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Dynamic Analysis of Piles under Rocking Motion

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Introduction

Vibration of piles under rocking/rotational mode coupled with lateral translation is a typical characteristic of piles supporting rotating machines and piles under earthquake forces. In many cases it has been found that this coupled motion is the critical and often governs design.

Researchers namely, Parmelee et al. (1964), Tajimi (1966), Penzien (1970), Novak (1974), Novak and El-Sharnouby (1983), Bannerjee and Sen (1987), Dobry and Gazetas (1988), have provided solution to this problem.

Of these solutions, Novak's method is very popular for its simplicity, though the method does not address a number of issues and has a few limitations, namely,

- The values are given for Poisson's ratio of 0.25 and 0.40 only. Thus for any intermediate values and values beyond 0.4 another set of interpolation/ extrapolation is necessary.
- Novak and El-Sharnouby (1983) have given stiffness and damping coefficients for soil having parabolic variation but in many cases the variation is linear.
- It does not address the case of partially embedded piles, which is of great practical importance for piles driven in Arctic condition (especially in Northern Siberia-which constitute of a large number of Oil and Natural Gas facilities).
- The dynamic bending moment and shear force induced on pile cannot be evaluated- for which the common practice is to restrict the moments and shears to 50% of its design capacity– which in many cases could be very conservative.
- The formulation is valid for long piles only (i.e. the failure takes place in pile before soil yields), it does not cater to the case when the pile is short, where the failure takes place by yielding of the soil prior to the structural failure of the pile.

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Again, the formulas given by Dobry and Gazetas (1988), based on more rigorous analysis, are also popular. However, they do not address the issues of partial embedment, dynamic bending moments and shear or the case if the pile is short.

The Analytical Solution

The analytical solution of the present paper takes into the cognizance of many of the limitations listed above and also arrives at a formulation which makes the design procedure chart/coefficient independent and thus easily amenable to analysis based on a simple spread-sheet. The method is an extension of the method proposed earlier by Chowdhury and Dasgupta (2006, 2008).

The present formulation is based Beredugo and Novak's (1972) method of solution for a rigid cylinder embedded in an elastic half space.

Shown in Figure 1 is a pile embedded in the ground consisting of a homogeneous elastic medium under plane strain condition. Also, it was assumed that the equation of beams on elastic foundation is applicable. The pile considered is long and slender.

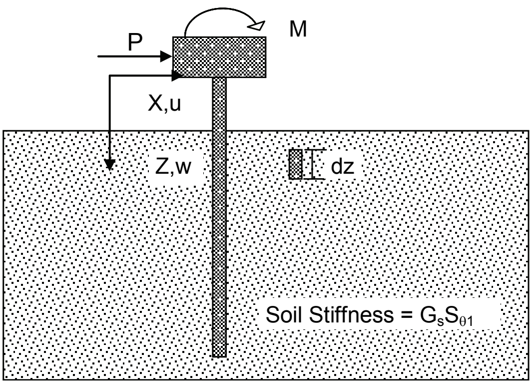

Fig. 1 Conceptual Model of Pile

Under static loading, the equation of equilibrium in the x-direction for such a beam on elastic foundation is given by

$$E_p I_p \frac{d^4u}{dz^4} = -k_s u \qquad (1)$$

where E_p = Young's modulus of the pile; I_p = moment of inertia of the pile cross section; k_s = elastic stiffness of the soil and is expressed as $G_s\ S_{\theta 1}$; G_s = dynamic shear modulus of the soil.

For a long pile under load or moment at its head, the deflection equation [solution of eqn. (1)] can be written as

$$u = e^{-pz}(C_0 \cos pz + C_1 \sin pz) \qquad (2)$$

in which $p = \sqrt[4]{G_s S_{\theta 1}/(E_p I_p)}$; $S_{\theta 1}$= Beredugo's constant which is basically frequency dependent and it was shown by Beredugo and Novak (1972) that $S_{\theta 1}$ can be taken as frequency independent for practical design problems and the analysis becomes quite simplified for rigid circular embedded footing.

Considering the pile head undergoing specified deflection and rotation as well as its head is fixed to the pile cap (same boundary condition considered by Novak, 1974), and using conditions, at $z = 0$, $u = u_0$ and $du/dz = \theta_0$, eqn. (2) may be rewritten as

$$\frac{u}{L} = e^{-pz}\left[\frac{u_0}{L}\cos pz + \left(\frac{u_0}{L} + \frac{\theta_0}{qL}\right)\sin pz\right] \tag{3}$$

where L = length of the pile.

For small rotation $\theta_0 \cong u_0/L$ and $\theta \cong u/L$, eqn. (3) may be written as

$$\theta = \theta_0 e^{-pz}\left(\cos pz + \left(1 + \frac{1}{pL}\right)\sin pz\right) \tag{4}$$

Using $\beta = pL$ in eqn. (4), the shape function in dimensionless form for any arbitrary loading can be written as

$$\phi(z) = e^{\frac{-\beta z}{L}}\left[\cos\frac{\beta z}{L} + \eta \sin\frac{\beta z}{L}\right] \tag{5}$$

where $\beta = \sqrt[4]{\dfrac{G_s S_{\theta 1} L^4}{E_p I_p}}$; and $\eta = 1 + \dfrac{1}{\beta}$.

Thus the shape function for rotational mode remains invariant with respect to the lateral motion of pile for the given boundary condition.

Differentiating eqn. (5) with respect to z, one can write

$$\phi'(z) = \frac{\beta}{L} e^{\frac{-\beta z}{L}}\left((\eta - 1)\cos\frac{\beta z}{L} - (1 + \eta)\sin\frac{\beta z}{L}\right) \tag{6}$$

The shape function of the pile in the fundamental mode is shown in Figure 2

The potential energy $d\Pi$ of an element of depth dz, shown in Figure1, under rotational mode is then given by (Craig, 1988)

$$dP = \frac{E_p I_p}{2}\left[\frac{dq}{dz}\right]^2 + \frac{K_q}{2} q^2 \tag{7}$$

where, E_p = Young's modulus of pile; I_p= moment of inertia of pile; K_θ= rotational stiffness of soil having units of kN/m; θ = rotation of pile in the x-z plane and about horizontal axis and may be written as $\phi(z)\ q(t)$.

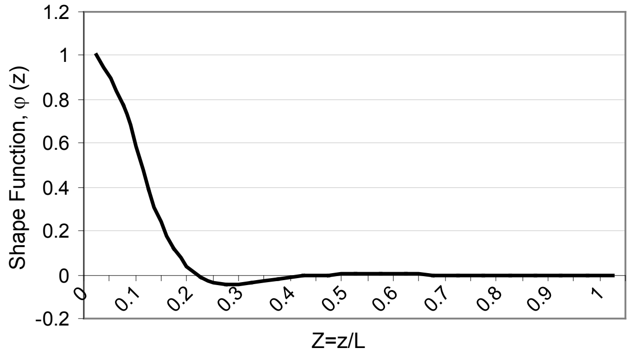

Fig. 2 Shape Function for the Pile for E_p/G_s =5000

For a rigid circular embedded footing with embedment D_f, the stiffness of the footing in the rotational mode may be expressed (Beredugo and Novak, 1972) as

$$K_\theta = G_b r_0^3 \left[C_{\theta 1} + \frac{G_s}{G_b}\frac{D_f}{r_0}\left(S_{\theta 1} + \frac{D_f^2}{3 r_0^2} S_{u1} \right) \right] \tag{8}$$

where, K_θ = foundation stiffness in horizontal direction; G_s = dynamic shear modulus of the soil along foundation surface; G_b = dynamic shear modulus of the soil at foundation base; r_0 = radius of the foundation; $C_{\theta 1}$ $S_{\theta 1}$, and S_{u1} = Berdugo's constants which are basically frequency dependent.

Ignoring the first term in eqn. (8) which represents the contribution of base resistance, and substituting the same in eqn.(7), for a cylindrical element of depth *dz*, embedded in soil, and also ignoring the term containing dz^2 which is again exceedingly small, the potential energy $d\Pi$ may be written as

$$d\Pi = \frac{E_p I_p}{2}\left[\frac{d\theta}{dz}\right]^2 + \frac{G_s r_0^2 S_{\theta 1} dz}{2}\theta^2 \tag{9}$$

The total potential energy over the whole length of the pile (L) is then given by

$$\Pi = \frac{E_p I_p}{2}\int_0^L \left[\frac{d\theta}{dz}\right]^2 dz + \frac{G_s r_0^2 S_{\theta 1}}{2}\int_0^L \theta^2 dz \tag{10}$$

Considering $\theta(z,t)= \phi(z)q(t)$, the rotational stiffness matrix may be written (Hurty and Rubenstein, 1967) as

$$K_{ij} = E_p I_p \int_0^L \phi_i'(z)\phi_j'(z)dz + G_s {r_0}^2 S_{\theta 1} \int_0^L \phi_i(z)\phi_\varphi(z)dz \qquad (11)$$

where the shape function of the problem is given by eqn. (5).

For the fundamental mode, the rotational stiffness of the pile may be obtained as

$$K_\theta = E_p I_p \int_0^L \phi'(z)^2 dz + G_s {r_0}^2 S_{\theta 1} \int_0^L \phi(z)^2 dz \qquad (12)$$

where *$\phi'(z)$* is as expressed in eqn. (6), and squaring the derivative,

$$\phi'(z)^2 = \frac{\beta^2}{L^2} e^{-\frac{2\beta z}{L}} \left(X - 2\eta \cos\frac{2\beta z}{L} + Y \sin\frac{2\beta z}{L} \right) \qquad (13)$$

where $X = 1 + \eta^2$; $Y = 1 - \eta^2$.

Again squaring eqn. (5) one can have

$$\phi(z)^2 = e^{-\frac{2\beta z}{L}} \left(\frac{X}{2} + \frac{Y}{2} \cos\frac{2\beta z}{L} + \eta \sin\frac{2\beta z}{L} \right) \qquad (14)$$

Substituting eqns. (13) and (14), eqn. (12), reduces to

$$\begin{aligned} K_\theta = {} & \frac{E_p I_p \beta^2}{L^2} \int_0^L e^{-\frac{2\beta z}{L}} \left(X - 2\eta \cos\frac{2\beta z}{L} + Y \sin\frac{2\beta z}{L} \right) dz \\ & + G_s {r_0}^2 S_{\theta 1} \int_0^L e^{-\frac{2\beta z}{L}} \left(\frac{X}{2} + \frac{Y}{2} \cos\frac{2\beta z}{L} + \eta \sin\frac{2\beta z}{L} \right) dz \end{aligned} \qquad (15)$$

Integrating eqn. (15) and after simplification, it may be expressed as

$$K_\theta = \frac{E_p I_p \beta^2}{L^2} \begin{bmatrix} X \frac{L}{2\beta}\left(1 - e^{-2\beta}\right) - 2\eta \frac{L}{4\beta}\left[e^{-2\beta}\left(\sin 2\beta - \cos 2\beta\right) + 1 \right] \\ + \frac{YL}{4\beta}\left(1 - e^{-2\beta}\left(\sin 2\beta + \cos 2\beta\right)\right) \end{bmatrix}$$

$$+ G_s {r_0}^2 S_{\theta 1} \begin{bmatrix} \frac{X}{2}\frac{L}{2\beta}\left(1 - e^{-2\beta}\right) + \frac{Y}{2}\frac{L}{4\beta}\left(e^{-2\beta}\left(\sin 2\beta - \cos 2\beta\right) + 1\right) + \\ \frac{\eta L}{4\beta}\left(1 - e^{-2\beta}\left(\sin 2\beta + \cos 2\beta\right)\right) \end{bmatrix} \qquad (16)$$

In eqn. (16), the values of *$e^{-2\beta}(\sin 2\beta + \cos 2\beta)$* and *$e^{-2\beta}(\sin 2\beta - \cos 2\beta)$* are ignored as they are exceedingly small and this also considerably simplifies the expression. Based on the above, simplified form may be written as

$$K_\theta = \frac{E_p I_p}{L}\left[\frac{X(1+\psi)(1-e^{-2\beta}) + Y\left(\frac{1}{2}+\frac{\psi}{4}\right) - \eta\left(1-\frac{\psi}{2}\right)}{2(\eta-1)}\right] \qquad (17)$$

and $$\psi = \frac{4G_s\lambda^2 S_{\theta 1}}{\pi E_p \beta^2} \qquad (17a)$$

where $\lambda = L/r_0$, the slenderness ratio of the pile and these are dimensionless quantities.

The accuracy of eqn. (17) will depend on the correct selection of $S_{\theta 1}$. For instance, for a rigid circular footing, Beredugo and Novak (1972) have furnished a frequency independent value of $S_{\theta 1}$= 2.5(for any value Poisson's ratio) which has been found to give adequate accuracy for practical engineering design.

Comparing the stiffness data with Novak (1974) and Dobry and Gazetas (1988), it is proposed that following values of $S_{\theta 1}$, given in Table-1 through Table-3, may be used for the calculation of dynamic response of pile under rocking mode.

Table-1 Suggested Value of $S_{\theta 1}$ for Poisson's Ratio of Soil =0.25

Poisson's Ratio	L/r_0 (Slenderness Ratio)	$S_{\theta 1}$(250)	$S_{\theta 1}$(500)	$S_{\theta 1}$(1000)	$S_{\theta 1}$(2500)	$S_{\theta 1}$(5000)	$S_{\theta 1}$(10000)
0.25	25	16.968	23.089	30.776	43.412	54.647	66.877
	40	17.358	23.656	31.586	44.678	56.390	69.253
	60	17.567	23.961	32.016	45.333	57.272	70.418
	80	17.674	24.110	32.225	45.648	57.688	70.958
	100	17.736	24.199	32.348	45.833	57.930	71.267

Note- The value in Parenthesis after $S_{\theta 1}$ depicts the value of E_p/G_s value of the soil

Table-2 Suggested Value of $S_{\theta 1}$ for Poisson's Ratio of Soil =0.40

Poisson's Ratio	L/r_0 (Slenderness Ratio)	$S_{\theta 1}$(250)	$S_{\theta 1}$(500)	$S_{\theta 1}$(1000)	$S_{\theta 1}$(2500)	$S_{\theta 1}$(5000)	$S_{\theta 1}$(10000)
0.40	25	18.037	24.623	32.937	46.707	59.054	72.614
	40	18.448	25.221	33.794	48.05	60.909	75.145
	60	18.671	25.543	34.249	48.748	61.851	76.393
	80	18.781	25.702	34.471	49.084	62.298	76.974
	100	18.847	25.795	34.603	49.281	62.557	77.307

Note- The value in Parenthesis after $S_{\theta 1}$ depicts the value of E_p/G_s value of the soil

Table-3 Suggested Value of $S_{\theta1}$ for Poisson's Ratio of Soil =0.50

Poisson's Ratio	L/r_0 (Slenderness Ratio)	$S_{\theta1}$(250)	$S_{\theta1}$(500)	$S_{\theta1}$(1000)	$S_{\theta1}$(2500)	$S_{\theta1}$(5000)	$S_{\theta1}$(10000)
	25	18.717	25.599	34.316	48.817	61.888	76.316
	40	19.141	26.217	35.202	50.21	63.813	78.946
0.50	60	19.37	26.55	35.674	50.936	64.794	80.247
	80	19.484	26.714	35.905	51.285	65.259	80.853
	100	19.552	26.811	36.041	51.49	65.531	81.203

Note:- The value in Parenthesis after $S_{\theta1}$ depicts the value of E_p/G_s value of the soil

For a particular pile having specific slenderness ratio and Poisson's ratio of the soil, the value of $S_{\theta1}$ may be selected from the above table and on substitution of the same in eqns. (3) and (17) gives the solution of pile stiffness in rocking mode.

Estimation of Mass Contribution of Pile

The mass matrix of the pile may be expressed as (Meirovitch, 1967)

$$M_z = m(z)\int \phi_i(z)\phi_j(z)dz \tag{18}$$

For the pile of length L, eqn. (18) can be expressed as

$$M_z = \frac{\gamma_p A_p}{g}\int \phi(z)^2 dz \tag{18a}$$

where, *m(z)* = mass per unit length of the pile $\cong \gamma_p A_p$; $\phi_i(z), \phi_j(z)$ are the shape function of the pile for different modes. For the fundamental mode $\phi_i(z) = \phi_j(z) = \phi(z)$, where $\phi(z)$ is as expressed in eqn (5)

For the present case of pile of length *L*, mass moment of inertia J_x is given by

$$J_x = \frac{M_z}{L}\int_0^L \left(\frac{r_0^{\,2}}{4} + z^2\right)dz \tag{19}$$

Substituting eqn. (18) one may now write

$$J_x = \frac{\gamma_p A_p r_0^{\,2}}{4g}\int_0^L j(z)^2 dz + \frac{\gamma_p A_p L^2}{g}\int_0^L \left(\frac{z}{L}\right)^2 j(z)^2 dz \tag{20}$$

where γ_p = unit weight of the pile material; A_p= cross sectional area of pile; *g* = acceleration due to gravity.

Substituting the values of $\phi(z)$ of eqn. (5) and after some simplification, J_x reduces to

$$J_x = \frac{\gamma_p A_p r_0^2 L}{16\beta g}\left[XF(\lambda) + \frac{Y}{2} + \eta\right] \tag{21}$$

in which $F(\lambda) = \left[\left\{1 - e^{-2\beta} + \frac{\lambda^2}{\beta^2} - 4\lambda^2 e^{-2\beta}\left(2 + \frac{1}{\beta} - \frac{2}{\beta^2}\right)\right\}\right]$; Y is given in eqn. (13) and λ = L/r_0, the slenderness ratio of the pile. F(λ)is a function of the slenderness ratio of the pile.

Eqn. (21) gives the inertial contribution of pile in the fundamental mode. Incidentally the inertial effect is usually ignored in design but could have significant effect if the number of pile is large in a pile group.

Radiation Damping for Pile under Rocking Mode

The damping of the pile embedded in soil will constitute of two parts: Material damping of the pile itself; and, radiation damping of the soil. It is obvious that the material damping of the pile will be much lower than that of the soil radiation damping. As the first step for calculating the soil damping one may ignore the material damping of the pile for the time being.

Material damping of soil is also a part of the vibration system. There is not much reliable data on this regard and could be obtained either based on laboratory test, else it may also be obtained for soils based on the results of Hardin (1991).

For a rigid footing embedded in soil for a depth D_f, Beredugo and Novak (1972) proposed an expression

$$C_\theta = r_0^4\sqrt{\rho G_s}\left[C_{\theta 2} + \frac{G_s}{G_b}\frac{D_f}{r_0}\left(S_{\theta 2} + \frac{D_f^2}{3r_0^2}S_{x2}\right)\right] \tag{22}$$

where, r_0 = radius of the foundation; G_b = dynamic shear modulus at foundation base; G_s = dynamic shear modulus of soil in which the foundation is embedded; D_f = depth of embedment; $C_{\theta 2}$, $S_{\theta 2}$ and S_{x2} = frequency independent.

Ignoring the first term in eqn. (22) which represents the contribution of base, damping for a cylindrical element of depth dz, embedded in soil, and also ignoring the term containing dz^2 which is again exceedingly small, the coefficient of damping may be written as

$$c(\theta) = r_0^3\sqrt{\rho G_s}S_{\theta 2}dz \tag{23}$$

For systems having continuous function, the damping is usually expressed as (Mario 1986),

$$C_\theta = c(\theta)\int \phi_i(z)\phi_j(z)dz \tag{24}$$

For the present case of pile of length L, eqn. (22) can be expressed as

$$C_\theta = r_0^3 \sqrt{\rho G_s} S_{\theta 2} \int_0^L \phi(z)^2 dz \tag{25}$$

Here $\phi_i(z), \phi_j(z)$ are the shape functions of the pile for different modes. For the fundamental mode $\phi_i(z) = \phi_j(z) = \phi(z)$, where $\phi(z)$ is as expressed in eqn (5), and hence

$$C_\theta = r_0^3 \sqrt{\rho G_s}\ S_{\theta 2} \int_0^L e^{-\frac{2\beta z}{L}} \left[\frac{X}{2} + \frac{Y}{2} \cos \frac{2\beta z}{L} + \eta \sin \frac{2\beta z}{L} \right] \tag{26}$$

Eqn. (26), on integration by parts and after some simplification, reduces to

$$C_\theta = r_0^3 \sqrt{\rho G_s}\ S_{\theta 2} L \left[X(1 - e^{-2\beta}) + Y/2 + \eta \right] / \left[4\beta \right] \tag{27}$$

Eqn. (27) expresses the soil damping for a single pile under rocking mode of vibration. Here the Factor $S_{\theta 2}$ is the damping coefficient, which is frequency dependent. Fortunately, the damping factor is required for calculation of the amplitude when the eigen solution of the problem is already done vis a vis, the dimensionless frequency number $a_0 = \omega r_0 / v_s$ term is known. Polynomial curve-fit for $S_{\theta 2}$ are available in terms of a_0 which can be used directly to arrive at these parameters.

Berdugo and Novak (1972) gave the following expression for the values of $S_{\theta 2}$

$$S_{\theta 2} = 0.0144 a_0 + 5.263 a_0^2 - 4.177 a_0^3 + 1.643 a_0^4 - 0.2542 a_0^5 \tag{28}$$

This value unlike other Beredugo's constant is independent of Poisson's ratio.

Material Damping of Pile

The structural stiffness contribution of the pile is given in the first part of eqn. (16), while that of the mass moment of inertia is given in eqn. (21). Thus, if C_c be the critical damping of the pile then it can be expressed as $C_c = 2\sqrt{KJ_x}$, where K and J_x are the stiffness and mass moment of inertia of the pile.

Depending on the material used for pile like RCC, steel etc, a suitable damping ratio (ζ) can be assumed. The damping (C_p) for the pile can be expressed as

$$C_p = \zeta C_c \tag{29}$$

This, when added to the radiation damping, calculated in eqn. (23), gives the complete damping quantity for the soil-pile system.

Piles with Different Boundary Conditions

Having established the stiffness, inertial and damping contribution of the pile in rocking mode based on minimization of the potential energy of the system, the method is extended to piles with other boundary conditions for which there is no standard solution available.

Partially Embedded Piles

This is a very common practice in Arctic and North Siberian condition, where due to environmental reason; the steel piles are driven into the ground when they protrude about 2-3m above the ground over which the pile cap and vibrating equipments are placed.

Let us consider the situation shown in Figure 3 for a partially embedded system.

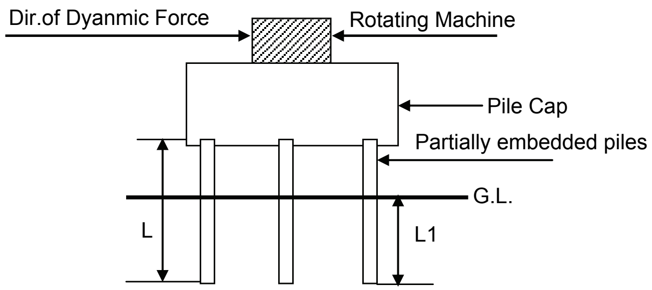

Fig. 3 Schematic Diagram of Partially Embedded Piles under Horizontal Load

If L denotes the full length of the pile and the length of embedment in soil is L_1, the constant β of eqn. (2) may be written as

$$\beta_e = \sqrt[4]{\frac{G_s S_{\theta 1} L_1^{\,4}}{E_p I_p}} \tag{30}$$

where subscript "*e*" represents embedment of the pile.

Thus, the shape function can be written as

$$\phi(z) = e^{-\frac{\beta_e z}{L_1}} \left(\cos\frac{\beta_e z}{L_1} + \eta \sin\frac{\beta_e z}{L_1}\right) \tag{31}$$

and

$$\phi'(z) = \frac{\beta_e}{L_1}\ e^{\frac{-\beta_e z}{L_1}}\left((\eta_e - 1)\cos\frac{\beta_e z}{L_1} - (1+\eta_e)\sin\frac{\beta_e z}{L_1}\right) \tag{32}$$

Square of the above is given by

$$\phi'(z)^2 = \frac{\beta_e^{\ 2}}{L_1^{\ 2}} e^{-\frac{2\beta_e z}{L}}\left(X_e - 2\eta_e\cos\frac{2\beta_e z}{L_1} + Y_e\sin\frac{2\beta_e z}{L_1}\right) \tag{33}$$

where $X = 1 + \eta_e^{\ 2}$; $Y = 1 - \eta_e^{\ 2}$ and $\eta_e = 1 + \frac{1}{\beta_e}$.

and

$$\phi(z)^2 = e^{-\frac{2\beta e z}{L_1}}\left(\frac{X_e}{2} + \frac{Y_e}{2}\cos\frac{2\beta_e z}{L_1} + \eta_e\sin\frac{2\beta_e z}{L_1}\right) \tag{33a}$$

Considering the fact that the embedment of a beam does not have any effect on the shape function of the system (Timoshenko, et al, 1990), the stiffness of the pile can be expressed as

$$K_q = E_p I_p \int_0^L \phi_l'(z)^2 dz + G_s S_{q1}\int_0^{L_1} \phi_l(z)^2 dz \tag{34}$$

$$\begin{aligned} K_\theta = {} & \frac{E_p I_p \beta_e^{\ 2}}{L_1^{\ 2}}\int_0^{\alpha L_1} e^{-\frac{2\beta z}{L1}}\left(X_e - 2\eta_e\cos\frac{2\beta_e z}{L_1} + Y_e\sin\frac{2\beta_e z}{L_1}\right)dz + \\ & G_s r_0^{\ 2} S_{\theta 1}\int_0^{L_1} e^{-\frac{2\beta_e z}{L}}\left(\frac{X_e}{2} + \frac{Y_e}{2}\cos\frac{2\beta_e z}{L_1} + \eta_e\sin\frac{2\beta_e z}{L_1}\right)dz \end{aligned} \tag{35}$$

Now considering $\alpha = L/L_1$ eqn. (35) may be expressed as

$$K_\theta = \frac{E_p I_p}{L_1}\frac{\left[X_e\left(\alpha + \psi_e - \alpha e^{-2\beta_e\alpha} - \psi_e e^{-2\beta_e}\right) + Y_e\left(\frac{\alpha}{2} + \frac{\psi_e}{4}\right) - \eta_e\left(\alpha - \frac{\psi_e}{2}\right)\right]}{2(\eta_e - 1)} \tag{36}$$

Here $\psi_e = \frac{4G_s\lambda_e^{\ 2}S_{\theta 1}}{\pi E_p \beta_e^{\ 2}}$ and $\lambda_e = \frac{L_1}{r_0}$.

Eqn. (36) gives the solution for stiffness of partially embedded piles. The correctness of the equation can be back checked by the fact that when the pile becomes fully embedded i.e.$L_1 = L$ one can have $\alpha \to 1$, $\beta_e = \beta$, $X_e = X$, etc. when eqn.(36) degenerates to eqn. (17), the stiffness for a full embedded pile.

The mass moment of inertia of pile remains same as stated in eqn. (21).

The damping matrix is given by the expression

$$C_\theta = r_0^{\ 3}\sqrt{\rho G_s}\, S_{\theta 2} L_1 [X_e(1 - e^{-2\beta e}) + Y_e/2 + \eta_e]/[4/(\eta_e - 1)] \tag{37}$$

Stiffness of the Pile for Varying Elastic Property

In the previous section, the calculation of stiffness as well as the damping of soil was based on the dynamic shear modulus of soil invariant with depth. While this could be possible for clayey soils, there are many cases when the dynamic shear modulus of the soil has been found to vary with depth. Generally this can be expressed as

$$G_s' = G_s(z/L)^m \tag{38}$$

where m = a number varying from 0-2 [considered 0 when G_s is constant with depth, assumed 1 for linear variation and 2 for parabolic distribution].

Thus for linearly varying soil the stiffness matrix can be written as

$$K_\theta = \frac{E_p I_p \beta^2}{L^2}\int_0^L e^{-\frac{2\beta z}{L}}\left(X - 2\eta\cos\frac{2\beta z}{L} - Y\sin\frac{2\beta z}{L}\right)dz + G_s r_0^{\ 2} S_{\theta 1}\int_0^L \left(\frac{z}{L}\right) e^{-\frac{2\beta z}{L}}\left(\frac{X}{2} + \frac{Y}{2}\cos\frac{2\beta z}{L} + \eta\sin\frac{2\beta z}{L}\right)dz \tag{39}$$

Integration of above by parts and ignoring the terms containing the factors $e^{-2\beta}.cos2\beta$, $\beta.e^{-2\beta}sin2\beta$ etc. which have extremely small values one can have

$$K_\theta = \frac{E_p I_p \beta}{2L}\left[X\left\{1 + \frac{\psi}{2\beta} - e^{-2\beta}\left(1 + \frac{\psi}{2\beta}(1+\beta)\right)\right\} + Y\left(\frac{3\psi}{8\beta} - \frac{1}{2}\right) - \eta\left(1 - \frac{\psi}{4\beta}\right)\right] \tag{40}$$

Here ψ is as defined in eqn. (17a), and the damping matrix for this is given by

$$C_x = \frac{r_0^{\ 3}\sqrt{\rho G_s}\, S_{\theta 2} L}{4\beta^2}\left[X\left[1 - e^{-2\beta}(1+\beta)\right] + \frac{3Y}{4} + \frac{\eta}{2}\right] \tag{41}$$

The mass coefficient remains same as expressed in eqn. (21).

When the dynamic shear modulus variation is parabolic with depth the stiffness equation of the pile is expressed as

$$K_\theta = \frac{E_p I_p \beta^2}{L^2}\int_0^L e^{-\frac{2\beta z}{L}}\left(X - 2\eta\cos\frac{2\beta z}{L} + Y\sin\frac{2\beta z}{L}\right)dz + G_s r_0^{\ 2} S_{\theta 1}\int_0^L \left(\frac{z}{L}\right)^2 e^{-\frac{2\beta z}{L}}\left(\frac{X}{2} + \frac{Y}{2}\cos\frac{2\beta z}{L} + \eta\sin\frac{2\beta z}{L}\right)dz \tag{42}$$

Eqn. (42) on successive integration by parts and simplifies to

$$K_\theta = \frac{E_p I_p \beta}{2L}\left[X\left\{\left(1+\frac{\psi}{8\beta^2}\right) - e^{-2\beta}\left(1+\frac{\psi}{2}\left(2+\frac{1}{\beta}-\frac{2}{\beta^2}\right)\right)\right\} + \frac{Y}{2} - \eta \right] \tag{43}$$

Eqn.(43) gives the stiffness expression of pile under parabolic variation of G along the length of pile. Here ψ is as defined in eqn (17a).

Proceeding in the manner stated earlier, the damping matrix may be expressed as

$$C_\theta = \frac{r_0^3\sqrt{\rho G_s}\, S_{\theta 2} L}{4\beta}\left[X\left(\frac{1}{4\beta^2} - e^{-2\beta}\left(2+\frac{1}{\beta}-\frac{2}{\beta^2}\right)\right)\right] \tag{44}$$

The mass coefficient remains same as in eqn. (21).

Dynamic Bending Moment and Shear Force in the Pile

For machine foundation subjected to a dynamic moment of $M_0 \sin\omega_m t$, and using, $MF = \left[\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}\right]^{-1}$, the amplitude of vibration is given by

$$\theta(t) = \left[(M_0 / K_\theta)\sin \omega_m t\right] MF \tag{45}$$

where ω_m= operating frequency of the machine; M_0 = unbalanced dynamic moment;

$r = \omega_m/\omega_n$ the ratio of operating and natural frequency; ζ= damping ratio of the system.

The peak amplitude is given by

$$\theta(t) = (M_0 / K_\theta) MF \tag{46}$$

The complete displacement function may be written as

$$\theta(z,t) = MF(M_0 / K_\theta) e^{-\frac{\beta z}{L}}\left(\cos\frac{\beta z}{L} + \eta \sin\frac{\beta z}{L}\right) \tag{47}$$

Thus bending moment is given by

$$M(x) = MF(E_p I_p M_0 / K_\theta)\frac{\beta}{L} e^{-\frac{\beta z}{L}}\left((1+\eta)\sin\frac{\beta z}{L} - (\eta - 1)\cos\frac{\beta z}{L}\right) \tag{48}$$

Considering maximum moment will be at the head at $z = 0$, the maximum dynamic moment may be given as

$$M_{max} = -MF(E_p I_p M_0 / K_\theta)\frac{\beta}{L}(\eta - 1) \tag{49}$$

The dynamic shear force is given by

$$V(z) = MF(E_p I_p M_0 / K_\theta)\frac{2\beta^2}{L^2} e^{-\frac{\beta z}{L}}\left(\sin\frac{\beta z}{L} - \eta\cos\frac{\beta z}{L}\right) \quad (50)$$

Dynamic Response of Short Piles under Rocking Mode

There are number of areas (e.g. Bonny river delta in Nigeria) where the top soil constitutes of very weak clay underlain by dense sand and the soil will yield much ahead of the pile itself. Broms (1965) has shown that the displacement curvature for such piles are completely different than that of long piles.

While a long pile embedded in soil behaves as a semi-infinite beam on elastic foundation, a short pile behaves as a beam of finite length on elastic foundation. Chowdhury and Dasgupta (2008) have given solution to the displacement curvature of such short beams on elastic foundation which is given by

$$u = C_0 \cosh pz \cos pz + C_1 \cosh pz \sin pz + C_2 \sinh pz \sin pz + C_3 \sinh pz \cos pz \quad (51)$$

where p is same as expressed in eqn. (2)

Expressing the above in terms of Puzrevsky function (Karnovsky and Lebed, 2001), eqn. (51) can be expressed as

$$u = C_0 V_0(pz) + C_1 V_1(pz) + C_2 V_2(pz) + C_3 V_3(pz) \quad (52)$$

where,

$$V_0(pz) = \cosh pz \cos pz\,;\ V_1(pz) = (\cosh pz \sin pz + \sinh pz \cos pz)/\sqrt{2} \quad (53)$$

$$V_2(pz) = \sinh pz \sin pz\,;\ V_3(pz) = (\cosh pz \sin pz - \sinh pz \cos pz)/\sqrt{2}$$

Puzrevsky function has some unique functional properties as defined below and shall be used for subsequent analysis for derivation of the stiffness, damping and mass of the piles.

$$V_0(0) = 1; V_0'(0) = 0; V_0''(0) = 0; V_0'''(0) = 0$$

$$V_1(0) = 1; V_1'(0) = p\sqrt{2}\,; V_1''(0) = 0; V_1'''(0) = 0$$

$$V_2(0) = 1; V_2'(0) = 0; V_2''(0) = 2p^2\,; V_2'''(0) = 0$$

$$V_3(0) = 1; V_3'(0) = 0; V_3''(0) = 0; V_3'''(0) = 2\sqrt{2}p^3$$

$$V_3'(pz) = p\sqrt{2}V_2(pz)\,;\ V_2'(pz) = p\sqrt{2}V_1(pz)\,;$$

$$V_1'(pz) = p\sqrt{2}V_0(pz)\,;\ V_0'(pz) = p\sqrt{2}V_3(pz) \quad (54)$$

For solution of the short pile, the model used is shown in Figure 4.

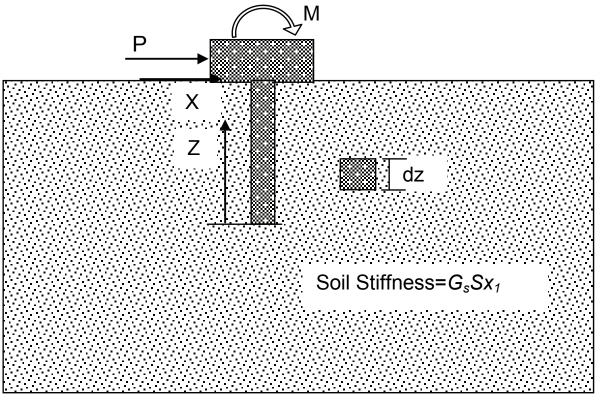

Fig. 4 Conceptual Model of Short Pile Under

For analysis similar to previous case, the pile is assumed fixed at the base and is also fixed at pile cap level and can undergo deflection and rotation at the pile head. Considering base of pile as z = 0, as shown in Figure 4, one can write, at *z = 0, u = 0* $\Rightarrow C_0 = 0$; and at *z = 0,* u' *= 0* $\Rightarrow C_1 = 0$ which gives

$$u = C_2V_2(pz) + C_3V_3(pz) \tag{55}$$

at the pile head, *z = L, u = 1* which gives

$$C_2V_2(pL) + C_3V_3(pL) = 1 \tag{56}$$

again at, *z = L*, u' *= 1* which gives

$$C_2V_2'(pL) + C_3V_3'(pL) = 1 \tag{57}$$

Using the derivative properties as shown above we have

$$C_2V_1(pL) + C_3V_2(pL) = \frac{1}{pL\sqrt{2}} \tag{58}$$

Expressing the above in matrix form, these conditions may be written as

$$[V][C] = [p] \tag{59}$$

which may be expressed as

$$\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \frac{1}{\Delta}\begin{bmatrix} V_2(pL) & -V_3(pL) \\ -V_1(pL) & V_2(pL) \end{bmatrix}\begin{Bmatrix} 1 \\ \dfrac{1}{pL\sqrt{2}} \end{Bmatrix} \tag{60}$$

where $\Delta = V_2^2(pL) - V_1(pL)V_3(pL)$ which gives

$$C_2 = \frac{1}{\Delta}\left[V_2(pL) - \frac{V_3(pL)}{pL\sqrt{2}}\right] \text{ and } C_3 = \frac{1}{\Delta}\left[\frac{V_2(pL)}{pL\sqrt{2}} - V_1(pL)\right] \quad (61)$$

Thus the displacement for the given boundary condition may be expressed as

$$u = \frac{1}{\Delta}\left[V_2(pL) - \frac{V_3(pL)}{pL\sqrt{2}}\right]V_2(pz) + \frac{1}{\Delta}\left[\frac{V_2(pL)}{pL\sqrt{2}} - V_1(pL)\right]V_3(pz) \quad (62)$$

Considering the fact that for a long pile, the shape function remains invariant for rocking mode with respect to lateral motion. For same boundary condition, it may be concluded that for a short pile also, the same condition would hold good. Thus, the shape function in dimensionless form in the rocking mode may be written as

$$\phi(z) = \frac{1}{\Delta}\left[V_2(\beta) - \frac{V_3(\beta)}{\beta\sqrt{2}}\right]V_2\left(\frac{\beta z}{L}\right) + \frac{1}{\Delta}\left[\frac{V_2(\beta)}{\beta\sqrt{2}} - V_1(\beta)\right]V_3\left(\frac{\beta z}{L}\right) \quad (63)$$

where the determinant Δ gets modified to, $\Delta = V_2^2(\beta) - V_1(\beta)V_3(\beta)$.

Considering $A = C_2/\Delta$ and $B = C_3/\Delta$, the shape function can now be expressed as

$$\phi(z) = AV_2\left(\frac{\beta z}{L}\right) + BV_3\left(\frac{\beta z}{L}\right) \quad (64)$$

Typical shape function for the short piles for E_p/G_s = 2500 is shown in Figure 5.

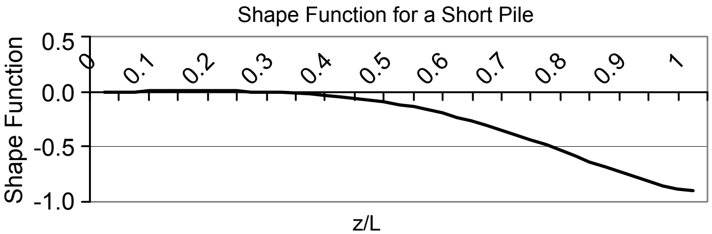

Fig. 5 Shape Function of Short Pile for E_p/G_s=2500

Differentiating eqn. (64) using the simplifications mentioned earlier, one can write

$$\phi'(z) = \frac{\beta\sqrt{2}}{L}\left[AV_1\left(\frac{\beta z}{L}\right) + BV_2\left(\frac{\beta z}{L}\right)\right] \quad (65)$$

Substituting the above functions in eqn. (8), the stiffness reduces to

$$K_\theta = \frac{2E_pI_p\beta^2}{L^2}\int_0^L\left[AV_1\left(\frac{\beta z}{L}\right)+BV_2\left(\frac{\beta z}{L}\right)\right]^2 + G_s r_0^2 S_{\theta 1}\int_0^L\left[AV_2\left(\frac{\beta z}{L}\right)+BV_3\left(\frac{\beta z}{L}\right)\right]^2 \quad (66)$$

The above is too complicated to solve in closed form and numerical integration may be used to arrive at a solution.

Considering $\xi = z/L$ and $L.d\xi = dz$; as $z \to L;\ \xi \to 1;$ as $z \to 0\ \xi \to 0;$ which gives

$$K_\theta = \frac{2E_pI_p\beta^2}{L^2}\int_0^1\left[AV_1(\beta\xi)+BV_2(\beta\xi)\right]^2 Ld\xi + G_s r_0^2 S_{\theta 1}\int_0^L[AV_2(\beta\xi)+BV_3(\beta\xi)]^2 Ld\xi \quad (67)$$

Substituting the value of Ψ in eqn. (67) it reduces to

$$K_\theta = G_s r_0^2 S_{\theta 1} L\left[\frac{2}{\psi}I_1 + I_2\right] \quad (68)$$

In which ψ is as defined in eqn (17a).

where,

$$I_1 = \int_0^1\left[AV_1(\beta\xi)+BV_2(\beta\xi)\right]^2 d\xi \text{ and } I_2 = \int_0^1\left[AV_2(\beta\xi)+BV_3(\beta\xi)\right]^2 d\xi \quad (69)$$

The integrals I_1 and I_2 can be solved by using Simpson's 1/3rd rule between limits 0-1 and can be back substituted in eqn. (68) to arrive at a stiffness value for the short pile.

As there is no theoretical or experimental benchmarking against which the stiffness values can be checked or compared, the use of this expression must always be backed up by dynamic field test of the piles to adjust the data (specially S_{x1} or E_p/G_s) to match the field observed value. The procedure is described as given hereunder.

Start the design with the following suggestive values of S_{x1} for various E_p/G_s values of Table-4.

Table 4 Suggested for S_{x1} for Short Piles (L/r ≤ 20) for Field Data Iteration

E_p/G_s	$S_{x1}(\nu=0.25)$	$S_{x1}(\nu=0.4)$	$S_{x1}(\nu=0.5)$
250	15.563	16.561	17.197
500	21.046	22.468	23.3.72
1000	27.873	29.860	31.135
2500	39.05	42.041	43.976
5000	49.07	53.014	55.576
10000	60.187	65.311	68.598

The values mentioned above, is based on formulation for long pile (with *L/r* <25), but may be used as a starting point for the iteration based on field observed data.

The mass moment of inertia of the pile for fundamental mode is given by

$$J_x = \frac{\gamma_p A_p r_0^2}{4g}\int_0^L \phi(z)^2 dz + \frac{\gamma_p A_p L^2}{g}\int_0^L \left(\frac{z}{L}\right)^2 \phi(z)^2 dz \tag{70}$$

$$J_x = \frac{\gamma_p A_p r_0^2 L}{4g} I_2 + \frac{\gamma_p A_p L^3}{g} I_3 \tag{71}$$

where

$$I_2 = \int_0^1 \left[AV_2(\beta\xi) + BV_3(\beta\xi)\right]^2 d\xi \text{ and } I_3 = \int_0^1 \xi^2 \left[AV_2(\xi) + BV_3(\xi)\right]^2 d\xi .$$

Finally,

$$J_x = \frac{M_p r_0^2}{4}\left[I_2 + 4\lambda^2 I_3\right] \tag{72}$$

where

$$M_p = \frac{\gamma_p A_p L}{g}$$

To start the design a value of $S_{\theta 1}$ is selected for the specific of E_p/G_s from Table-4 and the value of the frequency based on eqns. (68) and (72) is find out.

Let this be defined as ω_c, where the subscript *c* stands for the word "computed".

Let the field-tested natural frequency of the pile be ω_f, where $\omega_f \neq \omega_c$.

In most of cases it is seen (Prakash and Puri, 1995) that the field observed frequency value is different from the computed one and is usually less by about 30-40%. This is logical, for when the pile is bored or driven, the soil gets displaced and clayey soil due to its thixotropic property looses a part of its shear strength thus resulting in reduced dynamic shear modulus compared to the value observed during geotechnical investigation. Also, there could be cases when the field observed values be more than the computed one, especially in sandy soils, where the soil gets densified due to pile driving. The bottom line is that in very rare cases the computed and observed values would match.

Based on the above argument, the error (ε) in the analysis is given by $\varepsilon = \omega_c - \omega_f$. For $\varepsilon \to 0$ one can have, $\omega_c = \omega_f$ → $\omega_c^2 = \omega_f^2$.

using $\omega_c^2 = \frac{K}{M_x}$, it reduces to

$$\frac{4G_s S_{\theta 1} L}{M_p}\left[\frac{\frac{2}{\psi}I_1 + I_2}{I_2 + 4\lambda^2 I_3}\right] - \omega_f^2 = 0 \qquad (73)$$

Here ψ is as defined in eqn (17a).

It will be observed that all the factors β, I_1, I_2, I_3 in eqn.(73) is a function of E_p/G_s. The difference (which is the error ε) can now be set to zero or minimum by varying the value of E_p/G_s for which lim $\varepsilon \rightarrow 0$.

This can very easily be done by using the standard solver or goal seek in a spread sheet with boundary constraint that $S_{\theta 1} > 0$.

The above will automatically revise the value of E_p/G_s and upgrade the values of I_3, I_2 and I_1 (which are dimensionless functions), which may then be used to calculate the revised and exact stiffness and mass contribution of the pile which would closely simulate the field condition. The method has been explained in detail in Chowdhury and Dasgupta (2008).

Having established the mass and stiffness coefficients of the pile correctly based on field data, the damping may be established as

$$C_\theta = r_0^3 \sqrt{\rho G_s}\ S_{\theta 2} L I_2 . \qquad (74)$$

Here $S_{\theta 2}$ may be obtained from equation (28) after calculating the dimensionless frequency number $a_0 = \omega r_0 / v_s$

Group Effect of Piles

The formulation given in the preceding (for both long and short pile) is valid for single pile needs to be modified to consider the group effect when K_{group} is not necessarily $\sum_{i=1}^{n} K_{xi}$ where n = number of piles in a group. For such cases, the method proposed by Poulos (1971) is possibly the best-suited technique and can be used to modify the total stiffness of a pile group having n number of piles. Accordingly

$$K_{group} = \sum_{i=1}^{n} K_{xi} \Big/ \sum_{i=1}^{n} \alpha_{xi} \qquad (75)$$

where α_{xi} are the interaction factors provided by Poulos (1971).

Similarly the damping of the pile group may be obtained from the expression

$$C_{group} = \sum_{i=1}^{n} C_{xi} \Big/ \sum_{i=1}^{n} \alpha_{xi} \quad (76)$$

Effect of Pile Cap on Pile Stiffness

The pile cap has been found to affect the response of footing significantly. Before considering its effect within the proposed framework, it would be worthwhile to recapitulate the practice in vogue.

The sketch given in Figure 6 can represent the pile group with pile cap.

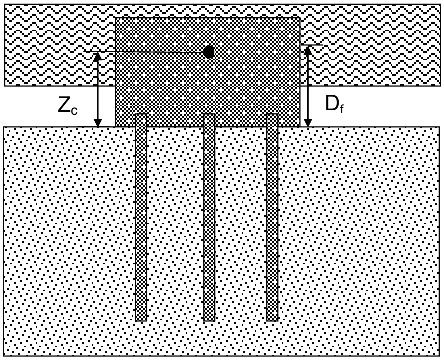

Fig. 6 Schematic Diagram of Pile and Pile-Cap with Embedment.

In such case usually the embedment stiffness $G_sS_fD_f$ is added to the pile group stiffness and the system is considered as a lumped mass single degree freedom system, the details of which are furnished in Novak(1974) and Prakash and Puri (1988).

In conventional formulation as the stiffness matrix is statically coupled, another set of stiffness $K_{x\theta}$ needs to be derived in addition to what has been derived above. To circumvent this issue, use of the following model is proposed as mentioned hereunder.

The Lagrange equation (Thompson, 1984) from the energy principle may used to derive the governing equations as follows

$$d(T+U) = \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right]dq_i = 0 \quad (77)$$

where

$T = f(q_1, q_2, q_3 ....q_n; \dot{q}_1, \dot{q}_2, \dot{q}_3, ...\dot{q}_n)$ and

$U = f(q_1, q_2, q_3, ...q_n)$.

where q's are the generalized coordinates, and dots indicate their time derivatives.

The kinetic energy, T for the system using the mathematical model of Figure 7, is given by

$$T = \frac{1}{2}M_x\dot{u}^2 + \frac{1}{2}J_x\dot{\theta}^2 + \frac{1}{2}M_f(\dot{u} + \dot{x} + Z_c\dot{\theta})^2 \tag{78}$$

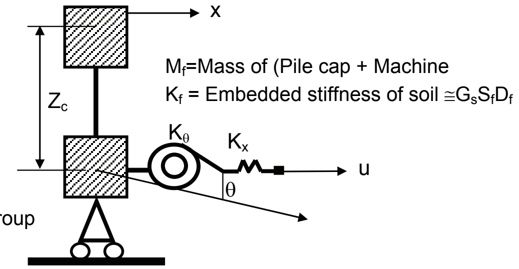

Fig. 7 Mathematical Model of Pile Group and Pile Cap under Coupled Sliding and Rocking Mode

The potential energy *U* is

$$U = \frac{1}{2}K_x u^2 + \frac{1}{2}K_\theta \theta^2 + \frac{1}{2}K_f x^2 \tag{79}$$

Differentiating

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = M_x\ddot{u} + M_f\left(\ddot{u} + \ddot{x} + Z_c\ddot{\theta}\right); \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = J_x\ddot{\theta} + M_f Z_c\left(\ddot{u} + \ddot{x} + Z_c\ddot{\theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = M_f\left(\ddot{u} + \ddot{x} + Z_c\ddot{\theta}\right) \tag{80}$$

and for potential energy

$$\frac{\partial U}{\partial u} = K_x U; \frac{\partial U}{\partial \theta} = K_\theta \theta \;\; and \;\; \frac{\partial U}{\partial x} = K_h x \tag{81}$$

where K_θ = rotational stiffness of pile group; *U* = potential energy of the system;

T = kinetic energy of the system.

Substituting the above values in eqn. (77) and writing in matrix form the governing equation may be deduced as

$$\begin{bmatrix} M_f & M_f & M_f Z_c \\ M_f & M_f + M_x & M_f Z_c \\ M_f Z_c & M_f Z_c & J_x + M_f Z_c^{\,2} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} K_f & 0 & 0 \\ 0 & K_x & 0 \\ 0 & 0 & K_\theta \end{bmatrix} \begin{Bmatrix} x \\ u \\ \theta \end{Bmatrix} = 0 \tag{82}$$

Eqn. (82) gives the complete free vibration equation of motion for pile plus pile cap with machine considering pile springs in translation and rocking modes.

Considering the equation to be dynamically coupled, the damping matrix can now be expressed as

$$[C] = \begin{bmatrix} C_f & 0 & 0 \\ 0 & C_x & 0 \\ 0 & 0 & C_\theta \end{bmatrix} \tag{83}$$

in which, M_f = mass of pile cap plus mass of machine; M_x = mass of pile group; J_x = mass moment of inertia of piles; Z_c= center of mass of foundation plus machine along vertical axes; K_h = lateral embedded stiffness of pile cap = $G_s\ S_{fx}\ D_f$; G_s = dynamic shear modulus of soil; S_{fx} = Berdugo's constant = 3.6,4, 4.1 for ν = 0.0, 0,25, 0.4 respectively; D_f = depth of embedment; K_x= lateral stiffness of pile group where

$$K_X = \frac{E_p I_p}{L^3}\left[\frac{\frac{5X}{4}(1-e^{-2\beta}) - \frac{3Y}{8} - \frac{3}{4}\eta}{(\eta-1)^3}\right] \tag{84}$$

Equation (84) has been derived in detail in Chowdhury and Dasgupta (2008).

It is to be noted that for calculations, the mass and mass moment of inertia of the pile group pile group, the mass and mass moment of inertia the mass and inertia of single pile has to be multiplied by the number of piles in the group. While for stiffness and damping the group stiffness and damping has to be derived according to eqns. (75) and (76).

Comparison of Results

The method proposed herein can very well be used for dynamic analysis of piles under horizontal force. However, the sanctity of the same will depend on how accurately the stiffness values have been evaluated. For this two RCC piles of radius 0.4m, 1.0m of length 40m has been has been checked with the reported results for comparison. The values K_θ [eqn. (16)] is shown in Figures. 8 and 9 for comparison.

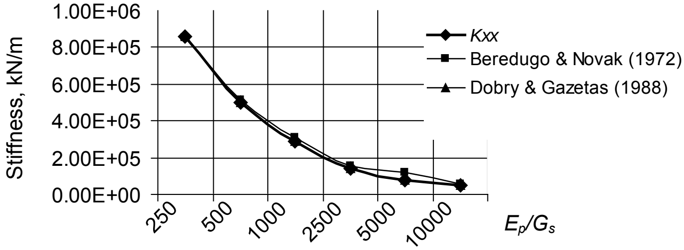

Fig. 8 Comparison of Stiffness Values for *R* =0.5m and Length=40 Meter

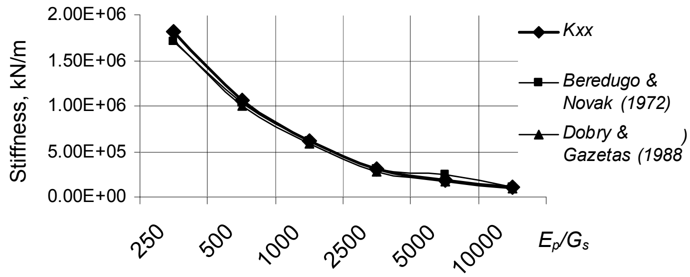

Fig.9 Comparison of Stiffness Values for *R* =1.0m and Length=40 Meter

The results clearly show that the values are in very good agreement for the base case and thus can well be used for other cases as mentioned above for which there are no direct solutions.

Finally the stiffness of a short pile is calculated, based on field observed data having the following properties.

1. Length of Pile= 10m, Diameter of pile=1.2 meter. Material of pile: RCC.
2. Method of installation- Bored Pile
3. Based on soil test, observed E_p/G_s =5000.
4. E_p considered $\cong 3\text{x}10^7$ kN/m^2.
5. Density of pile material =25 kN/m^3.
6. Field observed natural frequency of the pile is = 28 rad/sec (4Hz).
7. Poisson's ratio of soil considered = 0.4.

For the above conditions

Selected value of $S_{\theta 1}$ from Table-3 = 53.014

E_p/G_s = 5000 (given); β = 5.681. vide eqn. (5)

A = -0.000912 vide eqn. (64)

B = -0.003447 do

I_1 = 0.0277902, I_2 = 0.201259, I_3 = 0.16886 vide eqn. (72)

Computed Stiffness =7.78 x 10^5 kN/m

Computed natural frequency $\left(\sqrt{\dfrac{K_\theta}{J_x}}\right)$ = 39.96 rad/sec(6 Hz); Error(ε)=11.96

Setting the error(ε)=0 and running the goal seek/solver function in a spread sheet for changing E_p/G_s for boundary constraint $S_{\theta 1}$>0, we have the following upgraded data:

S_{x1} = 53.014; E_p/G_s = 3357; β = 6.2762; A = -0.0009; B = -0.00528

I_1 =0.00113, I_2 = 0.13898, I_3 =0.1168181.

Computed natural frequency based on above data=28 rad/sec (4Hz)

Revised Error (ε)=-0.000826

Thus based on the above data as per eqn. (68) the correct stiffness of the pile is given by K_{pile}=2.64 x 10^5 kN/m.

In case the above correction is already done for lateral pile stiffness and E/G value has been already modified to agree with the field observed data, the same can directly be used without carrying out the above mentioned modification again.

Discussions on the Results

Referring to Figures. 6 and 7 it is observed that the results are in excellent agreement with both Beredugo and Novak (1972) and Dobry and Gazetas (1988) stiffness values. Considering the base case being in such agreement, the other formulations can now be very easily adapted for which there are no standard solutions available.

The short pile case is basically a theoretical solution and needs significant field test and lab testing to arrive at a predefined $S_{\theta 1}$ value, which would make the method more powerful.

However in absence of such data the present algorithm as mentioned herein could become a very powerful tool for dynamic analysis of such piles for which no solution is available till date and yet remains a serious practical problem.

The method proposed would be valid for such piles while reported methods do not provide with any coefficients for piles with (L/r < 25).

Conclusion

A comprehensive analytical solution for dynamic analysis of long piles is presented herein, which is in very good agreement with the established formulation. Based on this, piles with boundary conditions like partially embedded and variation of G with depth can also be analyzed.

Considering the fact that the dynamic bending moment and shear force can also be obtained by this method, the standard practice of restricting the pile capacity to 50% of its capacity may be relaxed. It will be observed vide eqns. (45) and (47) that the moment and shear takes care of the dynamic magnification factor of the load as well as gives the complete distribution of the magnitudes along depth of the pile. This when combined with static load would give the complete design moment of the pile. Considering there is no uncertainty with this formulation, we can perhaps restrict the pile load limit to a higher value than 50% as in vogue presently. Short piles for which no established method exists can also be solved by the method presented herein.

Notations

The following symbols are used in this paper:

A_p = cross sectional area of the pile;

a_0 = dimensionless frequency number $\omega r_0/v_s$;

C_c = critical damping of the pile = $C_c = 2\sqrt{KJ_x}$;

C_p = damping for the pile;

$C_{\theta 1}$ $S_{\theta 1}$, and S_{u1} = Berdugo's constants which are basically frequency dependent

$C_{\theta 2}$, $S_{\theta 2}$ and S_{x2} = Berdugo's constants (frequency independent)

D_f = depth of embedment;

E_p = Young's modulus of pile;

G_s = dynamic shear modulus of the soil;

g = acceleration due to gravity;

G_b = dynamic shear modulus of soil at the foundation base;

$G' = G_s(x/L)^m$ = dynamic shear modulus of the soil, where m= a number varying from 0-2 [considered 0 when G_s is constant with depth, assumed 1 for linear variation and 2 for parabolic distribution];

Ip = moment of inertia of the pile cross section;

k_s = elastic stiffness of the soil and is expressed as $GS_{\theta 1}$;

K_θ = foundation stiffness in horizontal direction;

K and J_x = stiffness and mass moment of inertia of the pile;

L= length of the pile;

L_1= length of the embedment in soil;

m_p = mass of the pile;

M_z = mass variation along the pile length;

P_0 = amplitude of dynamic force;

r_0 = radius of the foundation;

$r = \omega_m/\omega_n$ = the ratio of operating and natural frequency;

$S_{\theta 1}$= Beredugo's constant can be taken as frequency independent;

u = displacement of the pile in the x direction;

$V(z)$= Puzrevsky functions;

α = Embedment ratio L/L_1;

γ_p = unit weight of the pile material;

θ = rotation of pile in the x-z plane and about horizontal axis = $\phi(z)\ q(t)$;

ζ = damping ratio;

Π = total potential energy over the length of the pile, L;

ρ = mass density of soil;

$\phi(z)$ = shape function;

ω_m = operating frequency of the machine;

ω_c = computed natural frequency of the pile;

ω_f = field-tested natural frequency of the pile.

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