Indian Geotechnical Journal, 37(1), 2007, 53-63

Relationships between Rate of Dilation, Peak and Critical State Friction Angles

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Introduction

One of the earliest observation to monitor a change in volume during shear for sands was recorded by Reynolds (1885). Reynolds showed that dense sands exhibit expansion in volume during shear, and in contrast, loose sands contract during shear deformation. Casagrande (1936) later demonstrated that irrespective of the initial relative density of sand, at considerable level of shearing strain, a state finally emerges when the material starts shearing at constant volume and at constant shear stress. This state is often termed as the critical and the corresponding values of internal friction angles and void ratio are often defined as the critical friction angles (ϕ_{cv}) and the critical void ratio (e_{cr}), respectively. Taylor (1948), followed by Skempton and Bishop (1950) attempted to separate the strength component (ϕ_{cv}) purely on an account of friction from that (ϕ_p-ϕ_{cv}) due to an expansion of the material; where ϕ_p is the angle of internal friction corresponding to peak stress ratio. Roscoe et al. (1958) using a novel shear apparatus (Roscoe 1953) later proved that the value of ϕ_{cv} depends only on particles' shape and material grading. The ideal method of determining ϕ_{cv} is by drawing a plot between ϕ_p and the corresponding rate of dilation (at ϕ_p) so that the value of ϕ_p associated with the zero rate of dilation, which is termed ϕ_{cv}, can be extrapolated (Bolton 1986). From an analysis of a number of test results, both in plane and triaxial conditions, Bolton (1986) has also concluded that the value of ϕ_{cv} remains essentially a function of mineralogy and can be determined experimentally within an error of about 1^o. Bolton (1986) recommended the value of ϕ_{cv} roughly equal to 33^o for quartz and 40^o for feldspar. It is, however, not fully clear whether plane strain condition yields slightly higher values of ϕ_{cv} than the triaxial state condition. For testing on Houston sand (quartz) no such difference was indicated by Schanz and Vermeer (1996). There seems to be nevertheless an agreement that the value of ϕ_{cv} does not depend on the stress level. However, this is not the case with the angle of friction in peak stress state and the corresponding rate of dilation; it should be mentioned that the rate of dilation also becomes maximum near the peak stress state. By carrying out experiments

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on steel shots, Bishop (1972) has shown that an increase in confining pressure leads to a reduction in the angle of shearing. Vesic and Clough (1968) and Billam (1972) have also reported the similar behavior regarding the effect of stress level on ϕ_p. A number of other researchers have also demonstrated that the Mohr-Coulomb failure envelope remains curved at low stress level, and the amount of curvature depends on the relative density of the sand, with dense sands possessing more curvature than loose sands (Vesic and Barksdale, 1963; Bishop 1966; Lee and Seed 1967; and Colliat-Dangus et al. 1988). It was also reported by Maeda and Miura (1999) that the value of the axial strain at failure increases with the increase in the level of confining stress. The original stress-dilatancy model (Rowe 1962) does not capture important behavioral features such as density and stress level dependencies. As shown by Bishop (1966), the stress-strain dilatancy behavior of sand varies remarkable with confining pressures. Bolton (1986) reviewed a large number of triaxial and plane strain test results and proposed a much simpler relationship among ϕ_p, ϕ_{cv} and ψ_p , which he found operationally equivalent to Rowe (1962)'s stress–dilatancy relationship; where ψ_p is the angle of dilatancy which indirectly quantifies the rate of dilation. Bolton (1986) provided the following simplified expressions:

$$\phi_p = \phi_{cv} + 0.8\psi_p \quad (1)$$

$$\phi_p = \phi_{cv} + 5I_R \text{ for plane strain condition} \quad (2)$$

$$\phi_p = \phi_{cv} + 3I_R \text{ for triaxial condition} \quad (3)$$

The quantity I_R, is referred to as dilatancy index and its magnitude is related to the relative density (R_d) and the effective stress ($p^/$) by the relationship

$$I_R = R_d\left(Q - ln(p^/)\right) - R \quad (4)$$

In the above equation $p^/$ is expressed in kPa and R_d in decimal; Q and R are constants.

It should be mentioned that in Equation (4), the effect of stress level is incorporated. Bolton (1986) recommended the values of R = 1 and Q = 10. Later, Salgado et al. (2000) recommended the value of Q = 9.0 and R = 0.49 based on his test results on clean Ottawa sand (without any percentage of silt).

The objective of the present study was to examine further these correlations. For this purpose a number of direct shear tests were conducted on Bangalore (quartz) sand by varying the magnitude of normal stress in between 50 kPa and 800 kPa. Four different relative densities of sands were employed. All the tests were continued upto a substantial value of the horizontal displacement so that the critical state was achieved in all the tests. The values of ϕ_p and ψ_p were determined in all the tests for different combinations of σ_v and R_d. All the test results were compared with the recommendations of Bolton (1986) and Salgado et al. (2000) so as to suggest correlations which provides better estimation with regard to the present experimental data.

Properties of Bangalore Sand

The Bangalore sand was found to comprise of generally sub-angular grains as can be seen from the microscopic picture of a portion of the sample (Figure 1). The grain size distribution of the material is shown in Figure 2. It can be noticed that the material comprises hardly of any fraction of silt. The average specific gravity (G) of the sand solids was found to be 2.67. The maximum and minimum unit weights of this sand were found to be 18.1 kN/m^3 and 14.3 kN/m^3, respectively. The values of the different parameters associated with the grain size distribution curve of the material are as follows: D_{10} = 0.23 mm, D_{30} = 0.40 mm, D_{50} = 0.62 mm, D_{60} = 0.78 mm, C_u = 3.40 and C_c = 0.90; D_{10}, D_{30}, D_{50} and D_{60} are the sizes corresponding to respective percentage of finer, and C_u and C_c are the uniformity coefficient and the coefficient of curvature of the material, respectively. As per the Indian Standard for classification of soils (IS 1498-1970, reaffirmed 2002), Bangalore sand was found to be poorly graded.

Fig. 1 A Microscopic View of a Portion of the Soil Sample

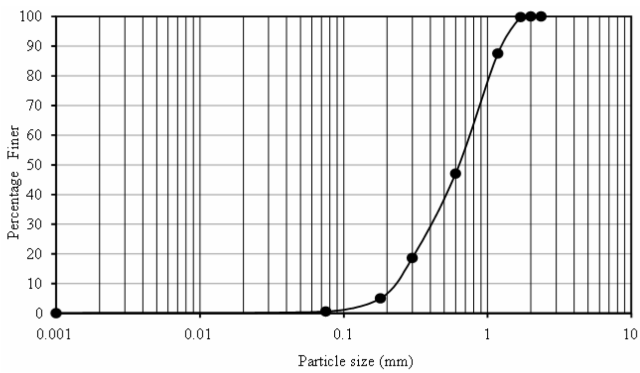

Fig. 2 Grain Size Distribution Curve of the Sample

Direct Shear Tests

A number of direct shear tests were conducted on chosen dry sand at four different values of unit weight, namely, 15.21 kN/m^3, 16.19 kN/m^3, 17.17 kN/m^3 and 17.66 kN/m^3; the corresponding relative densities of these samples were found to be 28.5 %, 55.6 %, 79.6 %, and 90.6 %, respectively. The size of the shear box was 60 mm x 60 mm and the sample height was kept equal to 30.77 mm for all the tests. All the samples were sheared at a uniform relative horizontal movement of 0.05 mm/minute between the upper and lower box. The vertical effective normal stress on all specimens was varied in between 50 kPa and 800 kPa. The samples of a given density were prepared by either raining the material from a constant height of fall (for loose to medium dense sand) or with the tamping technique using a fixed number of blows (for dense to very dense sand). All the tests were continued upto u/H = 30%; where H is the initial height of the sample and u is the horizontal displacement at any time.

Test Results

For all the tests, the variation of the horizontal (shear) force (P_h) and the corresponding change (v) in the vertical height of the sample with increase in the horizontal displacement (u) was continuously monitored at regular time interval; volumetric strain simply becomes equal to v/H. The corresponding test results are shown in Figures 3 - 6 in terms of (i) the variation of P_h/P_v with u/H, and (ii) the variation of v/H with u/H; where P_v is the magnitude of the vertical force. From these plots the values of friction angles (ϕ) and dilatancy angles (ψ) were determined using the following expressions:

$$\phi = a\tan^{-1}\left(\frac{P_h}{P_v}\right) \quad (5)$$

$$\psi = a\tan^{-1}\left(\frac{\delta v}{\delta u}\right) \quad (6)$$

The peak values of ϕ and ψ were designated by ϕ_p and ψ_p, respectively. The variation of ϕ_p and ψ_p with increase in σ_v is also illustrated in Figure 7. Following observations were drawn from Figures 3 - 7:

1. The peak values of ϕ and ψ invariably occur almost at the same value of the horizontal displacement. The magnitude of the u/H corresponding to ϕ_p increases with increase in σ_v. An increase in the relative density of the material causes a marginal decrease in the value of u/H associated with ϕ_p.
2. For a given relative density of the material, the behaviour of the material at low stress level always remains typically that of a dense sand which indicates a well defined peak corresponding to ϕ_p and then followed by a decrease in the shear stress which ultimately leads to the critical state of the material at very high values of horizontal displacement; in such cases the material initially shows a decrease in volume followed by an increase in volume. At low values of σ_v, the rate of dilation becomes maximum corresponding to ϕ_p and subsequently the value of dilatancy angle again decreases and finally becomes equal to zero in the critical state. On the contrary at very high values of σ_v, the behaviour of the material remains

similar to that of loose sand where the shear stress increases continuously to yield the critical state at very high values of horizontal displacement. In such cases the material experiences a continuous decrease in volume until reaching the critical state.

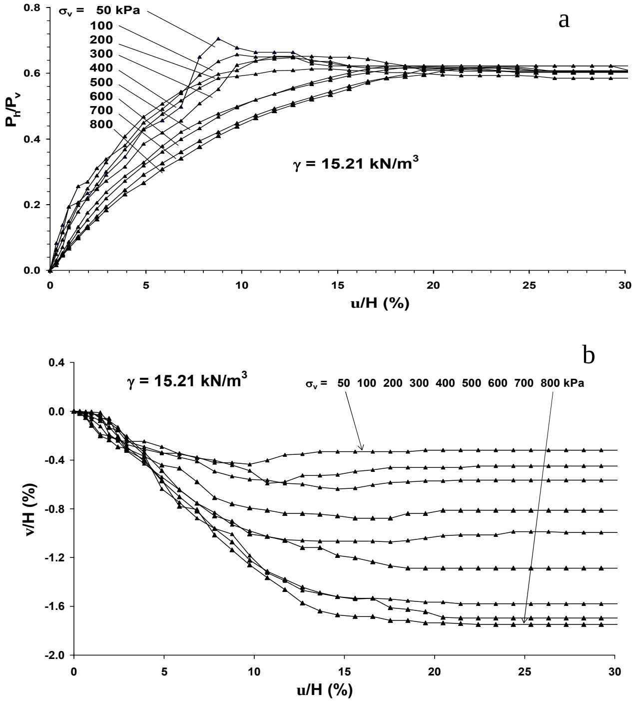

Fig. 3 For γ = 15.21 kN/m^3, the observed variation of (a) P_h/P_v with u/H, and (b) v/H with u/H

3. The values of ϕ_p as well as ψ_p decrease with increase in the value of σ_v. As compared to loose sand, the effect of σ_v on the changes in the values of ϕ_p and ψ_p was seen to be more significant in the case of dense sand.
4. For a given value of σ_v, an increase in the relative density of the material causes an increase in the values of both ϕ_p and ψ_p.

Correlation Between ϕ_p and ψ_p

For different chosen values of σ_v and relative density (R_d) of the material, the obtained values of ϕ_p were plotted against the corresponding values of ψ_p. All the data points are indicated in Figure 8. It can be noted that the relationship between ϕ_p and ψ_p can be best described by the following co-relation:

$$\phi_p = \phi_{cv} + 0.932\psi_p \tag{7}$$

It can be noticed from Figure 8 that the value of ϕ_{cv} for the chosen sand sample is found to be equal to 31.91^o (that is τ/σ_v = 0.62). It can also be noticed from Figures 3-6 that the value of τ/σ_v at very large value of u/H (25-30%)

remains very close to 0.62 indicating the achievement of the same critical state in all the tests.

A comparison of Equations (7) and (1) indicates that the recommendation of Bolton (1986) remains only marginally different from the present experimental finding. It should be mentioned that Bolton's expression relating ϕ_p, ϕ_c and ψ_p, is operationally indistinguishable from Rowe's stress–dilatancy relationship (1962).

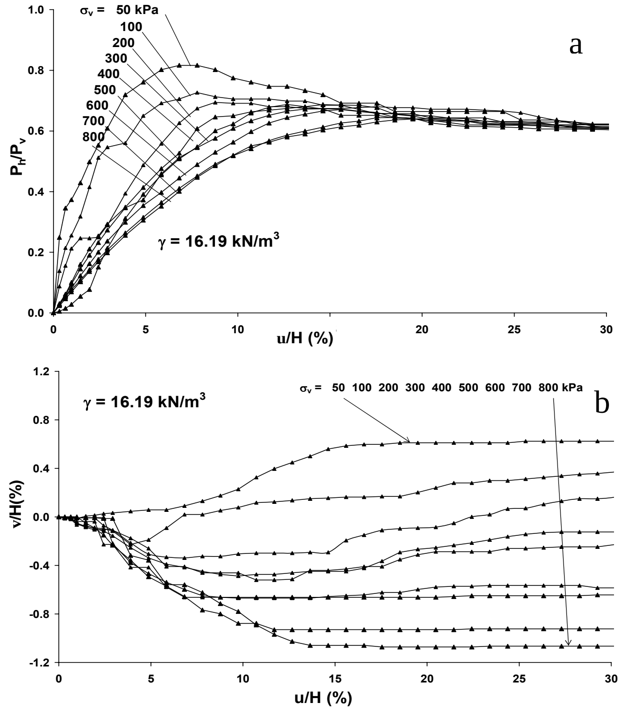

Fig. 4 For γ = 16.19 kN/m^3, the observed variation of (a) P_h/P_v with u/H, and (b) v/H with u/H

Correlation Between ϕ_p and σ_v

As seen earlier from Figure 7 that the value of ϕ_p reduces with increase in the value of σ_v. Bolton provided Equation (2) (for the plane strain case), where I_R (dilatancy index) is defined by Equation (4) with Q = 10 and R = 1. From the regression analysis, it was found that the following relationship holds quite good for the present data:

$$\phi_p = \phi_{cv} + 3.5 I_R \quad (8)$$

where $I_R = R_d(10 - \ln(\sigma_v)) - 1$ with σ_v expressed in kPa and R_d in decimal.

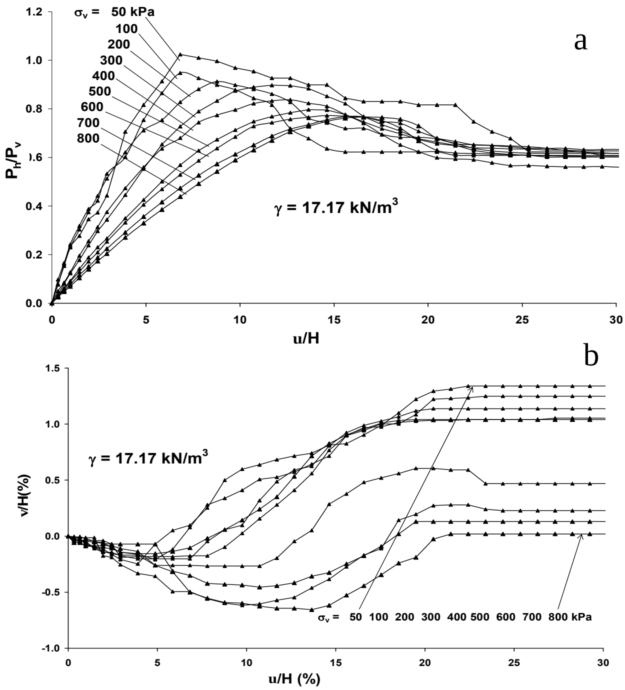

Fig. 5 For γ = 17.17 kN/m^3, the observed variation of (a) P_h/P_v with u/H, and (b) v/H with u/H

Experimentally measured values of ϕ_p were plotted against those estimated using (i) Bolton's recommendation (Equation 2), (ii) present correlation (Equation 8), and (iii) Salgadao et al. (2000)' recommendation and the corresponding comparison from three different correlations is shown in Figure 9 for all the data points. It can be noted the estimated values of ϕ_p from the recommendations of Bolton (1986) and Salgadao et al. (2000) are found to be slightly higher than those actually measured. On the other hand, the estimation from Equation (8) seems to be better.

Remarks

By knowing the mineral composition/particle shape and the grain size distribution curve of the material, it is possible to estimate an approximate value of ϕ_{cv} with an error of about 1^o. For a given relative density of the material, from the knowledge ϕ_{cv} and the stress level (σ_V), the secant value of ϕ_p for a dry cohesionless material can then be determined using any of the three relationships, namely, (i) recommendation of Bolton (1986) - (Equation 2), (ii) present recommendation- (Equation 8), and (iii) Salgado et al. (2000) recommendation- (Equation 2) but with the usage of Q=9.0 and R=0.49 in Equation (4). It should be mentioned that the three different recommendations may provide only a marginal difference in the value of ϕ_p and the average of the three can be adopted for carrying out the analysis where the effect of stress level on ϕ_p has to be taken into consideration. After determining the value of ϕ_p, Equation (1) or Equation (7) can then be used to find the value of the dilatancy angle (ψ_p). The knowledge of dilatancy angle becomes necessary where an analysis has to be carried with an employment of a non-associated flow rule while using the theory of plasticity.

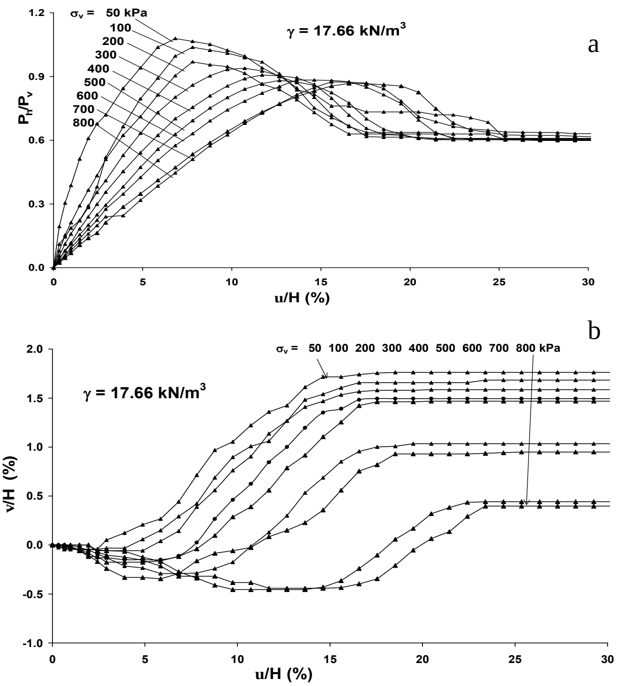

Fig. 6 For γ = 17.66 kN/m^3, the observed variation of the observed variation of (a) P_h/P_v with u/H, and (b) v/H with u/H

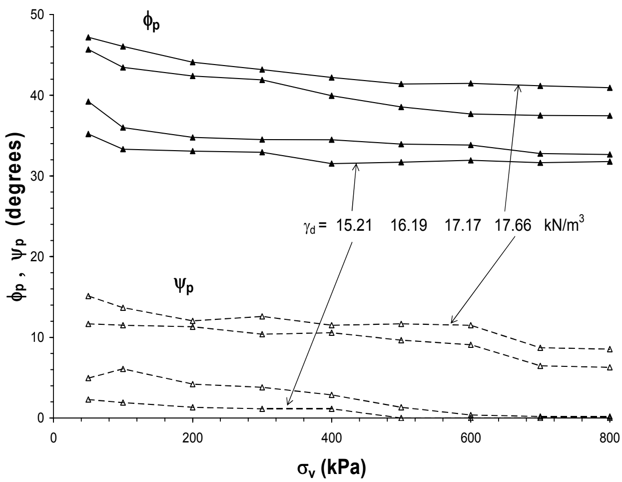

Fig. 7 The variation of ϕ_p and ψ_p with σ_v for all the tests

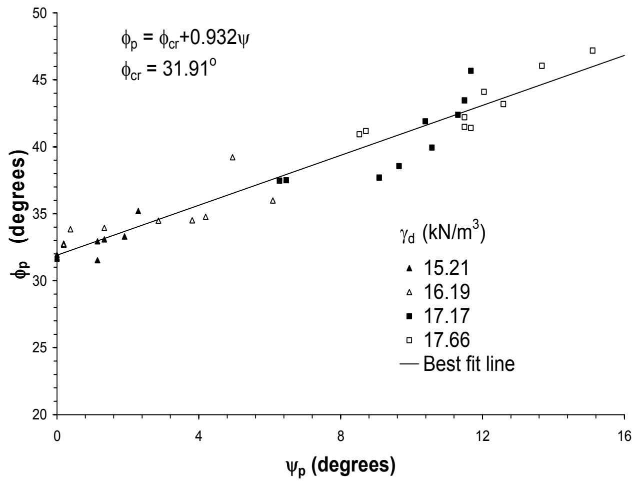

Fig. 8 The correlation between ϕ_p and ψ_p from all the test results

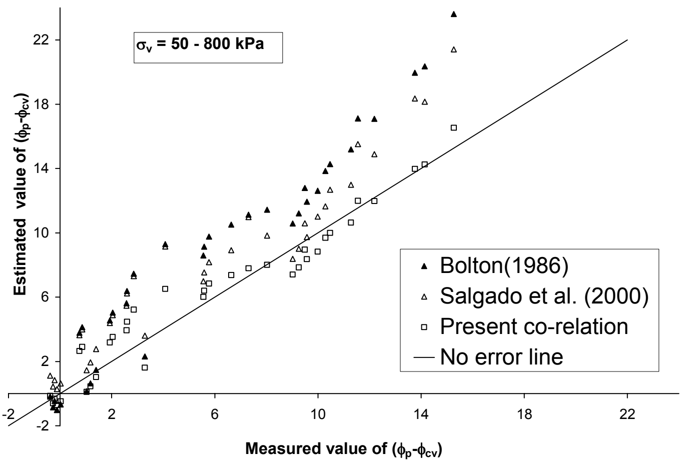

Fig. 9 The prediction of (ϕ_p-ϕ_{cv}) using different formulae against measured values of (ϕ_p-ϕ_{cv}) for all the tests

Conclusions

Based on a number of direct shear tests, an empirical relationship correlating ϕ_p, ϕ_{cv} and I_R, similar to that recommended by Bolton (1986) and Salgado et al. (2000), has been suggested. Using this relation from the knowledge of relative density (R_d) and critical state friction angle (ϕ_{cv}), the value of the peak friction angle can be determined for any required effective stress level (σ_v). Further, an expression correlating ψ_p with ϕ_{cv} and ϕ_p has also been provided on the basis of which the value of ψ_p can also be predicted. The suggested expressions are found to match well with the test results. The testing has clearly indicated that a decrease in σ_v leads to an increase in the values of

ϕ_p and ψ_p which necessitates the need of employing secant values of ϕ_p rather than using tangent ϕ_p, with some small value of an apparent cohesion, as is normally followed in practice to avoid complications in performing the analysis.

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