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Computation of Nγ for a Smooth Footing - Hill Mechanism

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Introduction

The bearing capacity of shallow foundations is one of the most studied areas in the field of soil-structure interaction problems. The foundation bases may be rough or smooth depending on the assumptions made in the analysis. Different failure mechanisms have been proposed for the analysis of bearing capacity of shallow foundations, the most common amongst them being the mechanisms postulated by Prandtl (1920), Terzaghi (1943) and that by Hill (1950). While the mechanism proposed by Terzaghi provided the first comprehensive bearing capacity analysis for a rough strip footing, the mechanisms postulated by Prandtl (1920) dealt with the strip footing with a smooth base. Hill (1950) proposed a different mechanism than that by Prandtl (1920) for the punch - indentation problem which is applicable for smooth footings. This mechanism gives a better estimate of bearing capacity factors in respect of cohesive – frictional (c-ϕ) ponderable (loaded with self weight) soils. Based on the mechanism postulated by Hill (1950), several theories have been proposed by researchers (Chen 1975; Michalowski 1997) to determine the bearing capacity factor- Nγ in respect of a ponderable soil.

In this paper, computations of bearing capacity factor Nγ based on the failure mechanism postulated by Hill for smooth footings, using limit equilibrium approach coupled with $K\ddot{o}tter's$ equation are presented and the results are compared with available solutions.

Brief Review of Hill Mechanism

Mechanism suggested by Hill for evaluating bearing capacity factor for the strip footing with a smooth base is the modification in the classical Prandtl's mechanism. Hill modified Prandtl's Mechanism as shown in Figure 1(a). His

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mechanism is symmetric about the axis of footing. It consists of: (i) a triangular wedge ABC with base angles, $\psi = \pi/4 + \phi/2$, and the interface between the wedge and the footing bottom is considered to be smooth, thus, allowing the horizontal movement of the wedge; (ii) a log spiral shear zone BCD of central angle $\pi/2$; and (iii) a passive Rankine triangular wedge BDE with base angles $\pi/4 - \phi/2$.

Formulation of the Problem

The analysis is primarily based on the computation of vertical (R_V) and horizontal (R_H) components of reaction (R) that acts on the curved part CD of the failure surface [Figure 1 (b)]. For this purpose $K\ddot{o}tter's$ equation (1903) is used.

$K\ddot{o}tter's$ *Equation*

For a cohesion-less soil medium (friction angle ϕ, unit weight γ) in passive state of equilibrium, $K\ddot{o}tter's$ equation gives the distribution of soil pressure p along the arc of the failure surface in the following form:

$$\frac{d\,p}{d\,s} + 2p\,\tan\phi\,\frac{d\,\alpha}{d\,s} - \gamma\,\sin(\alpha+\phi) = 0 \qquad (1)$$

in which, *dp* is differential reactive pressure on the elemental length *ds* of the failure surface, and α is angle made by the tangent to failure surface at the point of interest with the horizontal.

The applicability of $K\ddot{o}tter's$ equation to the analysis of the limit equilibrium problems has been demonstrated for the problems of a retaining wall using Coulomb's mechanism for the case of ponderable cohesion-less soil by Dewaikar and Halkude (2002).

Outline of the Proposed Analysis

For the analysis, the right half of the symmetry with width equal to 2B [Figure 1(a)] is considered. The forces that act on the failure wedge CDFB are vertical (R_V) and horizontal (R_H) components of reaction R on curved part CD of the failure surface, P_γ (passive thrust exerted on the Rankine's Wall) and *W* (self weight of the wedge CDFB). The analysis is primarily based on the computation of R_V and R_H using $K\ddot{o}tter's$ equation. After computing these reactions, free body diagram of wedge CDFB, is considered. In this free body diagram, $P_{p\gamma}$ is the only unknown force. Now, if the pole of the log spiral CD is correctly located, the calculated passive thrust $P_{p\gamma}$ from the force equilibrium conditions will be exactly the same, otherwise, they will be different. If they are different, trial location of the pole along the line BD is changed and for this new location of the pole, R_H, R_V, *W* and P_γ are again computed. Iterations are thus continued till the passive thrust $P_{p\gamma}$ obtained from both force equilibrium condition is the same to a specified decimal accuracy. After satisfying this condition, a unique value of $P_{p\gamma}$ is obtained, from which N_γ is computed.

The point of application of $P_{p\gamma}$ is computed using moment equilibrium condition in which moments of all the forces and reactions about the pole of log spiral are considered. It is convenient since the resultant reaction (R) always acts along the radial line of the log spiral and makes no contribution to the moment. Thus, in the proposed analysis, force equilibrium conditions are used to compute $P_{p\gamma}$ and moment equilibrium condition is used to compute its point of application.

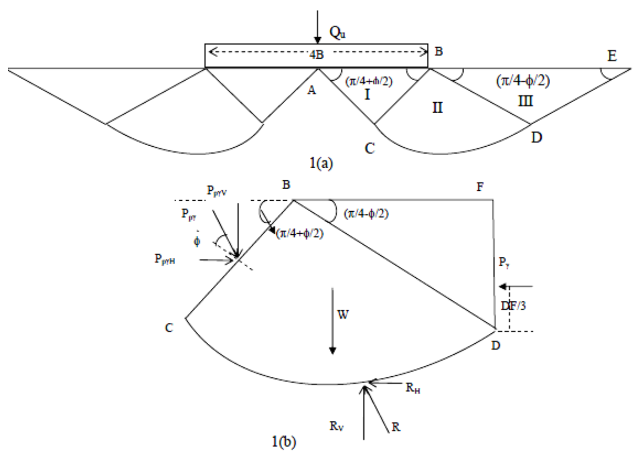

Fig. 1 (a) Failure Mechanism- Hill's Analysis;

(b) Free- body Diagram of Wedge CDFB

Integration of $K\ddot{o}tter's$ Equation over the Part DE of the Failure Surface

This part of the failure surface being straight, $d\alpha / ds = 0$ and $K\ddot{o}tter's$ equation reduces to the following form.

$$\frac{d\,p}{d\,s} = \gamma \sin(\alpha + \phi) \tag{2}$$

Integration of the above equation gives the soil reaction pressure at a point on the plane failure surface DE and the value of *p* at point D is calculated as

$$p = \gamma \sin(\pi / 4 + \phi / 2)\, DE \tag{3}$$

The distance DE depends upon the location of pole of the log spiral as shown in Figure 2. Referring to Figures 1 and 2, DE is calculated from geometry of the failure wedge. With this substitution Equation (3) becomes

$$p = \gamma \sin(\pi/4 + \phi/2)\, K\, r_0\, e^{\theta_m \tan\phi} \tag{4}$$

in which, r_0, θ_m, and θ_v are as shown in Figure 2 and K is given by the following expression:

$$K = \left(1 + \frac{\sin\theta_v}{e^{\theta_m \tan\phi}}\right) \tag{5}$$

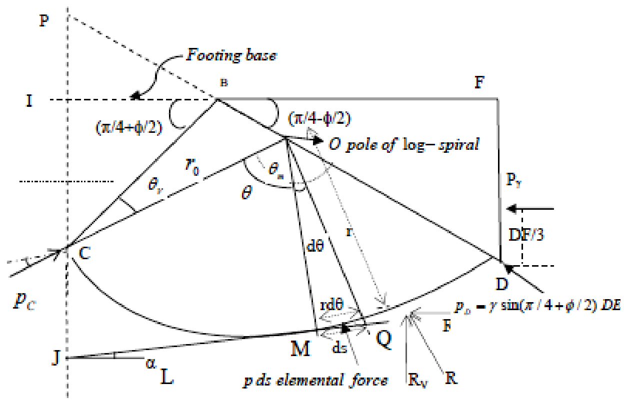

Fig. 2 Geometrical Relationships for Pole below Footing Base

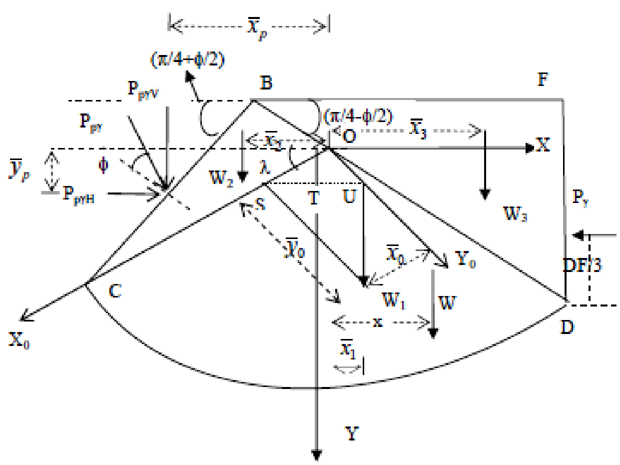

Fig. 3 Geometrical Relations for the Determination of Point of Application of Passive Thrust

Computation of Components of Reaction R on Curved Failure Surface CD

For this purpose, wedge CDFB as shown in Figure 1 (b) is referred. The magnitude of passive Rankine thrust, P_γ is given as:

$$P_\gamma = \frac{1}{2}\gamma\left(DF\right)^2\left(\frac{1+\sin\phi}{1-\sin\phi}\right) \tag{6}$$

Among the above forces, the known forces are P_γ and *W* and the forces R_H *and* R_V are calculated using $K\ddot{o}tter's$ equation for a curved failure surface (Figure 2).

Integration of the $K\ddot{o}tter's$ equation gives the pressure distribution on the curved failure surface (Mohapatro 2001) and is given as

$$p = \left\{\gamma\, r_0 K\cos\phi\sin\left(\pi/4+\phi/2\right)e^{(3\theta_m-2\theta)\tan\phi}\right\} + \frac{\gamma\, r_0\sec\phi}{\left(1+9\tan^2\phi\right)}\left[\begin{array}{l} e^{\theta\tan\phi}\left\{\begin{array}{l}3\tan\phi\sin\left(\theta-\theta_L+\phi\right)\\ -\cos\left(\theta-\theta_L+\phi\right)\end{array}\right\} - \\ e^{(3\theta_m-2\theta)\tan\phi}\left\{\begin{array}{l}3\tan\phi\sin\left(\theta_m-\theta_L+\phi\right)\\ -\cos\left(\theta_m-\theta_L+\phi\right)\end{array}\right\}\end{array}\right] \tag{7}$$

where $\theta_L = \pi/4+\phi/2+\theta_v$ and θ is as shown in Figure 2. In deriving the above expression, reactive pressure at point D as given by Equation (4) is used as a boundary condition.

Components of Resultant Reaction on the Failure Surface

The resultant reaction, R on the failure surface is given as

$$R = \int p\, ds \tag{8}$$

The vertical component, R_V of the reaction is obtained as (Figure 2),

$$R_V = \int_0^{\theta_m} p\, r_0 e^{\theta\tan\phi}\cos\left(\theta-\theta_L+\phi\right)\sec\phi\, d\theta \tag{9}$$

After substituting the value of *p* from Equation (7), the value R_V is obtained in the following form:

$$R_V = I_1 + I_2 + I_3 \tag{10}$$

Similarly, the horizontal component, R_H of the resultant reaction is given as (Figure 2)

$$R_H = \int_0^{\theta_m} p \sin(\theta - \theta_L + \phi) r_0 e^{\theta \tan\phi} \sec\phi \, d\theta \tag{11}$$

After substituting the value of p from Equation (7) and performing integration, R_H is obtained as

$$R_H = I_4 + I_5 + I_6 \tag{12}$$

The details of parameters I_1, I_2, I_3, I_4, I_5 and I_6 that appear in Equations (11) and (12) are reported in Appendix.

Self -weight of the Wedge CDFB

This is obtained by calculating weights, W_1 of part OCD, W_2 of part OCB and W_3, of part BDF (Figure 3). The required weight W (of part CDFB) is given as

$$W = W_1 + W_2 + W_3 \tag{13}$$

In which,

$$W_1 = \frac{1}{4}\frac{\gamma r_0^2}{\tan\phi}\left[e^{2\theta_m \tan\phi} - 1\right]; \quad W_2 = \frac{1}{4}\gamma r_0^2 \sin 2\theta_v;$$
$$W_3 = \frac{1}{4}\gamma r_0^2 K^2 e^{2\theta_m \tan\phi} \cos\phi \tag{14}$$

Computation of Passive Thrust $P_{p\gamma}$

The passive thrust is obtained by using the two equations of force equilibrium [Figure 1(b)]. The vertical force equilibrium is

$$R_V - P_{p\gamma}\cos(\pi/4 - \phi/2) - W = 0 \tag{15}$$

The horizontal force equilibrium is

$$P_{p\gamma}\sin(\pi/4 - \phi/2) - R_H - P_\gamma = 0 \tag{16}$$

The procedure of obtaining the desired value of $P_{p\gamma}$ has been described in the outline of the proposed analysis. The iterations were carried out till the computed values of $P_{p\gamma}$ from both the equations of force equilibrium matched with each other up to four decimal places.

Computation of N_γ

For this computation, Figure 4 is referred, which shows the free body diagram of the triangular wedge, ABC, subjected to forces Q_{ul} (kN/m), W_s

(Weight of the triangular soil wedge, ABC) and vertical component of the passive thrust, $P_{p\gamma}$. The vertical force equilibrium gives:

$$Q_{ul} + W_s = 2\,P_{p\gamma}\cos\left(\pi/4 - \phi/2\right) \tag{17}$$

After substituting the value of W_s in Equation (17) and dividing by 2B, it becomes:

$$\frac{Q_{ul}}{2B} = q_u = \frac{1}{2B}\left\{2\,P_{p\gamma}\cos\left(\pi/4-\phi/2\right) - \gamma\,B^2\tan\left(\pi/4+\phi/2\right)\right\} \tag{18}$$

Considering the total footing width equal to 4B, the ultimate bearing pressure, q_u for a strip footing on the surface of a cohesion-less soil, with unit weight of the soil γ is given by

$$q_u = 2\,\gamma\,B\,N_\gamma \tag{19}$$

On comparing Equations (18) and (19), the bearing capacity factor N_γ is finally obtained as

$$N_\gamma = \frac{1}{2}\left\{\frac{P_{p\gamma}}{\gamma\,B^2}\cos\left(\pi/4-\phi/2\right) - \frac{1}{2}\tan\left(\pi/4+\phi/2\right)\right\} \tag{20}$$

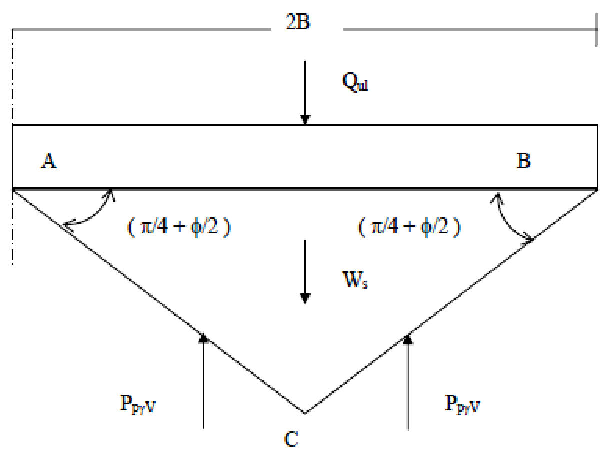

Fig. 4 Free –body Diagram Triangular Wedge ABC for Determination of N_γ

Point of Application of $P_{p\gamma}$

For this purpose, wedge CDFB as shown in Figure 3 is referred. This is obtained by calculating centroids, $\overline{x}_1$ of area OCD (weight W_1), $\overline{x}_2$ of area OCB (weight W_2), and $\overline{x}_3$ of area BDF (weight W_3). Then centroid of the wedge CDFB with reference to X, Y axes is given as

$$\bar{x} = \frac{W_1\bar{x}_1 + W_2\bar{x}_2 + W_3\bar{x}_3}{W_1 + W_2 + W_3} \tag{21}$$

The passive thrust, $P_{p\gamma}$ acts on the line, CB at point G with angle ϕ, to the normal. Unknown distance BG of point of application of $P_{p\gamma}$ from the pole O, can be obtained from moment equilibrium. Taking moment of all forces about the pole, O, the following equation is obtained after simplification.

$$P_{p\gamma}\left(BG\cos\phi - OB\sin\phi\right) = W\,\bar{x} + P_\gamma\left(\frac{2}{3}BD - OB\right)\sin\left(\pi/4 - \phi/2\right) \tag{22}$$

Results and Discussion

In Table 1, values of N_γ as obtained from the proposed analysis are presented for different values of the angle of internal friction (in the range of 20º-50º) along with those reported by other researchers. It is seen that, the N_γ values obtained by the proposed method are lower than those computed by Sokolovski (1965) based on method of characteristics, in the range of 5 to 13 per cent, for ϕ values greater than 20°.

The comparison with N_γ values reported by Chen (1975) which are based on the upper bound theory of limit analysis show a closer agreement to the values obtained by the proposed method. For $\phi \geq 25°$, they are lower than the proposed values in the range of 5 to 11.5 percent.

The N_γ values obtained by the proposed method are higher than those reported by *Bolton and Lau (1993)* based on the method of characteristics for $\phi \geq 25°$, in the range of 70 to 87 per cent. The proposed N_γ values are higher to the tune of 25 to 31 per cent than those reported by Michalowski (1997) using upper bound theory of limit analysis. Similarly, N_γ values as obtained by the proposed method are higher than the values reported by Frydman and Burd (1997), based on finite difference method coupled with associated flow rule, in the range of 34 to 61 per cent.

Thus, from above comparison, it is seen that the proposed N_γ values show a good agreement with the values reported by Sokolovski (1965) and Chen (1975); and so far as the computations using limit equilibrium method are concerned, the proposed N_γ values are the only available values in the literature, in the context of Hill's mechanism.

The position of the pole (corresponding to the unique failure surface) is indicated by the values of angle, θ_v (Figure 3), which is reported in Table 2 (refer column 2). From these values of angles (θ_v), it is seen that for general shear failure conditions, which exists beyond ϕ = 36°, the pole of the log spiral lies very close to the footing edge.

The location of point of application of the passive thrust ($P_{p\gamma}$) is possible using the proposed method only. It is expressed as the ratio of its vertical distance from Point C (Figure 3) and from the footing base. It increases with an increase in ϕ values (refer column 3 of Table 2).

TABLE 1: Comparison of Bearing Capacity Factor (N_γ) with other theories (Smooth Footing)

(L. B. = Limit Equilibrium; C = Characteristics; U. B. = Upper Bound; K = Kinematic; N. M. = Numerical Method)

ϕ (Deg)	N_γ					
	Proposed Analysis (L. B.) (2)	Sokolovski (1965) (C) (3)	Chen (1975) (U. B.) (4)	Bolton and Lau (1993) (C) (5)	Michalowski (1997) (K) (6)	Frydman and Burd (1997) (N. M.) (7)
20	1.5599	3.16	2.7	1.60	2.332	NA
25	6.5785	6.92	5.9	3.51	5.020	NA
30	13.9992	15.32	12.7	7.74	10.918	8.7
35	31.2645	35.18	28.6	17.80	24.749	20.7
40	75.7226	86.50	71.6	44.00	60.215	54.2
45	206.2721	NA	195.0	120.00	164.308	153.0
50	663.6980	NA	NA	389.00	519.089	NA

Table 2 Point of Application of $P_{p\gamma}$

ϕ(Deg.)	θ_v(Deg.)	$Ratio = \frac{B - x_p}{B}$
30	4.5243	0.1859
34	3.0690	0.2140
36	2.4932	0.2256
38	2.0038	0.2359
40	1.5919	0.2451
42	1.2480	0.2533
45	0.8442	0.2644
50	0.0403	0.2799

Conclusions

The following conclusions are drawn from the proposed analysis:

1. An analysis on the evaluation of bearing capacity factor, N_γ for a smooth footing based on Hill's mechanism is presented.
2. The concept of limit equilibrium condition coupled with $K\ddot{o}tter's$

equation identifies the unique failure surface consistent with the specified geometry of the failure surface.

3. Application of $Kötter's$ equation makes the analysis statically determinate and facilitates the computation of locating the point of application of passive thrust.
4. The N_γ values that are obtained from the proposed analysis based on the failure mechanism postulated by Hill with the limit equilibrium approach are the unique values since no other simplifying assumptions are made to evaluate them.
5. The N_γ values as obtained from the proposed analysis which effectively uses force and moment equilibrium equations show a good agreement with those reported by Sokolovski and Chen; and this establishes reliability of the proposed method of analysis.
6. The point of application of passive thrust $P_{p\gamma}$ can only be computed using the proposed method. It lies closer to the apex of the triangular wedge below the footing. Analysis also shows that the pole of the log-spiral lies at footing edge for practical range of ϕ values.

Appendix

Integration for R_V and R_H

$$I_1 = \gamma\, r_0^2\, K \sin(\pi/4+\phi/2)\, e^{3\theta_m \tan\phi} \begin{bmatrix} e^{-\theta_m \tan\phi} \sin(\pi/4-\phi/2) \\ -\sin(\pi/4-\theta_m-\phi/2) \end{bmatrix}$$

$$I_2 = \frac{\gamma\, r_0^2 \sec\phi}{4\left(1+9\tan^2\phi\right)} \begin{bmatrix} 3\tan\phi\left\{\sin 2\phi\, e^{2\theta_m \tan\phi} + \sin 2(\theta_m-\phi)\right\} \\ \dfrac{\left(e^{2\theta_m \tan\phi}-1\right)}{\sin\phi} + \left(e^{2\theta_m \tan\phi}\cos 2\phi - \cos 2(\phi-\theta_m)\right) \end{bmatrix}$$

$$I_3 = \frac{\gamma\, r_0^2 \sec\phi\, e^{3\theta_m \tan\phi}}{\left(1+9\tan^2\phi\right)} \left\{e^{-\theta_m \tan\phi} \sin(\pi/4-\phi/2) - \sin(\pi/4-\theta_m-\phi/2)\right\}$$
$$\left\{3\tan\phi \sin(\pi/4+\phi/2) - \cos(\pi/4+\phi/2)\right\}$$

$$I_4 = \gamma\, r_0^2\, K \sin(\pi/4+\phi/2)\, e^{3\theta_m \tan\phi} \begin{Bmatrix} \cos(\pi/4-\theta_m-\phi/2) \\ -e^{-\theta_m \tan\phi} \cos(\pi/4-\phi/2) \end{Bmatrix}$$

$$I_5 = \frac{\gamma\, r_0^2 \sec\phi}{4\left(1+9\tan^2\phi\right)} \begin{bmatrix} \sec\phi\left(e^{2\theta_m \tan\phi}-1\right) \\ -\tan\phi\left(e^{2\theta_m \tan\phi}\cos 2\phi - \cos 2(\phi-\theta_m)\right) \\ -\left\{\sin 2\phi\, e^{2\theta_m \tan\phi} + \sin 2(\theta_m-\phi)\right\} \end{bmatrix}$$

$$I_6 = \frac{\gamma\, r_0^2 \sec\phi\, e^{3\theta_m \tan\phi}}{\left(1+9\tan^2\phi\right)} \begin{Bmatrix} \cos\left(\pi/4-\theta_m-\phi/2\right) \\ -e^{-\theta_m \tan\phi}\ \cos\left(\pi/4-\phi/2\right) \end{Bmatrix}$$
$$\left\{\ 3\tan\phi \sin\left(\pi/4+\phi/2\right)-\cos\left(\pi/4+\phi/2\right)\right\}$$

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