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Dynamic Analysis of Piles under Lateral Loading

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Introduction

Vibration of piles under lateral or horizontal load is an important study for the pile supporting rotating machines and also structures under earthquake loading. In the majority of cases it has been found that of all the modes (like vertical, rocking, yawning, twisting etc.); the lateral vibration (coupled with rocking) is the most critical and often governs the design. Thus a study of such motion is of paramount importance for piles supporting important installations facilities in earthquake prone areas.

A number of researchers namely, Parmelee et al. (1964), Tajimi (1966), Penzien (1970), Novak et al. (1974, 1983), Banerjee and Sen (1987), Dobry and Gazetas (1988) only to name the notable ones, have proposed solutions to this problem. Of these solutions, Novak's method is the most popular in design offices for its simplicity in application. However, the method does not address a number of issues and has the following limitations.

- The method is coefficient based [function of the ratio of Young's modulus of pile (E_p), and dynamic shear modulus of soil (G_s)], as such for intermediate values one has to interpolate which may not always be very accurate.
- The values are given for Poisson's ratio of 0.25 and 0.40 only. Thus for any value between 0.25 and 0.40, or beyond 0.40 another set of linear interpolation/ extrapolation is necessary.
- Novak and El Sharnouby (1983) has given stiffness and damping coefficients for soil having parabolic profile but in many cases the variation is linear and no coefficients are available for this case.
- The method does not have a solution for partially embedded piles, which is of great practical importance for piles driven in arctic condition (for example, Northern Siberia which has large number of Oil and Gas facilities).
- The dynamic bending moment and shear force induced on pile cannot be evaluated.

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> Finally the formulation is valid for long piles (i.e. the failure takes place in the pile body before soil yields) and do not cater to short piles.

The simplified formulas given by Dobry and Gazetas (1988) are based on more rigorous analysis, but do not address the issues of partial embedment, dynamic bending moment and shear, short piles (i.e. L/r < 25), etc.

The Proposed Method

In the present paper an analytical solution has been proposed, which overcomes many of the limitations mentioned above and also arrive at a formulation making the design procedure independent of charts and coefficients and thus easily amenable to analysis using simple spreadsheet programs.

The present analysis is an extension in lateral direction of the theory as proposed by Chowdhury and Dasgupta (2006) for piles under vertical vibration.

Most of the work on dynamic analysis of pile is based on Baranov's (1967) solution on embedded foundation. The present formulation is based on Novak and Beredugo's (1972) usage of the above for a rigid cylinder embedded in elastic half space. Shown in Figure 1 is a pile embedded in homogeneous elastic medium and considered under plane strain condition.

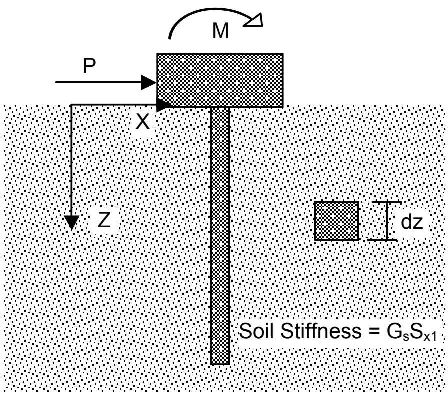

Fig. 1 Conceptual Model of Pile under Lateral Load

The pile is considered long and slender, to start with. Under static conditions, the equation of equilibrium in the x-direction [similar to beams on elastic foundation] is given by Timoshenko (1956)

$$E_p I_p \frac{d^4 x}{dz^4} = -k_s x \tag{1}$$

where, Ep = Young's modulus of the pile; Ip = moment of inertia of the pile cross section; k_s = elastic stiffness of the soil and is expressed as $G_s S_{x1}$; G_s = dynamic shear modulus of the soil; S_{x1} = Beredugo's constant which are basically frequency dependent.

However, it has been shown by Novak and Beredugo (1972) that considering this term frequency independent, no accuracy is lost for practical design problems and the analysis becomes quite simplified for rigid circular embedded footing. Elaboration about this parameter, in terms of piles, will be made later.

The general solution of Eqn.(1) may be written as

$$x = e^{-pz}(C_0 \cos pz + C_1 \sin pz) + e^{pz}(C_2 \cos pz + C_3 \sin pz) \tag{2}$$

where, $p = \sqrt[4]{\dfrac{G_s S_{x1}}{E_p I_p}}$. (2a)

For long piles under load or moment at its head, it is reasonable to assume that at significant distance from the pile head (where the load is applied), the curvature vanishes. This condition can only be satisfied when C_2 and C_3 in Eqn. (2) is considered insignificant. Hence the deflection equation can be taken as

$$x = e^{-pz}(C_0 \cos pz + C_1 \sin pz) \tag{3}$$

Considering the pile head undergoing specified deflection and rotation as well as its head is fixed to the pile cap (same boundary condition as considered by Novak (1974)), one can have [Figure 1], at z = 0, let x = $x_0 \Rightarrow C_0 = x_0$, which gives,

$$x = e^{-pz}(x_0 \cos pz + C_1 sin\ pz) \tag{4}$$

Again, at z = 0, $\dfrac{dx}{dz} = \theta_0$ one can have

$$C_1 = x_0 + \frac{\theta_0}{p} \tag{5}$$

Thus Eqn. (4) can now be represented as

$$x = e^{-pz}\left(x_0 \cos pz + \left(x_0 + \frac{\theta_0}{p}\right)\sin pz\right) \tag{6}$$

For magnitude of rotation being small $\theta_0 \approx x_0/L$, x may be written as

$$x = e^{-pz}\left(x_0 \cos pz + \left(x_0 + \frac{x_0}{pL}\right)\sin pz\right) \tag{6a}$$

$$\frac{x}{x_0} = e^{-pz}(\cos pz + \left(1 + \frac{1}{pL}\right)\sin pz) \tag{6b}$$

Now considering $\beta = pl$ and using Eqn. (6b), for any arbitrary loading, the generic shape function in dimensionless form can be represented as

$$\phi(z) = e^{\frac{-\beta z}{L}}(\cos\frac{\beta z}{L} + \left(1 + \frac{1}{\beta}\right)\sin\frac{\beta z}{L}) \tag{7}$$

in which

in which $\beta = \sqrt[4]{\frac{G_s S_{x1} L^4}{E_p I_p}}$, L being the length of the pile (7a)

Eqn. (7) can be further reduced to

$$\phi(z) = e^{\frac{-\beta z}{L}}(\cos\frac{\beta z}{L} + \eta\sin\frac{\beta z}{L}) \tag{8}$$

where

$$\eta = 1 + \frac{1}{\beta} \tag{9}$$

The generic shape function of the pile for the fundamental mode as in Eqn. (8) is shown in Figure 2 for a typical value of E_p/G_s=5000.

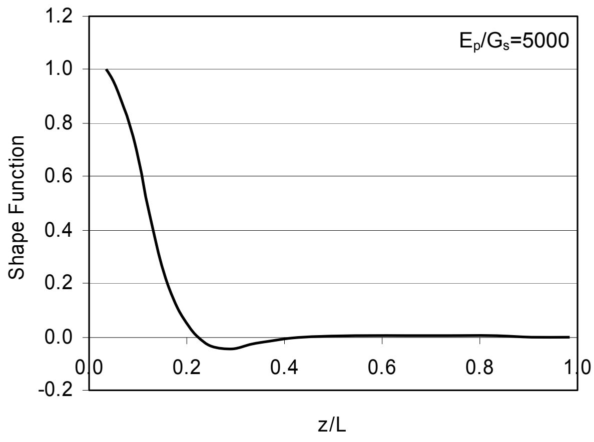

Fig. 2 Generic Shape Function Pile in the Horizontal Mode

The potential energy dΠ of an element of depth dz, shown in Figure 1, is then given by [Shames and Dym (1995)]

$$d\Pi = \frac{E_p I_p}{2}\left[\frac{d^2 v}{dz^2}\right]^2 + \frac{K_h}{2} v^2 \tag{10}$$

where, E_p = Young's modulus of pile; I_p = moment of inertia of pile; K_h = lateral dynamic stiffness of soil; v = displacement of the pile in the x direction and may be written as [ϕ(z) q(t)].

For a rigid circular embedded footing of embedment D_f, the stiffness of the footing may be expressed (Novak and Beredugo (1972)) as

$$K_h = G_b r_0 C_b + G_s D_f S_{x1} \tag{11}$$

where, K_h = foundation stiffness in horizontal direction; G_s = dynamic shear modulus of the soil along foundation surface; G_b = dynamic shear modulus of soil at the foundation base; r_0 = radius of the foundation; C_b and S_{x1} = constants which are basically frequency dependent.

Ignoring the first term in Eqn. (11) which represents the contribution of base resistance, and substituting the same in Eqn. (10), for a cylindrical element of depth dz, embedded in soil, the potential energy $d\Pi$ may be expressed as

$$d\Pi = \frac{E_p I_p}{2}\left[\frac{d_2 v}{dz^2}\right]^2 + \frac{G_s S_{x1} dz}{2} v^2 \tag{12}$$

The total potential energy over the length of the pile (L) is then given by

$$\Pi = \frac{E_p I_p}{2}\int_0^L \left[\frac{d^2 v}{dz^2}\right]^2 dz + \frac{G_s S_{x1}}{2}\int_0^L v^2 dz \tag{13}$$

Considering v(z,t) = ϕ(z) q(t), it can be proved (Hurty and Rubenstein (1967)) that,

$$K_{ij} = E_p I_p \int_0^L \varphi_i''(z)\varphi_j''(z)dz + G_s S_{x1}\int_0^L \varphi_i(z)\varphi_j(z)dz \tag{14}$$

where the shape function of the problem is given by Eqn. (8).

For the fundamental mode, stiffness of the pile is then given by

$$K = E_p I_p \int_0^L \varphi_i''(z)^2 dz + G_s S_{x1}\int_0^L \varphi_i(z)^2 dz \tag{15}$$

On double differentiation, Eqn. (8) reduces to

$$\phi''(z) = \frac{2\beta^2}{L^2} e^{\frac{-\beta z}{L}} (\sin\frac{\beta z}{L} - \eta\cos\frac{\beta z}{L}) \tag{16}$$

and

$$\phi''(z)^2 = \frac{4\beta^4}{L^4} e^{\frac{-2\beta z}{L}} \left(\frac{X}{2} - \frac{Y}{2}\cos\frac{2\beta z}{L} - \eta\sin\frac{2\beta z}{L}\right) \tag{17}$$

where,

$X = 1+\eta^2$; $Y = 1-\eta^2$ and η is given in Eqn. (9).

Again from Eqn. (8)

$$\phi(z)^2 = e^{\frac{-2\beta z}{L}} \left(\frac{X}{2} + \frac{Y}{2}\cos\frac{2\beta z}{L} + \eta\sin\frac{2\beta z}{L}\right) \tag{18}$$

Substituting Eqns. (17) and (18) in Eqn. (15), the stiffness reduces to

$$\begin{aligned} K = & \frac{4E_pI_p\beta^4}{L^4}\int_0^L e^{\frac{-2\beta z}{L}} \left(\frac{X}{2} - \frac{Y}{2}cos\frac{2\beta z}{L} - \eta sin\frac{2\beta z}{L}\right)dz \\ & + G_sS_{x1}\int_0^L e^{\frac{-2\beta z}{L}} \left(\frac{X}{2} + \frac{Y}{2}cos\frac{2\beta z}{L} + \eta sin\frac{2\beta z}{L}\right)dz \end{aligned} \tag{19}$$

Eqn. (19) on integration by parts and on simplification may be expressed as

$$\begin{aligned} K = & \frac{4E_pI_p\beta^4}{L^4}\left[\frac{X}{2}\times\frac{L}{2\beta}\left(1 - e^{-2\beta}\right) - \frac{Y}{2}\times\frac{L}{4\beta}\left[e^{-2\beta}\left(sin\,2\beta - cos\,2\beta\right) + 1\right]\right] \\ & - \frac{4E_pI_p\beta^4}{L^4}\left[\frac{\eta L}{4\beta}\left(1 - e^{-2\beta}\left(sin\,2\beta + cos\,2\beta\right)\right)\right] \end{aligned}$$

$$\begin{aligned} & + G_sS_{x1}\left[\frac{X}{2}\times\frac{L}{2\beta}\left(1 - e^{-2\beta}\right) + \frac{Y}{2}\times\frac{L}{4\beta}\left(e^{-2\beta}\left(sin\,2\beta - cos\,2\beta\right) + 1\right)\right] \\ & + G_sS_{x1}\left[\frac{\eta L}{4\beta}\left(1 - e^{-2\beta}\left(sin\,2\beta + cos\,2\beta\right)\right)\right] \end{aligned} \tag{20}$$

In Eqn. (20), $e^{-2\beta}(\sin 2\beta+\cos 2\beta)$ and $e^{-2\beta}(\sin 2\beta-\cos 2\beta)$ may be ignored as their values are exceedingly small(highest is of the order 10^{-3} and the lowest is 10^{-30} for E_p/G_s value varying from 250 to 10,000) and has practically no effect on the stiffness value and this also considerably simplifies the expression.

Based on the above simplification, Eqn. (20) may be rewritten as

$$K = \frac{4E_pI_p\beta^4}{L^4}\left[\frac{X}{2}\times\frac{L}{2\beta}\left(1-e^{-2\beta}\right)-\frac{Y}{2}\times\frac{L}{4\beta}-\frac{\eta L}{4\beta}\right]$$
$$+G_sS_{x1}\left[\frac{X}{2}\times\frac{L}{2\beta}\left(1-e^{-2\beta}\right)+\frac{Y}{2}\times\frac{L}{4\beta}+\frac{\eta L}{4\beta}\right] \tag{21}$$

$$\Rightarrow K = \frac{E_pI_p\beta^3}{L^3}\left[X\left(1-e^{-2\beta}\right)-\frac{Y}{2}-\eta\right]+\frac{G_sS_{x1}L}{4\beta}\left[X\left(1-e^{-2\beta}\right)+\frac{Y}{2}+\eta\right] \tag{22}$$

Taking $E_pI_p\beta^3/L^3$ as common in Eqn. (22) and substituting the value of β from Eqn. (7a), Eqn. (22) reduces to

$$K = \frac{E_pI_p\beta^3}{L^3}\left[\frac{5X}{4}\left(1-e^{-2\beta}\right)-\frac{3Y}{8}-\frac{3\eta}{4}\right]$$ which can be further simplified to

$$K = \frac{E_pI_p}{L^3}\left[\frac{\frac{5X}{4}(1-e^{-2\beta})-\frac{3Y}{8}-\frac{3}{4}\eta}{(\eta-1)^3}\right] \tag{23}$$

The accuracy of Eqn. (23) will be dependent on the correct selection of S_{x1}. For instance for rigid circular footing Novak and Berdugo (1972) has furnished a frequency independent value of S_{x1} = 4.0 to 4.1(depending on Poisson's ratio). This has been found to give adequate accuracy for practical engineering design. Comparing the stiffness data with Novak (1974) and Dobry and Gazetas (1988), it is proposed that the values of S_{x1} as furnished in Tables 1,2 and 3 be used for the calculation of dynamic response of the pile in the lateral direction.

The values of S_{x1} has been obtained by using similar technique used earlier by Lysmer and Richart (1966), for deriving equivalent stiffness and damping of circular footings for Lysmer's analog from a solution based on elastic half space theory as proposed by Bycroft(1956).

Table 1 Suggested Value of S_{x1} for different E_p/G_s value of soil with Poisson's Ratio = 0.25 (figures in the parentheses are E_p/G_s)

L/r_0 (Slenderness Ratio)	*S_{x1} (250)*	*S_{x1} (500)*	*S_{x1} (1000)*	*S_{x1} (2500)*	*S_{x1} (5000)*	*S_{x1} (10000)*
25	2.00	1.83	1.66	1.43	1.25	1.07
40	2.19	2.05	1.90	1.70	1.55	1.39
60	2.30	2.17	2.05	1.87	1.74	1.60
80	2.36	2.24	2.12	1.96	1.84	1.71
100	2.39	2.28	2.17	2.01	1.90	1.78

Table 2 Suggested Value of S_{x1} for different E_p/G_s value of soil with Poisson's Ratio = 0.40 (figures in the parentheses are E_p/G_s)

L/r_0 *(Slenderness Ratio)*	S_{x1} *(250)*	S_{x1} *(500)*	S_{x1} *(1000)*	S_{x1} *(2500)*	S_{x1} *(5000)*	S_{x1} *(10000)*
25	2.27	2.08	1.89	1.63	1.43	1.23
40	2.48	2.32	2.16	1.94	1.76	1.59
60	2.60	2.46	2.31	2.12	1.97	1.82
80	2.66	2.53	2.40	2.22	2.08	1.94
100	2.70	2.57	2.45	2.28	2.15	2.02

Table 3 Suggested Value of S_{x1} for different E_p/G_s value of soil with Poisson's Ratio = 0.50 (figures in the parentheses are E_p/G_s)

L/r_0 *(Slenderness Ratio)*	S_{x1} *(250)*	S_{x1} *(500)*	S_{x1} *(1000)*	S_{x1} *(2500)*	S_{x1} *(5000)*	S_{x1} *(10000)*
25	2.45	2.25	2.05	1.77	1.55	1.34
40	2.67	2.50	2.33	2.09	1.91	1.72
60	2.80	2.65	2.50	2.29	2.13	1.96
80	2.87	2.72	2.58	2.39	2.24	2.10
100	2.91	2.77	2.63	2.45	2.32	2.18

For a particular pile having specific slenderness ratio and Poisson's ratio of the soil, the value of S_{x1} can be selected from Table-1 and on substitution of the same in Eqn. (7a), Eqn. (23) gives the solution of pile stiffness in the lateral direction.

Estimation of Contribution of Pile Mass

The mass matrix of the pile may be expressed [Meirovitch(1967)] as

$$M_x = m(x)\int \phi_i(z)\phi_j(z)dz \tag{24}$$

For the present case of the pile of length L, Eqn. (24) can be expressed as

$$M_x = \frac{\gamma_p A_p}{g}\int_0^L \phi(z)^2\, dz \tag{25}$$

where, γ_p = unit weight of the pile material; A_p = cross sectional area of the pile; g = acceleration due to gravity.

or, $$M_x = \frac{\gamma_p A_p}{g}\int_0^L e^{\frac{-2\beta z}{L}}\left[\frac{X}{2}+\frac{Y}{2}\cos\frac{2\beta z}{L}+\eta\sin\frac{2\beta z}{L}\right] \qquad (26)$$

Eqn. (26) on integration and after simplification gives

$$M_x = \frac{\gamma_p A_p L}{4g}\left[\frac{X(1-e^{-2\beta})+\frac{Y}{2}+\eta}{\beta}\right] \qquad (27)$$

Eqn. (27) is the inertial contribution of the pile material for the fundamental mode. Incidentally, the inertial effect is usually ignored in design but could have significant effect if the number of piles is large in a pile group.

Radiation Damping for Pile under Lateral Load

Damping of the pile embedded in soil medium will be constituted of two parts: material damping of the pile itself and radiation damping of the soil. It is obvious that the material damping of the pile will be much lower than that of the soil radiation damping. Material damping of soil is also a part of the vibration system. However, it has been found that for translational vibration their effect is insignificant and may be neglected. Else if desired, their values may be obtained from resonant column test from the laboratory when the damping may be obtained from ratio of successive amplitudes. As the first step for calculating the total damping, one may ignore the material damping of the pile for the time being.

For a rigid footing embedded in soil for a depth D_f, Novak and Beredugo (1972) have proposed the expression

$$C_z = r_0\left(\sqrt{\rho_b G_b}\right)\bar{C}_b + r_0\left(\sqrt{\rho G_s}\right)\bar{S}_{x2}D_f \qquad (28)$$

where, r_0 = radius of the foundation; G_b = dynamic shear modulus at the foundation base; G_s = dynamic shear modulus of the soil in which the foundation is embedded; D_f = depth of embedment; $\bar{C}_b$ and S_{x2} = frequency independent constants as defined by Novak and Berdugo (1972).

With reference to Figure 1 for a pile element of length dz embedded in the soil, and ignoring the bearing effect, Eqn (28) may be expressed as

$$c(x) = r_0\left(\sqrt{\rho G_s}\right)S_{x2}dz \qquad (29)$$

For systems having continuous response function, the damping may be expressed as [Paz (1986)]

$$C_x = c(x)\int\phi_i(z)\phi_j(z)dz \qquad (30)$$

For the pile of length L, Eqn. (30) may be expressed as

$$C_x = r_0\left(\sqrt{\rho G_s}\right)S_{x2}\int_0^L \varphi(z)^2\,dz \tag{31}$$

or, $$C_x = r_0\left(\sqrt{\rho G_s}\right)S_{x2}\int_0^L e^{\frac{-2\beta z}{L}}\left[\frac{X}{2}+\frac{Y}{2}cos\frac{2\beta z}{L}+\eta sin\frac{2\beta z}{L}\right] \tag{32}$$

On integration and after simplification Eqn. (32) reduces to

$$C_x = r_0\left(\sqrt{\rho G_s}\right)S_{x2}L\left[\frac{X(1-e^{-2\beta})+\frac{Y}{2}+\eta}{4\beta}\right] \tag{34}$$

Eqn. (34) expresses the soil damping for a single pile under horizontal mode of vibration. The factor S_{x2} is a frequency dependent damping coefficient. Fortunately the damping factor is required for calculating the amplitude when the eigen solution of the problem is already done *vis a vis*, the dimensionless frequency number $a_0 = \omega r_0 / v_s$ term is known *a priori*. Polynomial fit curve for S_{x2} are available in terms of a_0 which can be used directly to obtain these parameters. S_{x2} for different Poisson's ratios are given in Table-4.

Table 4 Values of S_{x2} [Beredugo and Novak (1972)]

Poisson's ratio	S_{x2}
0.0	$7.334a_0 + \frac{0.8652a_0}{a_0 + 0.00874}$
0.25	$0.83a_0 + \frac{41.59a_0}{a_0 + 3.90}$
0.5	$0.96a_0 + \frac{56.559a_0}{a_0 + 4.68}$

Material Damping of Pile

The structural stiffness contribution of the pile is given in the first part of Eqn. (22), while that of the mass is given in Eqn. (27). Thus, if C_c be the critical damping of the pile then it can be expressed as $C_c = 2\sqrt{Km_p}$; K and m_p are being the stiffness and mass matrices of the pile.

Depending on the material used for pile (like RCC, steel etc.) a suitable damping ratio (ζ) can be assumed. The damping (C_p) for the pile can then be expressed as

$$C_p = \zeta C_c \tag{35}$$

This, when added to the radiation damping, calculated through Eqn. (34), gives the complete damping quantity for the soil-pile system.

Piles with other Boundary Conditions

Having established the stiffness, mass and damping of the pile in lateral direction based on minimization of the potential energy of the system, the above method can be extended for the piles with other boundary conditions for which there are no standard solutions available.

Partially Embedded Piles

In Arctic and North Siberian condition, due to environmental reasons, the steel piles are driven into the ground when they protrude about 2-3m above the ground over which the pile cap and vibrating equipments are placed. In such cases the existing solutions cannot be used. However, a solution of the same is proposed hereunder. Aschmatic representation of partially embedded pile is shown in Figure 3.

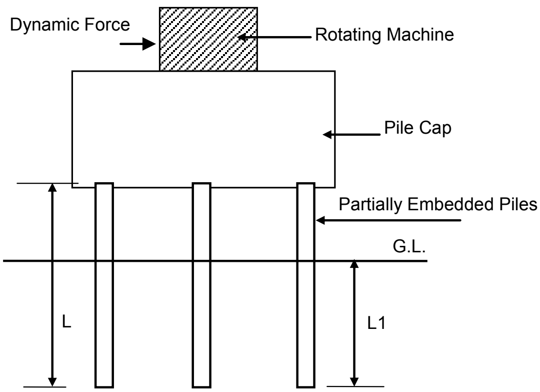

Fig. 3 Schematic Diagram of Partially Embedded Piles under Horizontal Load

Let L be the full length of the pile and the length of the embedment in soil be L_1.

For this case, one may write

$$\beta_e = \sqrt[4]{\frac{G_s S_{x_1} L_1^4}{E_p I_p}} \tag{36}$$

Here subscript “e” represents embedment of the pile.

The shape function can thus be represented by

$$\phi(z) = e^{\frac{-\beta_e z}{L_1}} (\cos\frac{\beta_e z}{L_1} + \eta \sin\frac{\beta_e z}{L_1}) \tag{37}$$

and $$\phi''(z) = \frac{2\beta_e^2}{L_1^2} e^{\frac{-\beta_e z}{L_1}} (\sin\frac{\beta_e z}{L_1} - \eta_e \cos\frac{\beta_e z}{L_1}) \tag{38}$$

and hence

$$\phi''(z)^2 = \frac{4\beta_e^4}{L_1^4} e^{\frac{-2\beta_e z}{L_1}} \left(\frac{X_e}{2} - \frac{Y_e}{2}\cos\frac{2\beta_e z}{L_1} - \eta_e \sin\frac{2\beta_e z}{L_1} \right) \tag{39}$$

where, $X_e = 1 + \eta_e^2$; $Y_e = 1 - \eta_e^2$ and $\eta_e = 1 + \frac{1}{\beta_e}$.

Now, considering the fact that the embedment of a pile does not have any effect on the shape function of the system (Timoshenko et. al. (1990)), the stiffness of the pile for the fundamental mode may be written as

$$K = E_p I_p \int_0^L \varphi_i^{2}(z)^2 dz + G_s S_{x_1} \int_0^{L_1} \varphi_i(z)^2 dz \tag{40}$$

Considering, $\alpha = L/L_1$, Eqn. (40) may be rewritten as

$$\begin{aligned} K = {} & \frac{4E_p I_p \beta_e^4}{L_1^4} \int_0^{\alpha L_1} e^{\frac{-2\beta_e z}{L1}} \left(\frac{X_e}{2} - \frac{Y_e}{2} cos\frac{2\beta_e z}{L_1} - \eta_e sin\frac{2\beta_e z}{L_1} \right) dz + \\ & G_s S_{x1} \int_0^{L_1} e^{\frac{-2\beta_e z}{L}} \left(\frac{X_e}{2} + \frac{Y_e}{2} cos\frac{2\beta_e z}{L_1} + \eta_e sin\frac{2\beta_e z}{L_1} \right) dz \end{aligned} \tag{41}$$

Eqn. (41) on integration by parts and after simplification, may be expressed

$$K = \frac{E_p I_p \beta_e^3}{L_1^3} \left[X_e\left(\frac{1}{4}+\alpha\right) + Y_e\left(\frac{1}{8}-\frac{\alpha}{2}\right) + \eta_e\left(\frac{1}{4}-\alpha\right) - X_e e^{-2\beta_e}\left(\frac{\alpha}{4}e^{-2\beta_e(1-\alpha)}+1\right) \right] \tag{42}$$

which can be further simplified to

$$K = \frac{E_p I_p}{L_1^3} \frac{\left[X_e\left(\frac{1}{4}+\alpha\right) + Y_e\left(\frac{1}{8}-\frac{\alpha}{2}\right) + \eta_e\left(\frac{1}{4}-\alpha\right) - X_e e^{-2\beta_e}\left(\frac{\alpha}{4}e^{-2\beta_e(1-\alpha)}+1\right) \right]}{(\eta_e - 1)^3} \tag{42a}$$

Eqn. (42a) gives the solution for stiffness of a partially embedded pile in the ground. The correctness of the equation can be back checked by the fact that when the pile becomes fully embedded i.e. for $L_1 = L$ $\alpha \rightarrow 1$, $\beta_e = \beta$, $X_e = X$ etc. when Eqn. (42a) degenerates to Eqn. (23).

Proceeding in an identical manner as done before, the mass and damping terms may be computed as

$$M_x = \frac{\gamma_p A_p L_1}{4g}\left[\frac{X_e\alpha(1-e^{-2\beta_e}) + \frac{Y_e\alpha}{2} + \eta_e\alpha}{1/(\eta_e - 1)}\right] \tag{43}$$

$$C_x = r_0\left(\sqrt{\rho G_s}\right)S_{x2}L_1\left[\frac{X_e(1-e^{-2\beta_e}) + \frac{Y_e}{2} + \eta_e}{4/(\eta_e - 1)}\right] \tag{44}$$

Pile Embedded in Soils with varying Elastic Property

In the previous section, the calculation of stiffness as well as the damping of soil was based on constant dynamic shear modulus of the soil. While this could be possible for clayey soils, there are cases where the dynamic shear modulus of the soil has been found to vary with depth. Generally this can be expressed as

$$G' = G_s(x/L)^m \tag{45}$$

where m = a number varying from 0-2 (considered 0 when G is constant with depth , assumed 1 for linear variation and 2 for parabolic distribution).

For a linearly varying soil the stiffness matrix can be written as

$$\begin{aligned} K = {} & \frac{4E_pI_p\beta^4}{L^4}\int_0^L e^{\frac{-2\beta z}{L}}\left(\frac{X}{2} - \frac{Y}{2}cos\frac{2\beta z}{L} - \eta sin\frac{2\beta z}{L}\right)dz \\ & + G_sS_{x1}\int_0^L\left(\frac{z}{L}\right)e^{\frac{-2\beta z}{L}}\left(\frac{X}{2} + \frac{Y}{2}cos\frac{2\beta z}{L} + \eta sin\frac{2\beta z}{L}\right)dz \end{aligned} \tag{46}$$

Integration of above and ignoring the terms containing the factor, $\beta e^{-2\beta}.cos2\beta$, $\beta.e^{-2\beta}sin2\beta$ etc., having extremely small contributions, Eqn. (46) reduces to

$$K = \frac{E_pI_p\beta^3}{L^3}\left[X\left(1-e^{-2\beta}\right) - \frac{Y}{2} - \eta\right] + \frac{G_sS_{x1}L}{4\beta^2}\left[X\left[1-e^{-2\beta}(1+\beta)\right] + \frac{3Y}{4} + \frac{\eta}{2}\right] \tag{47}$$

and can be further simplified to

$$K = \frac{E_p I_p \beta^3}{L^3}\left[X\left\{1 - e^{-2\beta}\left(1 + \frac{1}{4} + \frac{1}{4\beta}\right)\right\} - Y\left(\frac{1}{2} - \frac{3}{16}\beta\right) - \eta\left(1 - \frac{1}{8\beta}\right)\right] \quad (48)$$

The damping matrix for this case, proceeding in same manner as outlined earlier, can be represented by

$$C_x = \frac{r_0\left(\sqrt{\rho G_s}\right)S_{x2}L}{4\beta^2}\left[X\left[1 - e^{-2\beta}(1+\beta)\right] + \frac{3Y}{4} + \frac{\eta}{2}\right] \quad (49)$$

The mass coefficient remains the same as expressed in Eqn. (27).

When the dynamic shear modulus variation is parabolic with depth, the stiffness equation of the pile can be expressed as

$$K = \frac{4E_p I_p \beta^4}{L^4}\int_0^L e^{\frac{-2\beta z}{L}}\left(\frac{X}{2} - \frac{Y}{2}cos\frac{2\beta z}{L} - \eta sin\frac{2\beta z}{L}\right)dz + G_s S_{x1}\int_0^L\left(\frac{z}{L}\right)^2 e^{\frac{-2\beta z}{L}}\left(\frac{X}{2} + \frac{Y}{2}cos\frac{2\beta z}{L} + \eta sin\frac{2\beta z}{L}\right)dz \quad (50)$$

Eqn. (50) on integration and on subsequent simplification reduces to

$$K = \frac{E_p I_p \beta^3}{L^3}\left[X\left(1 - e^{-2\beta}\right) - \frac{Y}{2} - \eta\right] + \frac{G_s S_{x1} L}{4\beta}\left[X\left(\frac{1}{4\beta^2} - e^{-2\beta}\left(2 + \frac{1}{\beta} - \frac{2}{\beta^2}\right)\right)\right] \quad (51)$$

which can be further simplified to

$$K = \frac{E_p I_p \beta^3}{L^3}\left[X\left\{\left(1 + \frac{1}{16\beta^2}\right) - e^{-2b}\left(\frac{3}{2} + \frac{1}{4\beta} - \frac{1}{8\beta^2}\right)\right\} - \frac{Y}{2} - \eta\right] \quad (52)$$

Eqn. (52) gives the stiffness expression of pile under parabolic variation of G along the length of pile.

Proceeding in same manner as stated above the damping matrix may be expressed as

$$C_x = \frac{r_0\left(\sqrt{\rho G_s}\right)S_{x2}L}{4\beta}\left[X\left(\frac{1}{4\beta^2} - e^{-2\beta}\left(2 + \frac{1}{\beta} - \frac{2}{\beta^2}\right)\right)\right] \quad (53)$$

The mass coefficient remains the same as expressed in Eqn. (27).

Computation of Bending Moment and Shear Force

For machine foundation subjected to a lateral load of $P_0 \sin\omega_m t$, the amplitude of vibration is given by

$$v(t) = \frac{\frac{P_0}{K}\sin\omega_m t}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} \quad (54)$$

where, ω_m = operating frequency of the machine; P_0 = unbalanced dynamic force; r = ω_m/ω_n = the ratio of operating and natural frequency; ζ= damping ratio of the system.

Thus the peak amplitude is given by

$$v(t) = \frac{\frac{P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} \quad (55)$$

The complete displacement and bending moment functions are given by

$$v(z,t)= \frac{\frac{P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}\phi(z) \quad (56)$$

or

$$v(z,t)= \frac{\frac{P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} e^{\frac{-\beta z}{L}}\left(\cos\frac{\beta z}{L}+\eta\sin\frac{\beta z}{L}\right) \quad (57)$$

$$E_p I_p v'' = -M(z) = -\frac{\frac{E_p I_p P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}\frac{2\beta^2}{L^2}e^{\frac{-\beta z}{L}}\left(\sin\frac{\beta z}{L}-\eta\cos\frac{\beta z}{L}\right) \quad (58)$$

$$M_z = \frac{\frac{E_p I_p P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}\frac{2\beta^2}{L^2}e^{\frac{-\beta z}{L}}\left(\sin\frac{\beta z}{L}-\eta\cos\frac{\beta z}{L}\right) \quad (58a)$$

The maximum moment will be at the head i.e. at z = 0, and it can be expressed as

$$M_{\max} = \frac{\frac{2E_p I_p P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}\left(\frac{\beta(\beta+1)}{L^2}\right) \quad (59)$$

The shear force is given by

$$E_p I_p v''' = -V(z) = \frac{\frac{E_p I_p P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} \frac{2\beta^3}{L^3}\left[(\eta-1)\sin\frac{\beta z}{L}+(\eta+1)\cos\frac{\beta z}{L}\right] \quad \text{or}$$

$$V(z) = -\frac{\frac{E_p I_p P_0}{K}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} \frac{2\beta^3}{L^3}\left[(\eta-1)\sin\frac{\beta z}{L}+(\eta+1)\cos\frac{\beta z}{L}\right] \tag{60}$$

Dynamic Response of Short Piles in the Horizontal Mode

There are no solutions till date for this type of piles. Existing solutions are based on long piles with the implicit assumption that under ultimate load piles fail before the soil. However there are number of areas (e.g. Bonny river delta in Nigeria, where the topsoil constitute of very weak clay underlain by dense sand) where the soil will yield much before the pile. Broms (1965) has shown that the displacement curvatures for such piles are completely different than that of long piles.

While a long pile embedded in soil behaves as a semi-infinite beam on elastic foundation, a short pile behaves as a beam of finite length on elastic foundation. Bojtsov, et. al.(1982) has given solution to the generic displacement curves of such short beams on elastic foundation which is given by

$$x = C_0 \cosh pz \cos pz + C_1 \cosh pz \sin pz + C_2 \sinh pz \sin pz + C_3 \sinh pz \cos pz \tag{61}$$

where p is same as expressed in Eqn. (2a)

Expressing the above in terms of Puzrevsky function (Karnovsky and Lebed (2001)), Eqn. (61) can be expressed as

$$x = C_0 V_0(pz) + C_1 V_1(pz) + C_2 V_2(pz) + C_3 V_3(pz) \tag{62}$$

where,

$$V_0(pz) = \cosh pz \cos pz \tag{63}$$

$$V_1(pz) = \frac{1}{\sqrt{2}}(\cosh pz \sin pz + \sinh pz \cos pz) \tag{64}$$

$$V_2(pz) = \sinh pz \sin pz \tag{65}$$

$$V_3(pz) = \frac{1}{\sqrt{2}}(\cosh pz \sin pz - \sinh pz \cos pz) \tag{66}$$

Puzrevsky function, defined below, has some unique functional properties, which will be used for subsequent analysis for derivation of the stiffness, damping and mass of the piles.

$$V_0(0) = 1 ; V_0'(0) = 0 ; V_0''(0) = 0 ; V_0'''(0) = 0 \tag{67}$$

$$V_1(0) = 0 ; V_1'(0) = p\sqrt{2} , V_1''(0) = 0 ; V_1'''(0) = 0 \tag{68}$$

$$V_2(0) = 0 ; V_2'(0) = 0 ; V_2''(0) = 2p^2 , V_2'''(0) = 0 \tag{69}$$

$$V_3(0) = 0 ; V_3'(0) = 0 ; V_3''(0) = 0 ; V_3'''(0) = 2\sqrt{2}p^3 \tag{70}$$

$$V_3'(pz) = p\sqrt{2}V_2(pz) ; V_2'(pz) = p\sqrt{2}V_1(pz) ; \tag{71}$$

$$V_1'(pz) = p\sqrt{2}V_0(pz) ; V_0'(pz) = -p\sqrt{2}V_3(pz) \tag{72}$$

For a solution of the short pile one may use the model shown in Figure 4.

For the analysis similar to previous case, the pile may be assumed as fixed at the base and fixed at the pile cap level and can undergo deflection and rotation at pile head. Considering base of pile at z = 0 shown in Figure 4, one may write,

At z = 0, x = 0 $\Rightarrow C_0 = 0$ and at z = 0 x' = 0 $\Rightarrow C_1 = 0$ which gives,

$$x = C_2V_2(pz) + C_3V_3(pz) \tag{73}$$

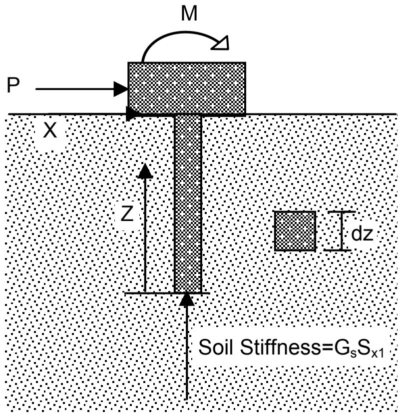

Fig. 4 Conceptual Model of Short Pile under Lateral Load

At the pile head, i.e. at z = L x = 1 gives

$$C_2V_2(pL) + C_3V_3(pL) = 1 \tag{74}$$

Again at z = L x' =1/L (as θ=δ/L) which gives

$$C_2V_2'(pL) + C_3V_3'(pL) = 1/\mathrm{L} \tag{75}$$

Using Eqns. (71) and (72), one may write

$$C_2V_1(pL) + C_3V_2(pL) = \frac{1}{pL\sqrt{2}} \tag{76}$$

The above may be expressed in matrix form as

$$[\mathbf{V}][\mathbf{C}] = [\mathbf{p}] \tag{77}$$

which can be further reduced to

$$[\mathbf{C}] = [\mathbf{V}]^{-1}[\mathbf{p}] \tag{78}$$

Performing the above operation gives

$$\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} V_2(pL) & -V_3(pL) \\ -V_1(pL) & V_2(pL) \end{bmatrix}\begin{bmatrix} 1 \\ \dfrac{1}{pL\sqrt{2}} \end{bmatrix} \tag{79}$$

where $\Delta = V_2^{\,2}(pL) - V_1(pL)V_3(pL)$ which implies

$$C_2 = \frac{1}{\Delta}\left[V_2(pL) - \frac{V_3(pL)}{pL\sqrt{2}}\right] \text{and } C_3 = \frac{1}{\Delta}\left[\frac{V_2(pL)}{pL\sqrt{2}} - V_1(pL)\right] \tag{80}$$

Thus, the displacement for the given boundary condition is then expressed as

$$x = \frac{1}{\mathit{\Delta}}\left[V_2(pL) - \frac{V_3(pL)}{pL\sqrt{2}}\right]V_2(pz) + \frac{1}{\mathit{\Delta}}\left[\frac{V_2(pL)}{pL\sqrt{2}} - V_1(pL)\right]V_3(pz) \tag{81}$$

Based on above considering β=pL, the generic shape function in dimensionless form is given by

$$\phi(z) = \frac{1}{\Delta}\left[V_2(\beta) - \frac{V_3(\beta)}{\beta\sqrt{2}}\right]V_2\left(\frac{\beta z}{L}\right) + \frac{1}{\Delta}\left[\frac{V_2(\beta)}{\beta\sqrt{2}} - V_1(\beta)\right]V_3\left(\frac{\beta z}{L}\right) \tag{82}$$

where the determinant Δ gets modified to $\Delta = V_2^{\,2}(\beta) - V_1(\beta)V_3(\beta)$.

Considering A = $\frac{1}{\Delta}\left[V_2(\beta) - \frac{V_3(\beta)}{\beta\sqrt{2}}\right]$ and B = $\frac{1}{\Delta}\left[\frac{V_2(\beta)}{\beta\sqrt{2}} - V_1(\beta)\right]$

the shape function can now be expressed as

$$\phi(z) = AV_2\left(\frac{\beta z}{L}\right) + BV_3\left(\frac{\beta z}{L}\right) \tag{83}$$

A typical shape function for the short piles E_p/G_s = 2500 is shown in Figure 5.

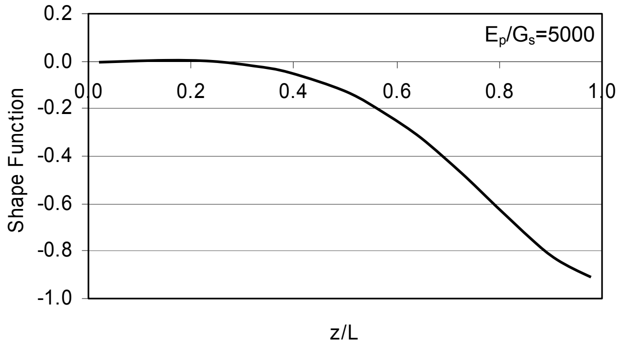

Fig. 5 Generic shape Function of Short Pile for E_p/G_s = 2500

Differentiating Eqn. (83) and using properties mentioned earlier one could have

$$\phi''(z) = \frac{2\beta^2}{L^2}\left[AV_0\left(\frac{\beta z}{L}\right) + BV_1\left(\frac{\beta z}{L}\right)\right] \tag{84}$$

Substituting the above functions in Eqn. (15), the stiffness can be expressed as

$$K = \frac{4E_pI_p\beta^4}{L^4}\int_0^L\left[AV_0\left(\frac{\beta z}{L}\right) + BV_1\left(\frac{\beta z}{L}\right)\right]^2 + GS_{x1}\int_0^L\left[AV_2\left(\frac{\beta z}{L}\right) + BV_3\left(\frac{\beta z}{L}\right)\right]^2 \tag{85}$$

Eqn. (85) is too complicated to solve in closed form as such numerical quadrature schemes may be used to obtain K.

Considering $\xi = \frac{z}{L}$ we have $L.d\xi = dz$ and as $z \to L;\ \xi \to 1;$ as $z \to 0\ \xi \to 0;$ which gives

$$K = \frac{4E_p I_p \beta^4}{L^4} \int_0^1 \left[AV_0(\beta\xi) + BV_1(\beta\xi)\right]^2 L d\xi + GS_{x1} \int_0^L \left[AV_2(\beta\xi) + BV_3(\beta\xi)\right]^2 L d\xi \quad (86)$$

Substituting the value of β [Eqn. (7a)] in Eqn. (86), the stiffness may be written as

$$K = GS_{x1} L[4 \int_0^1 \left[AV_0(\beta\xi) + BV_1(\beta\xi)\right]^2 d\xi + \int_0^1 \left[AV_2(\beta\xi) + BV_3(\beta\xi)\right]^2 d\xi] \quad (87)$$

➔ $$K = GS_{x1} L[4I_1 + I_2] \quad (88)$$

where

$$I_1 = \int_0^1 \left[AV_0(\beta\xi) + BV_1(\beta\xi)\right]^2 d\xi \text{ and } I_2 = \int_0^1 \left[AV_2(\beta\xi) + BV_3(\beta\xi)\right]^2 d\xi \quad (89)$$

The integrals I_1 and I_2 can very easily be solved by using Simpson's 1/3rd rule between limits 0-1 and can be back substituted in Eqn. (88) to compute the stiffness for the short pile.

However, one should note that there is no theoretical or experimental benchmarking against which the stiffness values can be checked or compared. So use of the expression *must always be backed up by dynamic field test of the piles* to adjust the data (especially S_{x1} or E_p/G_s) to match with the field observed values. In absence of comparative benchmarks the design may be initiated with the suggestive values of S_{x1} for various E_p/G_s given in Table-5.

Table 5 Suggested for S_{x1} for Short Piles (L/R ≤ 20) for Field Data Iteration

E_p/G_s	$S_{x1}(\nu=0.25)$	$S_{x1}(\nu=0.4)$	$S_{x1}(\nu=0.5)$
250	1.53	1.75	1.89
500	1.35	1.54	1.68
1000	1.17	1.34	1.46
2500	0.95	1.09	1.46
5000	0.95	1.09	1.46
10000	0.95	1.09	1.46

The values mentioned above, are based on the formulation for long pile (with L/r < 25) but may be used as a starting point for the iteration based on field observed data.

The mass of pile for the fundamental mode is given by

$$M_x = \frac{\gamma_p A_p}{g} \int_0^L \phi(z)^2 dz \text{ or } M_x = \frac{\gamma_p A_p L}{g} \int_0^1 \left[A V_2(\beta\xi) + B V_3(\beta\xi) \right]^2 d\xi \quad (90)$$

$$M_x = \frac{\gamma_p A_p L}{g} I_2 \quad (91)$$

To start the design a value of S_{x1} is selected for a specific Ep/Gs from Table-5 and found out the value of the frequency based on Eqns. (88) and (90). Let this be defined as ω_c where the subscript c stands for the word "computed". Let the field-tested natural frequency of the pile be ω_f , where $\omega_f \neq \omega_c$.

In most of cases it has been seen (Jadi. H (1999)) that the field observed frequency value deviates from the computed ones and is usually varies by about 30-40%. This is logical, for when the pile is bored or driven the soil gets displaced and clayey soil may loose a part of its shear strength thus resulting in reduced dynamic shear modulus compared to the value observed during geotechnical investigation. There could be cases where the field observed values might be more than the computed ones, especially in sandy soil where the soil gets densified due to pile driving. The bottom line is that in rare cases the computed and observed values would match very closely.

Based on the above argument the error(ε) in the analysis is then given by

$\varepsilon = \omega_c - \omega_f$ and for $\varepsilon \rightarrow 0$, we have $\omega_c = \omega_f$ → $\omega_c^2 = \omega_f^2$.

Considering $\omega_c^2 = \dfrac{K}{M_x}$ and using Eqns. (88) and (91), one can have,

$$\frac{G_s S_{x1} g}{\gamma_p A_p} \left[4 \frac{I_1}{I_2} + 1 \right] - \omega_f^{\ 2} = 0 \quad (92)$$

It will be observed that all the factors β, I_1, I_2 in Eqn. (92) is a function of E_p/G_s. The difference (= the error ε) can now be set to zero or minimum by varying the value of E_p/G_s for which lim $\varepsilon \rightarrow 0$.

This can very easily be done by using the standard solver or goal seek in a spreadsheet with boundary constraint that $S_{x1} > 0$.

The solver basically uses an algorithm called generalized reduced gradient technique (GRG2) used for constrained optimization (Lasdon et al. (1978)). The procedure begins with the nonlinear optimization technique with equality constraints. The necessary slack and surplus variables are added as x_s or x_s^2 to any inequality constraints, and the problem is to optimize y(**x**), subject to f_i(**x**) = 0 for i = 1, 2, ..., j, where j is the number of constrained equations and n is the number of independent variables where n > m.

This is a very standard technique used in all nonlinear programming and is used routinely as a mathematical tool in many standard commercially available software like MS excel, MATLAB etc having varied applications in engineering, science and economics modeling.

Use of the above will automatically revise the value of E_p/G_s and upgrade the values of I_2 and I_1(dimensionless but a function of E_p/G_s), which may then be used to calculate the revised and exact stiffness and mass contribution of the pile which would closely simulate the field condition.

The steps are furnished in detail in Appendix A as to how the data are updated and corrected for the example cited in example mentioned below.

Having established the mass and stiffness coefficients of the pile correctly based on field data the damping may now be established as

$$C_x = r_0\left(\sqrt{\rho G_s}\right)S_{x2}L\ I_2 \tag{93}$$

where I_2 is the corrected upgraded value and S_{x2} is as obtained from Table 4.

Comparison of Results

The method proposed herein can very well be used for dynamic analysis of piles under horizontal force. However, the sanctity of the same will depend on how accurately the stiffness values have been evaluated. For this two RCC piles of radius 0.3m, 0.6m of length 30m has been has been checked with the reported results for comparison. The values K_{xx} [Eqn. (23)] is shown in Figure 6 for comparison.

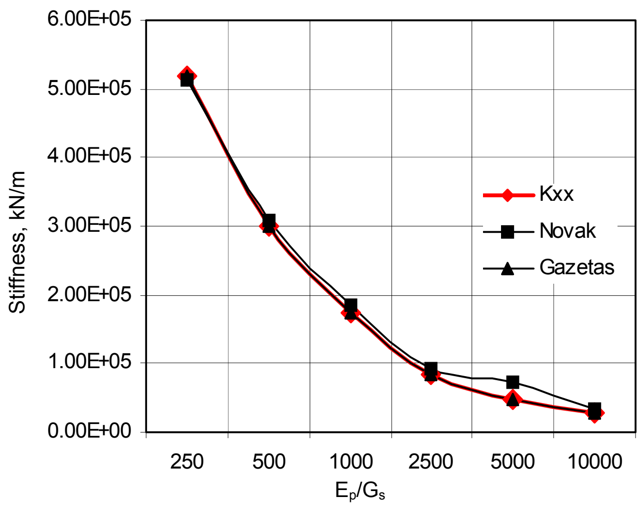

Fig. 6 Comparison of Stiffness Values for r = 0.3m and Length = 30 m

Next, the results of uncoupled horizontal frequency of a real time compressor foundation weighing 400kN supported on 9 RCC piles of length 36m and diameter 1.8 m. The pile cap size is 7mX5mX2m. The piles are spaced at distance of 3.0m. The natural frequencies of the foundation are compared for E_p/G_s value varying from 250-10,000. Weight of the compressor is 400 kN.

The results clearly shows that the values are in very good agreement for the base case and thus can well be used for other cases as mentioned above for which there are no direct solutions. Finally, the stiffness of a short pile has been computed. This is based on the field observed data having the following properties:

Length of Pile = 10m, diameter of pile = 1.2m, material of pile = RCC.

method of installation- Bored Pile

based on soil test, observed E_p/G_s = 5000.

E_p considered = $3X10^7$ kN/m^2.

unit weight of pile material = 25 kN/m^3.

field observed natural frequency of the pile is = 58rad/sec (9Hz).

Poisson's ratio of soil considered = 0.4.

For the above conditions: Selected value of S_{x1} from Table-5 =1.09

E_p/G_s= 5000 (given),

β = 2.1512. from Eqn. (7a)

A= 0.50135 ; B=0.02705 from Eqn. (82)

I_1= 0.23802, I_2 = 0.9035 from Eqn (89)

Computed natural frequency $\left(\sqrt{\frac{K_p}{M_p}}\right)$ = 68.26 rad/sec (11Hz)

➔Error(ε) = 10.26

Setting the error (ε) = 0 and running the solver function in a spread sheet for changing E_p/G_s for boundary constraint S_{x1}>0, the following upgraded data have been obtained:

S_{x1} = 1.09; E_p/G_s = 7246; β =1.96064; A=0.65984; B = -0.04832; I_1 = 0.27266 and I_2 = 0.949504.

Computed natural frequency based on above data =58 rad/sec(9Hz).

Revised Error(ε) = $-2.79X10^{-7}$

Thus based on the above data as per Eqn. (88), the correct stiffness of the pile is deduced as K_{pile} = 9.206×10^4 kN/m.

It is to be noted here that the E_p/G_s value has increased from 5000 to 7246 meaning thereby that the soil had lost some of its initial strength due to boring of the pile-which is quite logical.

Results and Discussions

Referring to Figures 6 and 7, it may be observed that the stiffness values are in excellent agreement with the existing solutions. Assuming that the base case being in such agreement, the other formulations can now be very easily adapted for which there are no solutions available.

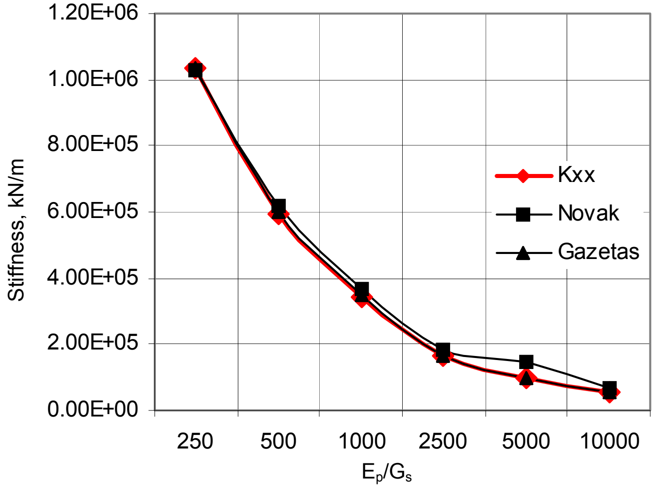

Fig. 7 Comparison of Stiffness Values for r = 0.6m and Length = 30m

Table 6 shows the uncoupled horizontal frequency, the result speaks for itself, for the frequency based on proposed stiffness matches with the existing results almost exactly. However, the natural frequency at E_p/G_s = 5000 based on Novak's solution reflects an error that could set in due to linear interpolation.

Table 6 Comparison of Natural Frequency for a Compressor Foundation

E_p/G_s	Frequency (rad/sec) with $K_{proposed}$	Frequency (rad/sec) with K_{Novak}	Frequency (rad/sec) with $K_{Gazzetas}$
250	252.64	251.44	252.57
500	192.10	194.85	192.07
1000	146.14	150.76	146.07
2500	101.79	107.21	101.71
5000	77.30	94.66**	77.35
10000	58.87	63.98	58.82

** The stiffness value was linearly interpolated from Novak's table for E_p/G_s= 5000

The short pile case is basically a theoretical solution and needs significant field test and lab testing to arrive at a predefined S_{x1} value, which would make the method more powerful. However in absence of such data the present solution could become a very powerful tool for the dynamic analysis of such piles for which no solution is available and yet remains a serious practical problem. The proposed method would be far more rational then using Gazetas'/Novak's formula which is anyway not valid for such piles while Novak's method does not provide with any coefficients (L/r < 25) for the same.

Conclusion

A comprehensive analytical solution for dynamic analysis of long piles has been presented and is in good agreement with the existing solutions. Based on this, piles with boundary conditions like partial embedment and soils with varying G_s can also be analyzed.

The solutions have been worked out for various values of E_p/G_s and ν varying from 250-10000 and 0.25-0.5 respectively. Poisson's ratio, ν is insensitive to soil type, confining pressure and void ratio but depends very much on the degree of saturation and drainage condition.

For saturated clays and sands below WT, $\nu \approx 0.5$

Nearly saturated clays, above WT, $\nu \approx 0.4$

Wet silty sand (S_r = 50 to 90%), $\nu \approx 0.35$

Nearly dry sands, stiff clays and rocks, $\nu \approx 0.25$

This range is sufficient to cater to all type of soils i.e. from very soft clay to reasonably stiff medium dense sand. Details of G_s and E_p values are reported elsewhere (Fang (1997)).

It may be noted that both Novak and Gazetas has worked out the stiffness and damping of pile only. No direct formula for bending moment or shear force has been derived or suggested by them. It is for this the industrial practice of design of machine foundations resting on pile is restricted to resonance and amplitude check only and no calculation is usually done for deriving the dynamic moments and shears induced in the pile due to dynamic loads. To circumvent this deficiency and uncertainty, the pile load is usually restricted to only 50% of its static capacity. Considering the fact that the dynamic bending moment and shear force can also be obtained by this method, and that too analytically inducing no numerical error the standard practice of restricting the pile capacity to 50% of its capacity will not be necessary. It will be observed from Eqns. (58) and (60) that the moment and shear take care of the dynamic magnification factor of the load at the same time gives a complete distribution of its magnitudes along the depth of the pile. This when combined with static load would give the design moment for the pile.

Short piles for which no established method exists also can be solved by the present method.

The analysis being totally closed form no elaborate software is required and can directly be used for design office use by generating a simple spreadsheet.

Notations

The following symbols are used in this paper:

A_p = cross sectional area of the pile

a_0 = dimensionless frequency number $\omega r_0 / v_s$

C_b and S_{x1} = Beredugo's frequency dependent constant

$\overline{C}_b$ and S_2 = frequency independent constants

C_c = critical damping of the pile = $C_c = 2\sqrt{Km_p}$

C_p = damping for the pile

D_f = depth of embedment

E_p = Young's modulus of pile

G_s = dynamic shear modulus of the soil

g = acceleration due to gravity

G_b = dynamic shear modulus of soil at the foundation base

$G' = G(x/L)^m$ = dynamic shear modulus of the soil, where m = a number varying from 0-2 [considered 0 when G_s is constant with depth, assumed 1 for linear variation and 2 for parabolic distribution]

I_p = moment of inertia of the pile cross section

K= stiffness of the pile

k_s = elastic stiffness of the soil and is expressed as GS_{x1}

K_h = lateral dynamic stiffness of soil

L= length of the pile

L_1= length of the embedment in soil

m_p = mass of the pile

M(x) = mass variation along the pile length

P_0 = amplitude of dynamic force

r_0 = radius of the foundation

r = ω_m/ω_n = the ratio of operating and natural frequency

S_{x2} = frequency dependent damping coefficient factor

v = displacement of the pile in the x direction = $\phi(z)$ q(t)

V(z)= Puzrevsky functions

α = embedment ratio L/L_1

γ_p = unit weight of the pile material

ζ = damping ratio

Π = total potential energy over the length of the pile, L

ρ = mass density of soil

$\phi(z)$ = shape function

ω_m = operating frequency of the machine

ω_c = computed natural frequency of the pile

ω_f = field-tested natural frequency of the pile.

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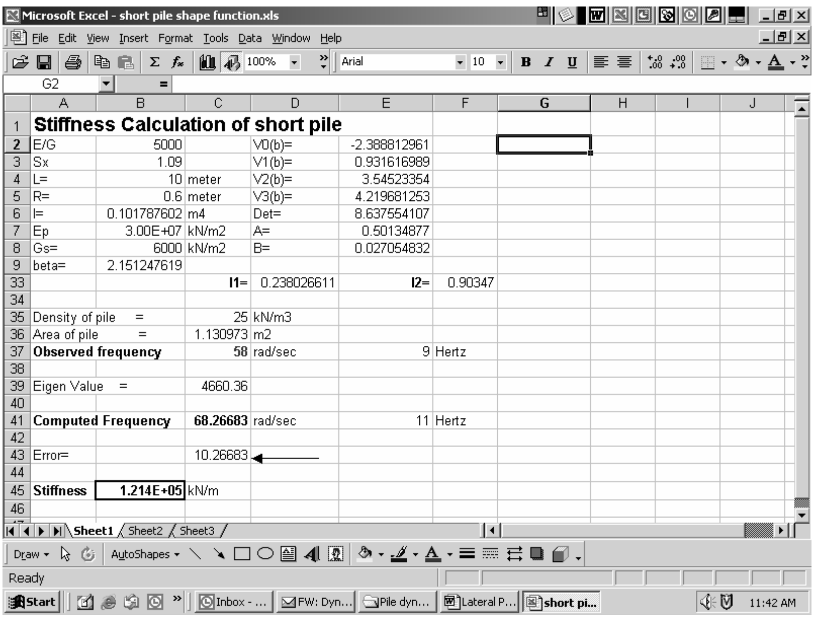
Appendix A

Computer run steps for short pile based field observed data

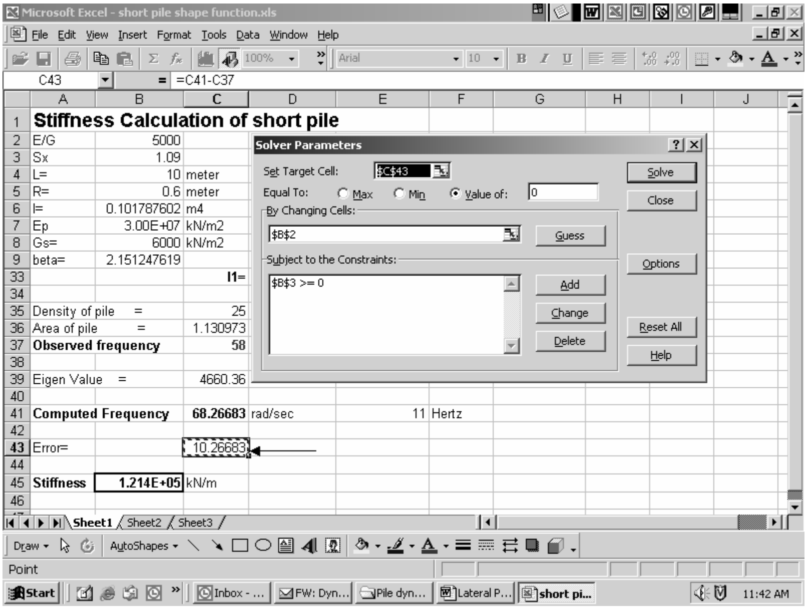
The appendix shows the computer run for the evaluation of the stiffness of the pile in lateral direction in three steps.

> Stiffness and frequency calculation of pile based on theoretical data and calculating the error based on field observed data.



	A	B	C	D	E	F
1	**Stiffness Calculation of short pile**					
2	E/G	5000		V0(b)=	-2.388812961	
3	Sx	1.09		V1(b)=	0.931616989	
4	L=	10	meter	V2(b)=	3.54523354	
5	R=	0.6	meter	V3(b)=	4.219681253	
6	I=	0.101787602	m4	Det=	8.637554107	
7	Ep	3.00E+07	kN/m2	A=	0.50134877	
8	Gs=	6000	kN/m2	B=	0.027054832	
9	beta=	2.151247619				
33			**I1=**	0.238026611	**I2=**	0.90347
34						
35	Density of pile =		25	kN/m3		
36	Area of pile =		1.130973	m2		
37	**Observed frequency**		**58**	rad/sec	9	Hertz
38						
39	Eigen Value =		4660.36			
40						
41	**Computed Frequency**		**68.26683**	rad/sec	11	Hertz
42						
43	Error=		10.26683			
44						
45	**Stiffness**	**1.214E+05**	kN/m			
46						

> The data screen just prior to run of the solver with command to change E_p/G_s value keeping the S_{x1} value >0.



> Final value of the stiffness and frequency of the pile after solver has optimized the data.