the beam, contrary to the conventional assumptions. Comparative results have been presented to highlight the deviations in the behaviour obtained from the assumptions of uniform and varying modulus of subgrade reactions. Typical results considering varying modulus of subgrade reaction shows maximum deviation of 45%-50% in non-dimensional flexural responses when compared to the results obtained assuming uniform subgrade modulus along the length of the footing.

Introduction

Flexural analysis of foundations on granular soils is carried out by modeling the same either as beams or as plates on elastic medium. Since long, this has been an interesting and widely studied research area in geotechnical engineering. One of the approaches of tackling such problems is to model the soil by closely spaced discrete springs. This gave rise to the development of the concept of subgrade modulus [Winkler (1867)]. In developing and solving the resulting differential equations, most of the studies have been carried out either by assuming the modulus of subgrade reaction to be constant, or of some known distribution along the span of the foundation. It is well known that modulus of elasticity is a function of the confining pressure, initial void ratio and the deviatoric stress, although it is established that confining pressure and deviatoric stress are inter-dependent, and not independent parameters.

Apart from parameters such as the width and depth of foundation, modulus of subgrade reaction depends on elastic parameters of the soil, and hence, likely to be a function of the confining pressure. Depending on the loading conditions, the confining pressure varies along the length of the footing; thus, the elastic modulus, and subsequently the modulus of subgrade reaction are also likely to vary spatially, even for homogeneous soils. Thus, there is a need and scope to look into the issue more critically and check the validity of the assumed distributions. Such a study is undertaken and reported in this paper.

Modeling the behavior of strip and combined footings resting on soils has largely been carried out as beams on elastic foundation using Winkler's assumption. Even though the original Winkler-model is subjected to several limitations, with appropriate choice of the parameters, the model is found to be quite efficient and reasonably correct in analyzing and predicting the behavior of long beams resting on such foundations. Several researchers [Filolenko-Borodich (1940), Hetenyi (1946), Pasternak (1954), Kerr (1964)] proposed improvements on the Winkler model to remove its inherent deficiencies. These models have been extended to analyze the response of footings resting on reinforced foundation beds [Ghosh and Madhav (1994), Shukla and Chandra (1994), Maheshwari *et al*. (2004), Deb *et al*. (2005)]. However, in all these studies, the authors used constant modulus of subgrade reaction along the length of the footing, as it had been done in the original model. However, this assumption may not be realistic.

Makhlouf and Stewart (1965) showed that elastic modulus of granular soil is a function of confining pressure, and not a function of maximum principal stress. Janbu (1963) proposed the following relationship between the elastic modulus of soil and the effective confining stress, as follows.

1 Assistant Professor, Indian Institute of Technology Guwahati, Guwahati, 781039, E-mail: arindam.dey@iitg.ernet.in

2 Professor, Indian Institute of Technology Kanpur, Kanpur, 208016, India, E-mail: pkbd@iitk.ac.in

3 Professor, University of Kwazulu-Natal, Durban, 4041 South Africa, E-mail: chandra@ukzn.ac.za

$$E' = \frac{E}{p_a} = m\left(\sigma_h'\right)^{1-n} \quad (1)$$

where, E' is the elastic modulus of the subgrade (E) in a non-dimensional form and $\sigma_h' = (\sigma_h / p_a)$ is the non-dimensional effective confining stress. p_a is the atmospheric pressure, and m and n are modulus number and a dimensionless number respectively. Depending on the type and characteristics of the granular soil, the parameters m and n may vary significantly. Some data have been reported as follows: The parameter n has a magnitude of 0.5 for medium dense sands [Lade and Nelson (1987)]; for sands and silty sands, the magnitude of m vary from 50 to 500 [Janbu (1963), Lade and Nelson (1987)].

Janbu's expression [Equation 1] has been used to determine the regression parameters (*m* and *n*) for the reported data by Makhlouf and Stewart, 1965, (obtained from the laboratory triaxial tests on sands of two different relative densities). For the dense sand, *m*=18 and *n*=0.6212 has been determined, while for the loose sand, the parameters are found to be *m*=22.17 and *n*=0.7185. Based on the idea developed about the magnitudes of *m* and *n* from Figure 1, in the present study, typical values of *m*=25 and *n*=0.5 have been chosen for the compacted overlying bed. Although the chosen values may be artefacts and need to be determined from experimental procedures for each type of soil, still they prove to be handy enough to develop a generalized technique to determine the variation of subgrade modulus and contact stress in unison along the span of the footing resting on reinforced foundation bed.

From several plate load tests, the modulus of subgrade reaction (k) is observed to be a function of shape and size of the plate, depth of embedment and the type of soil [Terzaghi (1955)]. Several researchers [Biot (1937), Vesic (1961), Barden (1963), Vlasov and Leontiev (1966), Fletcher *et al.* (1971), Horvath (1983)], have correlated the modulus of subgrade reaction with the Elastic modulus and Poisson's ratio. Daloglu and Vallabhan (2000) suggested that the value of modulus of subgrade reaction should vary along the domain of a slab depending on the material and geometric properties of the plate; however, it was cautioned that it is not easy to determine this variation. Dey (2005) proposed a parabolic distribution of subgrade modulus along the width of footings without any proper justification. Therefore, in this paper, an attempt has been made to present a study on the determination of the variation of subgrade modulus beneath beams resting on elastic foundations, subjected to different but practically possible loading conditions.

Statement of The Problem

Figure 1(a) depicts a surface footing resting on a compacted granular bed, underlain by a poor granular deposit and subjected to a generalized loading system. A layer of reinforcement (geogrid/geomat/geocell) has been placed at the interface of the granular media. Figure 1(b) provides the schematic diagram of the proposed problem. The granular strata are represented by Winkler medium and both the footing and the reinforcement are idealized as elastic beams of length $2l_f$ and $2l_r$, possessing flexural rigidities of $E_f I_f$ and $E_r I_r$ respectively. The reinforcement is placed at a depth H beneath the footing, which is same as the thickness of the overlying compacted granular bed. The unit weight of the overlying and underlying granular strata are considered to be γ_1 and γ_2 respectively. The generalized distribution of the contact pressures at the footing-soil and reinforcement-soil interfaces are denoted by $p_f(x)$ and $p_r(x)$ respectively. The reinforcement is subjected to a uniform surcharge $\gamma_1 H$ all over its length effective due to the overburden pressure from the overlying granular bed. This eventually induces a horizontal frictional force along the entire span of the reinforcement with a maximum magnitude at the mid-span and a minimum value of 0 (zero) at its edge. The reinforcement is considered inextensible. However, full mobilization of the frictional force is assumed owing to a very small magnitude of deflection. Three different loading conditions commonly encountered in geotechnical practice has been considered as has been shown in Figure 2, and are described as follows:

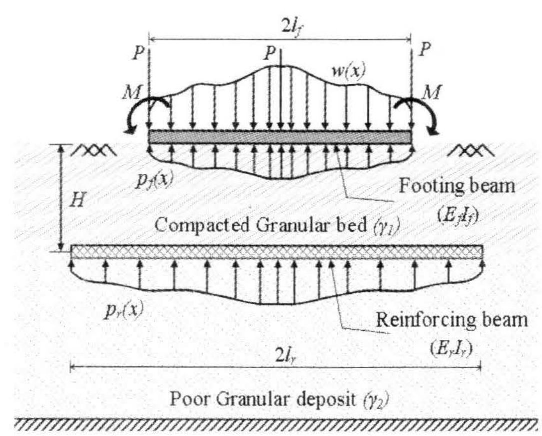

Fig. 1a Definition Sketch of the Problem: Beam on Reinforced Elastic Foundations

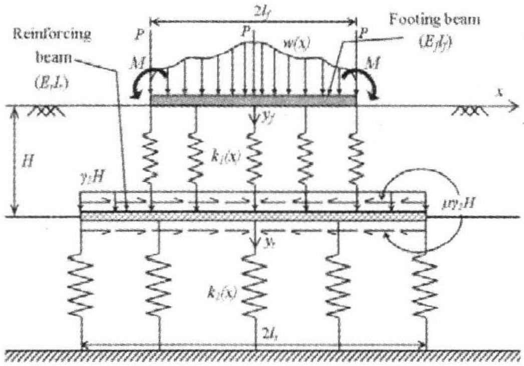

Fig. 1b Schematic Diagram of the Problem: Beam on Reinforced Elastic Foundations

Case A: Concentrated load acting at the centre of a strip footing.

Case B: Two concentrated loads acting at the edges of a combined footing.

Case C: Footing acted upon by a generalized loading system as in an aqueduct.

Analysis

Assumptions

The analysis is based on the following assumptions:

(a) Plane strain condition persists along the section normal to the footing, and

(b) Due to symmetry in loading and geometry, it is sufficient to analyze only half of the footing.

Governing differential equations

Referring to Figure 1(b), the governing differential equation for the footing is expressed as follows [Hetenyi (1946), Maheshwari *et al.* (2004)]:

$$E_f I_f \frac{d^4\left[y_f(x)\right]}{dx^4} = -p_f(x) = -\left[k_1(x)\right]\left\{\left[y_f(x)\right]-\left[y_r(x)\right]\right\} \quad \forall 0 \le x \le l_f \tag{2}$$

where, $k_1(x)$ is the distribution of subgrade modulus along the width of the footing, and $y_f(x), y_r(x)$ is the spatial deflection of the footing and reinforcement respectively.

Similarly, the governing differential equations for the flexural response of the reinforcement are expressed as:

$$\left.\begin{aligned} E_r I_r \frac{d^4 y_r(x)}{dx^4} - T(x)\frac{d^2\left[y_r(x)\right]}{dx^2} - \frac{d\left[T(x)\right]}{dx}\frac{d\left[y_r(x)\right]}{dx} \\ = \gamma_1 H - \left\{\left[p_r(x)\right]-\left[p_f(x)\right]\right\} \\ = \gamma_1 H - \left\{\left[k_1(x)\right]+\left[k_2(x)\right]\right\}\left[y_r(x)\right]+\left[k_1(x)\right]\left[y_f(x)\right] \end{aligned}\right\} \quad \forall 0 \le x \le l_f \tag{3a}$$

$$\left.\begin{aligned} E_r I_r \frac{d^4\left[y_r(x)\right]}{dx^4} - \left[T(x)\right]\frac{d^2\left[y_r(x)\right]}{dx^2} - \frac{d\left[T(x)\right]}{dx}\frac{d\left[y_r(x)\right]}{dx} \\ = \gamma_1 H - \left[p_r(x)\right] \\ = \gamma_1 H - \left[k_2(x)\right]\left[y_r(x)\right] \end{aligned}\right\} \quad \forall l_f \le x \le l_r \tag{3b}$$

where, $k_2(x)$ is the distribution of modulus of subgrade reaction along the length of reinforcement, and $T(x)$ is the mobilized tension along the length of the reinforcement, which is expressed as follows:

$$T(x) = (\mu_1 + \mu_2)\gamma_1 H (l_r - x) \tag{3c}$$

where, μ_1 and μ_2 are the coefficients of interface friction at the top and bottom of the reinforcement-soil interface respectively. Ideally, since the characteristic and properties of the granular medium at the top and bottom of the reinforcement are different, the coefficient of the interface friction at the top and bottom of the reinforcement-soil interface should be also different. However, as the coefficient of interface friction between the granular soil and geogrids vary between 0.3 - 0.5, which is not so significantly different, it is assumed that the coefficient of interface friction at the top and bottom of reinforcement-soil interface are identical and equal, and is denoted by μ, and Equation (3c) is modified and expressed as follows:

$$T(x) = 2\mu\gamma_1 H (l_r - x) \tag{3d}$$

The above equation indicates that the tension mobilized in the reinforcement is minimal at the edges and maximal at the mid-span. The above equations are expressed in their non-dimensional form and are utilised to determine the flexural response of the footing resting on reinforced foundation beds as described in a later section.

Boundary and Continuity Conditions

The boundary conditions for the footing subjected to different loading conditions, as shown in Figure 2, are described as follows:

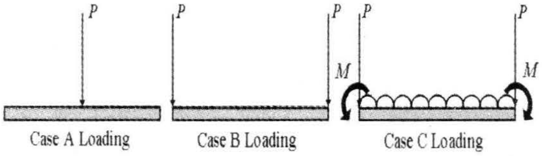

Fig 2. Different Types of Load Distribution Considered in the Study

Case A Loading condition

(a) At the mid-span of the footing, the slope is zero and the shear force is half of the applied load i.e.

$$\left.\frac{d\left[y_f(x)\right]}{dx}=0\right|_{x=0} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=-\frac{P}{2}\right|_{x=0} \tag{4a}$$

(b) At the edge of the footing, the bending moment and the shear force are zero i.e.

$$\left.-E_f I_f \frac{d^2\left[y_f(x)\right]}{dx^2}=0\right|_{x=l_f} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=0\right|_{x=l_f} \tag{4b}$$

Case B Loading condition

(a) At the mid-span of the footing, the slope and the shear force are zero i.e.

$$\left.\frac{d\left[y_f(x)\right]}{dx}=0\right|_{x=0} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=0\right|_{x=0} \tag{5a}$$

(b) At the edge of the footing, the bending moment is zero and the shear force is equal to the applied concentrated edge load i.e.

$$\left.-E_f I_f \frac{d^2\left[y_f(x)\right]}{dx^2}=0\right|_{x=l_f} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=P\right|_{x=l_f} \tag{5b}$$

Case C Loading condition

(a) At the mid-span of the footing, the slope and the shear force are zero i.e.

$$\left.\frac{d\left[y_f(x)\right]}{dx}=0\right|_{x=0} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=0\right|_{x=0} \tag{6a}$$

(b) At the edge of the footing, the bending moment is equal to the concentrated edge moment (M) and the shear force is equal to the applied concentrated edge load (P) i.e.

$$\left.-E_f I_f \frac{d^2\left[y_f(x)\right]}{dx^2}=-M\right|_{x=l_f} \quad and \quad \left.-E_f I_f \frac{d^3\left[y_f(x)\right]}{dx^3}=P\right|_{x=l_f} \tag{6b}$$

The above boundary conditions are expressed in non-dimensional form and have been used in the iterative procedure to determine the flexural response of the footing as described later.

As the flexural response of the reinforcement is governed by two differential equations [Equations (3a) and (3b)] within the ranges of $0 \le x \le l_f$ and $l_f \le x \le l_r$, continuity is to be established at the junction of the domains governed by the two equations. Thus, at an infinitesimal distance (ε) left and right of the point, situated at a distance of l_f from the mid-span of the reinforcement (denoting the junction of the domains governed by the two equations as mentioned above), continuity is established in terms of deflection, slope, bending moment and shear force, which is expressed in non-dimensional form as follows:

Deflection is equal:

$$\left.\left[y_r'(x_n)\right]\right|_{x_n=1-\varepsilon}=\left.\left[y_r'(x_n)\right]\right|_{x_n=1+\varepsilon} \tag{7a}$$

Slope is equal:

$$\left.\left\{\frac{d\left[y_r'(x_n)\right]}{dx_n}\right\}\right|_{x_n=1-\varepsilon}=\left.\left\{\frac{d\left[y_r'(x_n)\right]}{dx_n}\right\}\right|_{x_n=1+\varepsilon} \tag{7b}$$

Bending moment and Shear force is equal:

$$\left\{E_r I_r \frac{d^2\left[y_r'(x_n)\right]}{dx_n^2}\right\}\Bigg|_{x_n=1-\varepsilon} = \left\{E_r I_r \frac{d^2\left[y_r'(x_n)\right]}{dx_n^2}\right\}\Bigg|_{x_n=1+\varepsilon} \quad (7c)$$

$$\left.\begin{aligned} &\left\{E_r I_r \frac{d^3\left[y_r'(x_n)\right]}{dx_n^3} - T'(x_n)\frac{d^2\left[y_r'(x_n)\right]}{dx_n^2}\right\}\Bigg|_{x_n=1-\varepsilon} \\ &= \left\{E_r I_r \frac{d^3\left[y_r'(x_n)\right]}{dx_n^3} - T'(x_n)\frac{d^2\left[y_r'(x_n)\right]}{dx_n^2}\right\}\Bigg|_{x_n=1+\varepsilon}\end{aligned}\right\} \quad (7d)$$

Solution Technique

The differential equations governing the flexural responses of the footing and the reinforcement are expressed in finite difference form using the Central Difference Scheme. The footing and the reinforcement are discretized into n_f and n_r number of nodes. The finite difference form of the governing equation for the footing [Equation (2)] is expressed as follows:

$$\left[\frac{y'_{f,i-2} - 4y'_{f,i-1} + \left(6 + h_n^4 k'_{1,i}\right) y'_{f,i} - 4y'_{f,i+1} + y'_{f,i+2}}{h_n^4}\right] = Rk'_{1,i} y'_{r,i} \quad \forall 0 \le x_n \le 1 \quad (8)$$

where, $y'_{f,i}$, $y'_{r,i}$ is the non-dimensional deflection of the footing and reinforcement at the i^{th} node respectively, and $k'_{1,i}$ is the non-dimensional modulus of subgrade reaction of the overlying compacted granular stratum at the i^{th} node.

Similarly, the finite difference forms of the governing differential equations for the reinforcement [Equations (3a) and (3b)] are expressed as follows:

$$C^a_{r,i-2} y'_{r,i-2} + C^a_{r,i-1} y'_{r,i-1} + C^a_{r,i} y'_{r,i} + C^a_{r,i+1} y'_{r,i+1} + C^a_{r,i+2} y'_{r,i+2} = \gamma_1 H_n + k'_{1,i} y'_{f,i} \quad (9a)$$

$$C^b_{r,i-2} y'_{r,i-2} + C^b_{r,i-1} y'_{r,i-1} + C^b_{r,i} y'_{r,i} + C^b_{r,i+1} y'_{r,i+1} + C^b_{r,i+2} y'_{r,i+2} = \gamma_1 H_n \quad (9b)$$

$C^a_{r,i-2} - C^a_{r,i+2}$ and $C^b_{r,i-2} - C^b_{r,i+2}$ are the derived non-dimensional coefficients are expressed as follows:

$$\left.\begin{aligned}
C^a_{r,i-2} &= C^b_{r,i-2} = \frac{1}{h_n^4} \\
C^a_{r,i-1} &= C^b_{r,i-1} = -\frac{4}{h_n^4} - \frac{2\mu H_n \gamma_1'' R\left(l_n - x_{n,i}\right)}{h_n^2} - \frac{\mu H_n \gamma_1'' R}{h_n} \\
C^a_{r,i} &= \frac{6}{h_n^4} + \frac{4\mu H_n \gamma_1'' R\left(l_n - x_{n,i}\right)}{h_n^2} + Rk'_{1,i}\left\{1 + \frac{k'_{r,i}}{Rk_r}\right\} \\
C^b_{r,i} &= \frac{6}{h_n^4} + \frac{4\mu H_n \gamma_1'' R\left(l_n - x_{n,i}\right)}{h_n^2} + Rk'_{2,i} \\
C^a_{r,i+1} &= C^b_{r,i+1} = -\frac{4}{h_n^4} - \frac{2\mu H_n \gamma_1'' R\left(l_n - x_{n,i}\right)}{h_n^2} + \frac{\mu H_n \gamma_1'' R}{h_n} \\
C^a_{r,i+2} &= C^b_{r,i+2} = \frac{1}{h_n^4}
\end{aligned}\right\} \quad (10)$$

where the non-dimensional coefficients are expressed as follows: h_n is the length of mesh segment, H_n is the relative depth of reinforcement below the footing or relative thickness of the overlying compacted granular bed, γ_1'' is the relative unit weight of the compacted granular fill, R is the relative flexural rigidity of footing and reinforcement, l_n is the relative length of the footing and reinforcement, $x_{n,i}$ is the non-dimensional distance of i^{th} node from the mid-span of the footing, $k'_{1,i}, k'_{2,i}$ is the non-dimensional modulus of subgrade reaction of the overlying compacted stratum and the underlying poor granular stratum at the i^{th} node, and $k'_{r,i}$ is the non-dimensional relative stiffness of the granular media at the i^{th} node.

In order to obtain the flexural response of the footing and the reinforcement, the finite difference form of the governing non-dimensional equations [Equations (8), (9a) and (9b)] are solved by the method of successive approximations as suggested in standard references [Whittaker and Robinson (1924), Hetenyi (1946)]. In order to start the iteration for the solution, a deflection profile is assumed for the footing and reinforcement $\left[\left[y_f'(x_n)\right]_{previous} \text{ and } \left[y_r'(x_n)\right]_{previous}\right]$ as an initial guess. With the assumed profile, Equation (8) is written for each node of the discretized footing (with proper incorporation of the boundary conditions) to obtain a set of equations, which are solved by Gauss-Seidel iterative technique to determine the approximate deflection profile of the footing $\left[\left[y_f'(x_n)\right]_{current}\right]$. The approximate deflection profile of footing is further used to solve the set of equations governing the flexural response of reinforcement [Equations (9a) and (9b)] (written for each node of the discretized reinforcement) in order to determine the deflection profile of the reinforcement $\left[\left[y_r'(x_n)\right]_{current}\right]$. This completes one cycle of iteration. After each complete cycle of iteration, the RMS error in the non-dimensional deflection profiles of the footing and reinforcement $\left[RMS_{y,f}, RMS_{y,r}\ resp.\right]$ between successive iterations are calculated and summed up, to obtain the total RMS error $\left[RMS_{y,total}\right]$ of the reinforced foundation system between successive cycles. The above mentioned RMS errors are computed as follows:

$$RMS_{y,f} = \sqrt{\frac{\sum_{x_n=0}^{1}\left\{\left[y_f'(x_n)\right]_{current} - \left[y_f'(x_n)\right]_{previous}\right\}^2}{n_f}} \quad (11a)$$

$$RMS_{y,r} = \sqrt{\frac{\sum_{x_n=0}^{l_n}\left\{\left[y_r'(x_n)\right]_{current} - \left[y_r'(x_n)\right]_{previous}\right\}^2}{n_r}} \quad (11b)$$

$$RMS_{y,total} = RMS_{y,f} + RMS_{y,r} \quad (11c)$$

The method of successive approximations is stopped when the magnitude of the total RMS error becomes smaller than the desired tolerance level (10^{-5} for this case). Once the final deflection profile of the footing and reinforcement are determined by the above procedure, the same is used to compute the bending moment, shear force and the contact pressure along the length of the footing and the reinforcement.

Results and Discussion

Typical Distributions of Contact Pressure and Modulus of Subgrade Reaction for Beams on Elastic Foundation Bed

In order to initiate the iterative process to obtain the final solution, as available from various literatures, initial guesses for normalized contact pressures along the span of the footing $\left[p_f'(x)\right]$ are considered and is depicted in Figure 3. The representative polynomial forms are described as:

$$\begin{aligned}
&(a)\ p_f'(x_n) = 1 - x_n \\
&(b)\ p_f'(x_n) = -1.71x_n^3 + 1.82x_n^2 + 0.32x_n + 0.454 \\
&(c)\ p_f'(x_n) = 14.32x_n^5 - 28.41x_n^4 + 19.78x_n^3 - 5.62x_n^2 + 0.576x_n + 0.341 \\
&(d)\ p_f'(x_n) = -4.54x_n^5 + 9.13x_n^4 - 3.29x_n^3 - 2.45x_n^2 + 0.242x_n + 1 \\
&(e)\ p_f'(x_n) = -0.9x_n^2 + 6\times10^{-7}x_n + 1 \\
&(f)\ p_f'(x_n) = 1
\end{aligned} \quad (12)$$

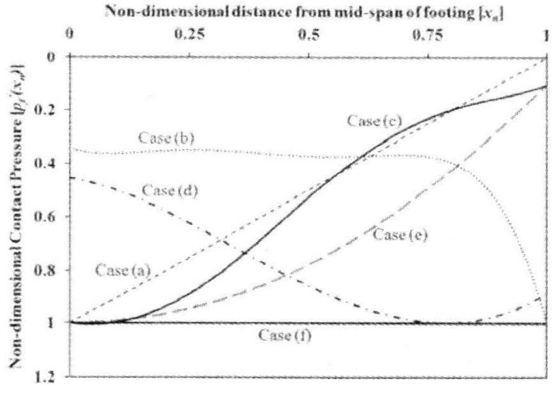

Fig. 3 Initial Non-Dimensional Contact Pressure Distributions

The normalized confining pressure along the span of the footing $\left[\sigma_h'(x_n)\right]$ is computed using the following expression:

$$\sigma_h'(x_n) = K_0\left[p_f'(x_n)\right] \quad (13)$$

wherein, the coefficient of earth pressure at rest (K_0) is determined using Jaky's expression as:

$$K_0 = 1 - \sin\phi' \quad (14)$$

where, ϕ' is the effective angle of internal friction of the granular medium. A value of 30° is assigned for the same in this problem.

Thereafter, the distribution of non-dimensional elastic modulus along the span of the footing $\left[E'(x_n)\right]$ is determined by using Equation (1), and using it in conjunction to the correlation proposed by Vesic (1961), the distribution of normalized modulus of subgrade reaction along the length of footing $\left[k_1'(x_n)\right]$ is obtained as:

$$k_1^{'}(x_n) = 0.818 \frac{1}{1-\nu_s^2}\left\{\frac{l_f^4}{b_f^4}\right\}\left\{\left[E^{'}(x)\right]\frac{p_a l_f^4}{E_f I_f}\right\}^{1.83} \quad (15)$$

where, b_f is the width of the beam, and ν_s is the Poisson's ratio of the soil. In this problem, the width of the beam is considered as unity and the Poisson's ratio for sand is taken as 0.35.

The distribution of normalized modulus of subgrade reaction along the length of beam as obtained from Equation (15) is used in conjunction to the method of successive approximation (described earlier) in order to determine the deflection profile of the footing. Similar to the RMS errors for deflection, the RMS errors of the contact stress distributions $\left(RMS_{p,f}\right)$ from the two consecutive iterations is computed as follows:

$$RMS_{p,f} = \sqrt{\frac{\sum_{x_n=0}^{1}\left\{\left[p_f^{'}(x_n)\right]_{current} - \left[p_f^{'}(x_n)\right]_{previous}\right\}^2}{n_f}} \quad (16)$$

If the magnitude of total RMSE is observed to be greater than the desired tolerance level (considered 10^{-3} for the present problem), the entire process described above is repeated based on the current distribution of contact pressure along the footing. At the end of the numerical procedure, the final distribution of modulus of subgrade reaction beneath the footing is determined based on the final distribution of contact pressure of the beam. Thus, in the procedure developed herein, the distribution of modulus of subgrade reaction and the contact pressure along the length of the footing are both obtained as a part of the solution process contrary to the conventional approach of assuming the modulus of subgrade reaction and then determining the contact pressure beneath the footing.

Figure 4 depicts a typical representation [Case A Loading condition (Figure 2) and Case (c) contact stress distribution (Figure 3) of as the initial guess] of the reduction in the magnitude of $RMS_{p,f}$ with the number of iterations, thus revealing the termination of the iterative procedure when the $RMS_{p,f}$ falls below 10^{-3}. It is observed that 80 iterations are required to achieve the final solution for this typical case, whereas based on the results from other cases (not presented here for the sake of brevity), the number of iterations required to achieve the final solution ranged from 80-120. For a typical case [Case A loading and Case (d) contact stress distribution], Figure 5 shows the transition in the nature of the contact pressure with the number of iterations required to achieve the final solution. For the sake of brevity, other detailed results are not presented herein. A typical distribution of contact pressure and the modulus of subgrade reaction studied for a particular loading condition are depicted in Figure 6. The figures reveal that regardless of the choice of the initial contact pressure distribution, the final distribution is unique for each of the loading conditions encountered in the problem.

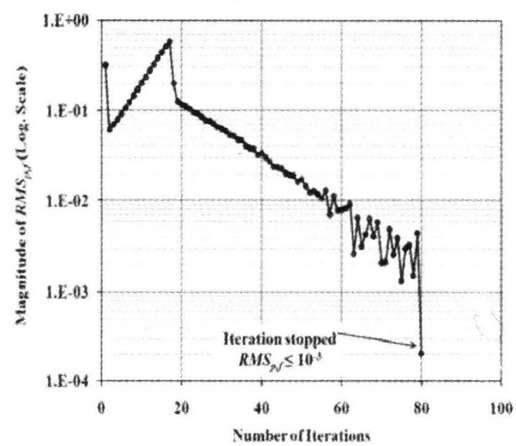

Fig. 4 Variation in the Magnitude of RMSE with the Number of Iterations [Case A Loading Condition and Case (c) Distribution of Contact Pressure as the Initial Guess

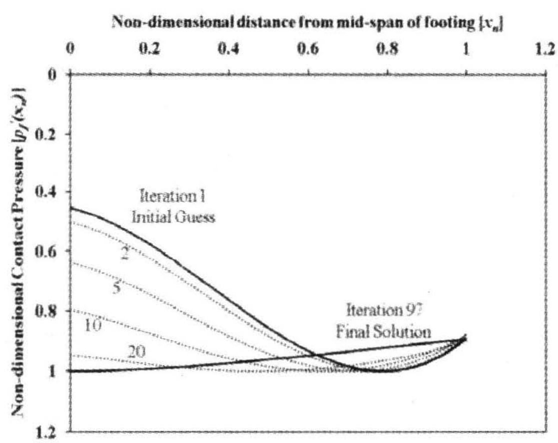

Fig. 5 Variation of Non-Dimensional Contact Pressure along the Footing with the Number of Iterations [Case A Loading Condition and Case (d) Distribution of Contact Pressure as the Initial Guess]

The distribution of non-dimensional modulus of subgrade reaction (expressed in a range of zero to one) beneath the footing, as obtained from the final distribution of contact pressure beneath the footing, is depicted in Figure 7.

It is noted here that the modulus of subgrade reaction is not uniform beneath beams on reinforced elastic foundation and exhibits considerable variations both spatially and with the change of loading conditions.

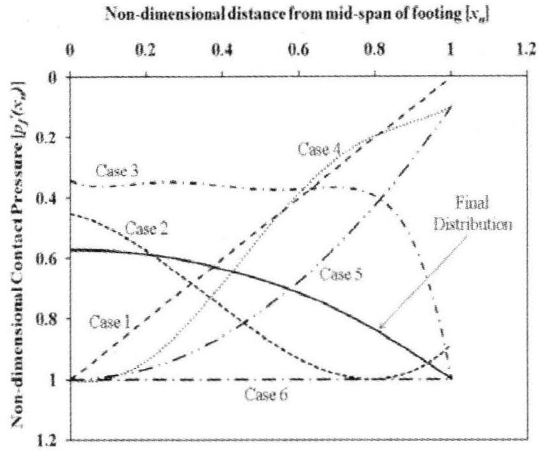

Fig. 6 Initial and Final Distributions of Contact Pressure for Case C Loading: Beams on Reinforced Elastic Foundations

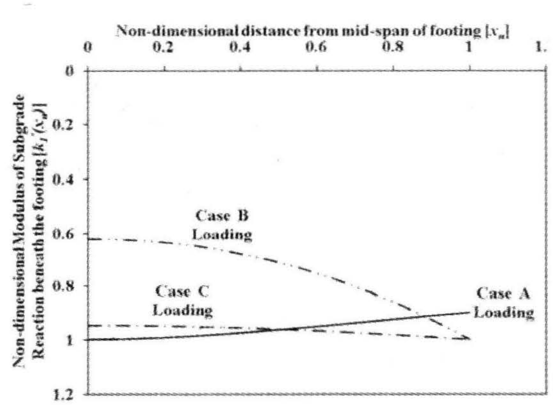

Fig. 7 Variation of Subgrade Modulus Beneath Beam on Reinforced Elastic Foundation Subjected to Various Loading Conditions

The polynomial fits (with $R^2 > 0.995$) representing the distribution of sub grade modulus along the length of the beam resting on reinforced elastic foundation subjected to various loading conditions are expressed as follows:

Case A Loading:

$$k'(x_n) = 0.081x_n^3 - 0.181x_n^2 - 0.01x_n + 1 \qquad (17a)$$

Case B Loading:

$$k'(x_n) = 0.428x_n^2 - 0.024x_n + 0.596 \qquad (17b)$$

Case C Loading:

$$k'(x_n) = -0.016x_n^3 + 0.076x_n^2 - 0.003x_n + 0.943 \qquad (17c)$$

Typical Flexural Responses for Beams on Reinforced Elastic Foundation Beds with Constant and Variable Subgrade Modulus

Figures 8 (a-c) depict typical variation in the flexural response of a footing resting on reinforced elastic foundations and subjected to Case A loading condition. It is observed that the assumptions of constant and variable modulus of subgrade reaction produce flexural responses, which are quite different from each other. Instead of considering uniform modulus, accounting of variable subgrade modulus results in a difference of 46%, 56% and 57% in the non-dimensional deflection, bending moment and shear force in the footing. Therefore, it is extremely important to consider a realistic distribution of modulus of subgrade reaction and verify with proper experimentation to gain an insight into the problem of behavior of reinforced foundation beds, especially its flexural characteristics.

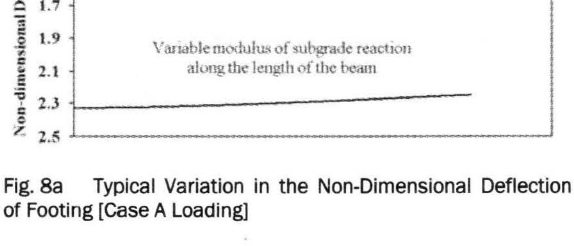

Fig. 8a Typical Variation in the Non-Dimensional Deflection of Footing [Case A Loading]

Comparison of the Contact Pressure Distribution beneath the Footing Resting on Unreinforced and Reinforced Elastic Foundations

Considering the footing resting on unreinforced and reinforced elastic foundations, the nature of contact stress distributions beneath the same are highlighted in Figures 9(a-c).

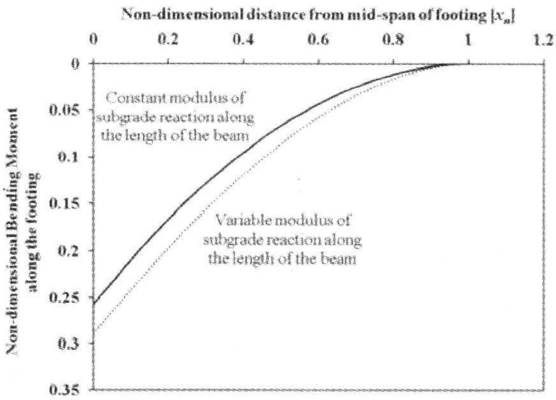

Fig. 8b Typical Variation in the Non-Dimensional Bending Moment of Footing [Case C Loading]

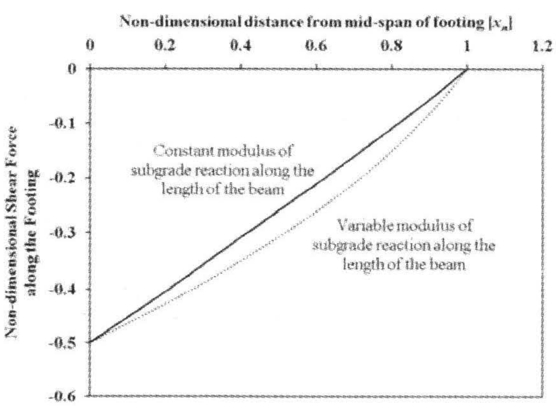

Fig. 8c Typical Variation in the Non-Dimensional Shear Force of Footing [Case C Loading]

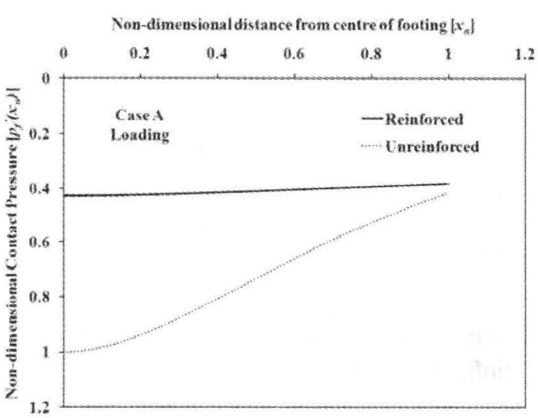

Fig. 9a Comparison of Non-Dimensional Contact Stress Distributions [Case A Loading]

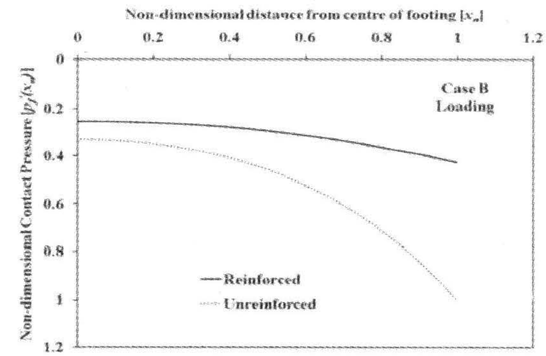

Fig. 9b Comparison of Non-Dimensional Contact Stress Distributions [Case B Loading]

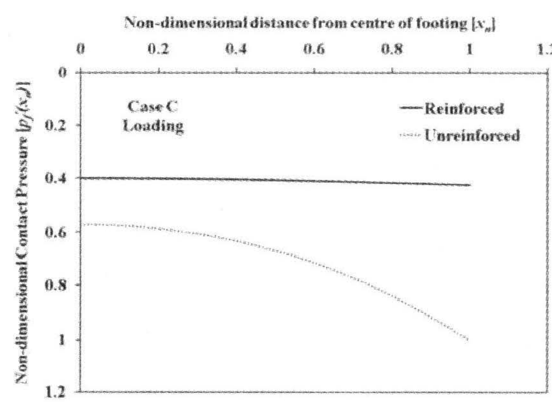

Fig. 9c Comparison of Non-Dimensional Contact Stress Distributions [Case C Loading]

Compared to the beam resting on unreinforced foundation system, it is observed that for all the cases of loading that have been considered in the study, the contact stress distribution obtained for reinforced foundation system shows more uniformity along the length of the footing. This may be stated as an effect of provision of the reinforcement, which induces reduction of differential stress distribution along the length of the footing to achieve a more uniform spatial stress distribution. It is also observed that the magnitude of the contact stress beneath the beams on reinforced elastic foundation is sufficiently reduced due to the provision of reinforcement.

Conclusions

Based on the studies reported above the following conclusions are made:

- Conventionally, the analysis of beams on elastic foundation is carried out by assuming a particular distribution of contact pressure at the soil-structure interface. However, it is more prudent and rational to determine the same as a part of an analysis, as very often, the assumed contact pressure distribution may vary significantly from the actual one, which should be unique for the parameters under consideration. One such parameter is the modulus of subgrade reaction, which is very often assumed constant along the length of the beam, but in fact, shows a considerable spatial variation.

- In the developed generalized analysis procedure to find the flexural response of beam on elastic foundation, no a-priori assumptions regarding the distribution of contact pressure and modulus of subgrade reaction need to be made; rather these are the outcome of the analysis itself.

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