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Vibration of an Elastic Halfspace under Rectangular Loading

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Introduction

The literature of soil dynamics is quite diversified and it encompasses as diversified fields as earthquake engineering, machine foundation and protective structures such as radar installation. It is well known that the vibration, whether originating as impulse from a forge hammer or repeated vibrations from reciprocating engines are transmitted as waves through the ground. These waves in turn excite other structures distant from the source as well as the foundation from which vibrating energy is emanating. The mathematical aspects of vibrations, namely wave propagation and ground motion, natural frequency, response and settlement, cannot be treated out of context to the whole of discipline of soil dynamics to which it belongs. The design of a footing subjected to dynamic loading requires not only the consideration of the mathematical aspects of vibrations, but also of the art aspect, based on field observations, of wave propagation, resonance, particle motion, settlement, and changes in behavior of soils when subjected to both impulse and cyclic loading.

Thomson and Kobori (1962, 1963), using Fourier theorem, presented solutions of the dynamic response of rectangular footing on an elastic half space and subjected to uniformly distributed dynamic load. Compliance functions were obtained at the centre of the footing for different frequency, length to width ratios and for a particular Poisson ratio, $\nu = 1/4$. Elorduy, Nieto and Szekely (1967) reported an extensive study on the dynamic response of a rigid rectangular footing under the action of vertical periodic loading, resting on the surface of an elastic homogeneous halfspace. Wong and Luco (1976) proposed an approximate numerical procedure for calculating harmonic force-displacement relationship for a rigid foundation of arbitrary shape placed on an elastic halfspace. These results include the coupling compliances between the horizontal and rocking motions. Dasgupta and

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Dynamic behaviour of buildings and other foundation structures depends on the relative stiffness of the ground and that of the structure. If the stiffness of the structure is small in comparison to that of the ground, influence of the ground on the dynamic behavior of the structure is negligible and an infinitely rigid base can be assumed. The dynamic stiffness of the ground is measured by the ground compliance displacement produced by the loading of the structure. It depends on the elastic properties of the ground, the shape of the loading area and the frequency of the load.

Six co-ordinates comprising two orthogonal horizontal translations, a vertical translation, rocking about two mutually perpendicular horizontal axes and torsion about the vertical axis, can describe the motion of a footing block. The vertical translation mode and rotational mode occur as uncoupled motions when a complete symmetry exists.

The purpose of this study is to present a method that will permit the evaluation of force-displacement relationship and thereby to find the compliance functions for only three modes of vibration (vertical, horizontal and rocking). The approach is restricted to the case of rectangular foundations placed on the surface of elastic medium. Displacements have been computed for any point on the surface. These are useful for experimental studies, as measurements are made at any point on the surface, and may be useful for studies related to different contact stress conditions.

Analysis

The problem of establishing the dynamic stiffness of the ground has been of interest to those designing foundations for machines with oscillatory forces. In mathematical terms it is known as halfspace problem and Reissner (1936) initiated the solution. His study considered the loading under the circular foundation to be uniform, and found the displacement at the centre to be linearly proportional to the load with amplitude and phase varying with non-dimensional frequency factor a_0. Many authors studied the phenomena independently and made extensions to the above work. In all these studies, the average displacement at the center of the circular slab is expressed in terms of non-dimensional frequency factor a_0. Normally a circular slab has been chosen since its mathematical analysis, although complicated, yields results capable of analytical and numerical solution. The solution procedure followed herein was initiated by Thomson and Kobori (1962).

For a homogenous isotropic elastic body (Fig. 1), the displacement vector satisfies the following equation:

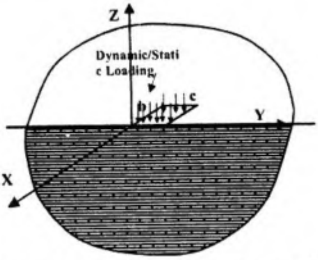


Fig.1 Rectangular Footing on an Elastic Halfspace

$$(\lambda+\mu)\left\{\frac{\partial\Delta}{\partial x},\frac{\partial\Delta}{\partial y},\frac{\partial\Delta}{\partial z}\right\}+\mu\nabla^2\{u,v,w\}=\rho\frac{\partial^2}{\partial t^2}\{u,v,w\} \tag{1}$$

where, $\Delta=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}=$ *dilatation;* $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=$ *Laplacian operator.*

By eliminating the displacement components u, v and w, the wave equation for the dilatation is obtained as

$$\left(\nabla^2-\frac{1}{c_l^2}\frac{\partial^2}{\partial t^2}\right)\Delta=0 \tag{2}$$

where $c_l=\sqrt{(\lambda+2\mu)/\rho}=$ *dilatational wave velocity.*

To solve eqn. (2), a triple Fourier Transform of Δ on x, y and t has been introduced as follows

$$f^3(\Delta)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(x,y,z,t)\,e^{-i(\beta x+\gamma y+\omega t)}dxdydt=\overline{\Delta} \tag{3}$$

Its inverse is then given by $f^{-3}(\overline{\Delta})=\Delta$. In addition, it can be shown that

$$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Delta}{\partial x^2},\frac{\partial^2\Delta}{\partial y^2},\frac{\partial^2\Delta}{\partial t^2}\right)e^{-i(\beta x+\gamma y+\omega t)}dxdydt=-\left(\beta^2,\gamma^2,\omega^2\right)\overline{\Delta}(\beta,\gamma,z,\omega) \tag{4}$$

So the triple Fourier Transform of eqn. (2) becomes

$$\left[\frac{d^2}{dz^2}-\left\{\left(\beta^2+\gamma^2\right)-h^2\right\}\right]\overline{\Delta}=0 \tag{5}$$

with the solution

$$\overline{\Delta}=Ae^{-\alpha_l z}+A'e^{+\alpha_l z} \tag{6}$$

where $\alpha_l^2 = \beta^2 + \gamma^2 - h^2; \quad h^2 = \omega^2 / c_l^2$.

In eqn. (6), in order to eliminate the physically inconsistent solution for an exponentially increasing $\overline{\Delta}$ with z, one has to set $A' = 0$. Thus, the solution of eqn. (6) reduces to

$$\overline{\Delta} = Ae^{-\alpha_l z} \tag{7}$$

Inverting eqn. (7)

$$\Delta(x,y,z,t) = f^{-3}\left(Ae^{-\alpha_l z}\right) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\beta,\gamma,\omega)\, e^{-\alpha_l z + i(\beta x + \gamma y + \omega t)} d\beta d\gamma d\omega \tag{8}$$

If one introduces the multiple Fourier Transform of the displacement components u, v and w in eqn. (1) and makes use of the solution for eqn. (8), eqn. (1) becomes

$$\left(\frac{d^2}{dz^2} - \left(\beta^2 + \gamma^2 - k^2\right)\right)\left(\overline{u}, \overline{v}, \overline{w}\right) = \left(\frac{k^2}{h^2} - 1\right) Ae^{-\alpha_l z}\left(i\beta, i\gamma, \alpha_l\right) \tag{9}$$

where

$$\frac{k^2}{h^2} - 1 = \frac{c_l^2}{c_2^2} - 1 = \frac{\lambda + \mu}{\mu};\ k^2 = \frac{\omega^2}{c_2^2} : c_2 = \sqrt{\frac{\mu}{\rho}} = shear\ wave\ velocity.$$

Using, $\alpha_2^2 = \beta^2 + \gamma^2 - k^2$, $\overline{u}, \overline{v}$ and $\overline{w}$ in eqn. (9) are given by

$$\left(\overline{u}, \overline{v}, \overline{w}\right) = \left(i\beta, i\gamma, \alpha_l\right)\frac{A}{h^2} e^{-\alpha_l z} + \left(B, C, D\right) e^{-\alpha_2 z} \tag{10}$$

where, and their inverse can be written as

$$\left(u, v, w\right) = f^{-3}\left\{\left(i\beta, i\gamma, \alpha_l\right)\frac{A}{h^2} e^{-\alpha_l z} + \left(B, C, D\right) e^{-\alpha_2 z}\right\} \tag{11}$$

The general solution for displacements is then expressed by

$$\left(u, v, w\right) = f^{-3}\left\{\left(i\beta, i\gamma, \alpha_l\right)\frac{A}{h^2} e^{-\alpha_l z} + \left(B, C, -\frac{i}{\alpha_2}(\beta B + \gamma C)\right) e^{-\alpha_2 z}\right\} \tag{12}$$

where A, B and C must be determined from the boundary conditions.

Assuming that the boundary conditions are specified in terms of stresses that are determined from the displacements of eqn. (12), the stresses may be obtained as

$$\begin{aligned}
\tau_{xz}\big]_{z=0} &= -\mu f^{-3}\left\{\frac{2i\alpha_l\beta}{h^2} Ae^{-\alpha_l z} + \left(\frac{1}{\alpha_2}\left(\beta^2 + \alpha_2^2\right)B + \frac{\beta\gamma}{\alpha_2}C\right)e^{-\alpha_2 z}\right\} \\
\tau_{yz}\big]_{z=0} &= -\mu f^{-3}\left\{\frac{2i\alpha_l\gamma}{h^2} Ae^{-\alpha_l z} + \left(\frac{\beta\gamma}{\alpha_2}B + \frac{1}{\alpha_2}\left(\gamma^2 + \alpha_2^2\right)C\right)e^{-\alpha_2 z}\right\} \\
\sigma_z\big]_{z=0} &= -\mu f^{-3}\left\{\left(2\alpha_2 + k^2\right)\frac{A}{h^2} e^{-\alpha_l z} - 2i\left(\beta B + \gamma C\right)e^{-\alpha_2 z}\right\}.
\end{aligned} \tag{13}$$

Ground Compliance of a Rectangular Foundation

In a dynamic problem, the stress distribution under the foundation is not known since it depends on the displacement which is yet unknown. Thus one has to adopt the procedure, used by others, of assuming a stress distribution under the foundation and solving for the corresponding displacement.

If one designates the stress distribution under the foundation as $q_j(x, y, t)$ where the subscript j defines the type loading, its Fourier Transform is given by

$$f^3 q_j(x, y, t) = \bar{q}_j(\beta, \gamma, \omega) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(\xi, \eta, \zeta) e^{-i(\beta\xi + \gamma\eta + \omega\zeta)} d\xi d\eta d\zeta \quad (14)$$

Further more, by assuming the function of the stress distribution to be separable, i.e.,

$q_j(x, y, t) = q_j(x, y) Q(t)$, the Fourier Transform is

$$f\left[Q(t)\right] = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int Q(\zeta) e^{-i\omega\zeta} d\zeta = \bar{Q}(\omega) \quad (15)$$

and the inverse is given by

$$q_j(x, y, t) = \frac{\bar{Q}(\omega)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_j(\xi, \eta) e^{-i(\beta\xi + \gamma\eta)} d\xi d\eta. \quad (16)$$

Present attempt is to obtain dynamic compliance at any point of the rectangular footings for only three modes of vibrations. The loadings considered are of the type

1. Vertical loading;
2. Horizontal loading;
3. Loading produced by the rocking of the foundation about its centre-line.

Vertical loading

The boundary stresses are defined as

$$\tau_{xz} = 0; \quad \tau_{yz} = 0; \quad \sigma_z = q_v(x, y) Q(t) \quad (17)$$

Taking triple Fourier Transform of the stresses given by eqn. (13) and substituting them in eqns. (16) and (17), three equations involving the arbitrary functions A, B and C are

$$\frac{2i\alpha_1\beta}{h^2}Ae^{-\alpha_1 z}+\left(\frac{1}{\alpha_2}\left(\beta^2+\alpha_2^{\,2}\right)B+\frac{\beta\gamma}{\alpha_2}C\right)e^{-\alpha_2 z}=0 \tag{18}$$

$$\frac{2i\alpha_1\gamma}{h^2}Ae^{-\alpha_1 z}+\left(\frac{\beta\gamma}{\alpha_2}B+\frac{1}{\alpha_2}\left(\gamma^2+\alpha_2^{\,2}\right)C\right)e^{-\alpha_2 z}=0 \tag{19}$$

$$\left(2\alpha_2+k^2\right)\frac{A}{h^2}e^{-\alpha_1 z}-2i\left(\beta B+\gamma C\right)e^{-\alpha_2 z}=-\frac{\overline{Q}(\omega)}{2\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q_n(\xi,\eta)e^{-i(\beta\xi+\gamma\eta)}d\xi d\eta \tag{20}$$

Assuming $q_n\left(\xi,\eta\right)$ to be a uniform stress, $-q_0$ = constant, in which case the right side of eqn. (20) becomes

$$\frac{q_0\overline{Q}(\omega)}{2\pi\mu}\int_{-b}^{b}e^{-i\beta\xi}d\xi\int_{-c}^{c}e^{-i\gamma\eta}d\eta=\frac{4q_0}{2\pi\mu}\frac{\sin\beta b\,\sin\gamma c}{\beta\gamma}\overline{Q}(\omega) \tag{21}$$

Eqn. (20) can now be written as

$$\left(2\alpha_2+k^2\right)\frac{A}{h^2}e^{-\alpha_1 z}-2i\left(\beta B+\gamma C\right)e^{-\alpha_2 z}=\frac{4q_0}{2\pi\mu}\frac{\sin\beta b\,\sin\gamma c}{\beta\gamma}\overline{Q}(\omega) \tag{22}$$

Solving eqns. (18), (19) and (22), A, B and C are obtained as

$$\begin{aligned}A&=\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\,\sin\gamma c}{\beta\gamma}\right)\frac{h^2\left[2\left(\beta^2+\gamma^2\right)-k^2\right]}{F(\beta,\gamma)}\overline{Q}(\omega)\\ B&=-i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\,\sin\gamma c}{\beta\gamma}\right)\frac{2\alpha_1\alpha_2\beta}{F(\beta,\gamma)}\overline{Q}(\omega)\\ C&=-i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\,\sin\gamma c}{\beta\gamma}\right)\frac{2\alpha_1\alpha_2\gamma}{F(\beta,\gamma)}\overline{Q}(\omega)\end{aligned} \tag{23}$$

where $F(\beta,\gamma)=\left[2\left(\beta^2+\gamma^2\right)-k^2\right]-4\alpha_1\alpha_2\left(\beta^2+\gamma^2\right).$

The compliance in the vertical direction at the centre of the rectangular footing can be determined from eqn. (12) as

$$w=f^{-3}\left[\frac{\alpha_1 e^{-\alpha_1 z}}{h^2}-\frac{i\left(\beta B+\gamma C\right)}{\alpha_2}e^{-\alpha_2 z}\right] \tag{24}$$

For z = 0

$$w=f^{-3}\left[\frac{\alpha_1 A}{h^2}-\frac{i\left(\beta B+\gamma C\right)}{\alpha_2}\right]=f^{-3}[\overline{w}] \tag{25}$$

$$w=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{w}e^{i(\beta x+\gamma y+\omega t)}d\gamma d\beta d\omega \tag{26}$$

Again

$$\beta B + \gamma C = \beta\left[-i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma}\right)\frac{2\alpha_1\alpha_2\beta}{F(\beta,\gamma)}\overline{Q}(\omega)\right]$$

$$+\gamma\left[-i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma}\right)\frac{2\alpha_1\alpha_2\gamma}{F(\beta,\gamma)}\overline{Q}(\omega)\right]$$

$$= -i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma}\right)\frac{2\alpha_1\alpha_2\gamma}{F(\beta,\gamma)}\left[\beta^2+\gamma^2\right]\overline{Q}(\omega) \qquad (27)$$

which results in

$$\overline{w}(\beta,\gamma,\omega) = -\frac{4q_0\alpha_1 k^2}{2\pi\mu}\left(\frac{\sin\beta b\sin\gamma c}{\beta\gamma}\right)\frac{\overline{Q}(\omega)}{F(\beta,\gamma)} \qquad (28)$$

and $$w = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\sin\gamma c}{\beta\gamma}\right)\frac{\alpha_1 k^2}{F(\beta,\gamma)}\overline{Q}(\omega)e^{i(\beta x+\gamma y+\omega t)}d\gamma d\beta d\omega \qquad (29)$$

Omitting $e^{i\omega t}$

$$\frac{w\big]_{z=0}}{p_v Q(t)} = \frac{-1}{\pi^2 bc\mu}\int_0^{\infty}\int_0^{\infty}\frac{\alpha_1 k^2}{F(\beta,\gamma)}\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma}\cos\beta x\cos\gamma y d\beta d\gamma \qquad (30)$$

The ground compliance for vertical dynamic load obtained was in terms of double infinite integrals and for numerical evaluation of this integral some simplification is necessary and this can be achieved by the transformation of coordinates to reduce one of the infinite integral to finite one and render the integrals in a form more suitable for computation. Omitting Q (*t*), eqn. (30) can be written in the form

$$\frac{wc\mu}{p_v} = \frac{1}{\pi^2 b}\int_0^{\infty}\int_0^{\infty}\frac{\alpha_1 k^2}{F(\beta,\gamma)}\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma}\cos\beta x\cos\gamma y d\beta d\gamma \qquad (31)$$

defining $\beta = r'\cos\theta;\ \gamma = r'\sin\theta;\ \beta^2+\gamma^2 = r'^2;\ d\beta d\gamma = r'dr'd\theta$ and substituting $h = \omega/c_1 = (c_2/c_1)(\omega/c_2) = nk;$ where, $n = \sqrt{(1-2\nu)/(2(1-\nu))}$, the finite integral form of eqn. (31) can be obtained as

$$\frac{wc\mu}{p_v} = \frac{1}{\pi^2 b}\int_0^{\infty}\int_0^{\frac{\pi}{2}}\left[\frac{k^2\sqrt{r'^2-n^2k^2}}{F(r',k)}\right]\left[\frac{\sin(r'b\cos\theta)\ \sin(r'c\sin\theta)}{r'^2\sin\theta\cos\theta}\right] \qquad (32)$$
$$\left[\cos(r'x\cos\theta)\cos(r'y\sin\theta)\right]r'd\theta dr'$$

substituting $r' = rk$ *and* $(\omega b)/c_2 = bk = a_0$, eqn. (32) becomes

$$\frac{wc\mu}{p_v} = \frac{1}{\pi^2 b} \int_0^{\infty} \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{r^2 - n^2}}{F(r)} \right] \left[\frac{\sin(ra_0 \cos\theta)\, \sin\left(r\frac{c}{b} a_0 \sin\theta \right)}{r \sin\theta \cos\theta} \right] \left[\cos\left(r\frac{x}{b} a_0 \cos\theta \right) \cos\left(r\frac{y}{c}\frac{c}{b} a_0 \sin\theta \right) \right] dr d\theta \tag{33}$$

where $F(r) = (2r^2 - 1)^2 - 4r^2\sqrt{r^2 - n^2}\sqrt{r^2 - 1}$.

Eqn. (33) can be computed from the evaluation of the Cauchy principal value of the integral. The zero in the denominator occurs when $r = z_0$ which corresponds to

Eqn. (33) can be computed from the evaluation of the Cauchy principal value of the integral. The zero in the denominator occurs when $r = z_0$ which corresponds to the simple poles of the Rayleigh function F(r). In order to avoid standing waves, or to insure that only outgoing waves are present, it is necessary to subtract one half of the

$$\frac{-i}{\pi a_0} \int_0^{\frac{\pi}{2}} \frac{\sqrt{z_0^2 - n^2}}{F'(z_0)} \frac{\sin(z_0 a_0 \cos\theta)\, \sin\left(z_0 \frac{c}{b} a_0 \sin\theta \right)}{z_0 \sin\theta \cos\theta} d\theta. \tag{34}$$

Horizontal Loading

In this case a horizontal shear load PH is applied to the foundation in the direction of X, leading to the boundary conditions

$$\sigma_z = 0; \tau_{yz} = 0; \tau_{xz} = q_H(x, y, t) = -q_0 Q(t)$$

Assuming again that the distribution of shear stress under the foundation is to be uniform and is equal to $-q_0$ and proceeding as in the vertical case, the quantities A, B and C are

$$A = i\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma} \right) \frac{2\beta\alpha_2 h^2}{F(\beta,\gamma)} \bar{Q}(\omega)$$

$$B = \frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma} \right) \frac{(2\alpha_2^2 + k^2)(\gamma^2 + \alpha_2^2) - 4\alpha_1\alpha_2\gamma^2}{\alpha_2 F(\beta,\gamma)} \bar{Q}(\omega)$$

$$C = -\frac{4q_0}{2\pi\mu}\left(\frac{\sin\beta b\ \sin\gamma c}{\beta\gamma} \right) \frac{(2\alpha_2^2 + k^2 - 4\alpha_1\alpha_2)}{\alpha_2 F(\beta,\gamma)} \bar{Q}(\omega)$$

The compliance in the horizontal direction can be found out by substituting values of A, B and C in eqn. (12) and solving the equation similarly as in the vertical case. The final expression for the ground compliance in the horizontal case is given by

$$\frac{u]_{z=0}}{p_H Q(t)} = \frac{1}{\pi^2 bc\mu} \int_0^\infty \int_0^\infty \left[\frac{F(\beta,\gamma) - \beta^2 \left[k^2 - 4\alpha_2(\alpha_1 - \alpha_2) \right]}{\alpha_2 F(\beta,\gamma)} \right] \left[\frac{\sin\beta b \sin\gamma c}{\beta\gamma} \cos\beta x \cos\gamma y \right] d\beta d\gamma \tag{35}$$

Making the substitutions similarly as in the vertical case to convert the infinite integral to finite one, the simplified expression for ground compliance for horizontal case then becomes

$$\frac{uc\mu}{p_H} = \frac{1}{\pi^2 a_0} \int_0^{\frac{\pi}{2}} \int_0^\infty \left[\frac{F(r)\sin^2\theta - (r^2-1)\cos^2\theta}{\sqrt{r^2-1}F(r)} \right] \left[\frac{\sin(ra_0\cos\theta)\,\sin\left(r\frac{c}{b}a_0\sin\theta\right)}{r\sin\theta\cos\theta} \right] \left[\cos\left(r\frac{x}{b}a_0\cos\theta\right)\cos\left(r\frac{y}{c}\frac{c}{b}a_0\sin\theta\right) \right] dr d\theta \tag{36}$$

While evaluating the numerical values singularities will occur due to nature of the Rayleigh function one at the Rayleigh pole and the other at $r = 1$. Following the similar arguments given for the vertical case, to avoid error due to these singularities, the above equation should be added to the following quantity

$$-\frac{i}{\pi a_0} \int_0^{\frac{\pi}{2}} \left[\frac{(1-i)\sin^2\theta}{\sqrt{2}} - \frac{\sqrt{z_0^2-1}}{f'(z_0)}\cos^2\theta \right] \left[\frac{\sin(z_0 a_0\cos\theta)\,\sin\left(z_0\frac{c}{b}a_0\sin\theta\right)}{z_0\sin\theta\cos\theta} \cos\left(z_0\frac{x}{b}a_0\cos\theta\right)\cos\left(z_0\frac{y}{c}\frac{c}{b}a_0\sin\theta\right) \right] d\theta. \tag{37}$$

Rocking loading

In this case foundation is assumed to undergo rotation about the x axis. The shear stress under the foundation is assumed to be zero as in the case (a) and the normal stress is assumed to increase linearly with y, the boundary conditions for the shear is same as vertical loading and additional boundary condition is given by

$\sigma_z]_{z=0} = -q_0 \frac{y}{c} Q(t)$ and the total moment M_R is $M_R = \frac{4}{3} q_0 bc^2$.

Substituting these boundary conditions in eqn. (13), the values of A, B and C can be found out. Substituting the values of A, B and C in eqn. (24) and proceeding as in the vertical case, the final expression for vertical compliance at any point can be obtained as

$$w = -\frac{q_0 Q(t) i}{\pi^2 \mu c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\alpha_1 k^2}{F(\beta,\gamma)} \frac{\sin\beta b}{\beta} \left(\frac{\sin\gamma c}{\gamma^2} - \frac{c\cos\gamma c}{\gamma} \right) e^{i(\beta x + \gamma y)} d\beta d\gamma \tag{38}$$

On simplification, the above equation reduces to

$$w=\frac{3M_RQ(t)}{\pi^2\mu bc^3}\int_0^\infty\int_0^\infty\frac{\alpha_1 k^2}{F(\beta,\gamma)}\frac{\sin\beta b\sin\gamma y}{\beta\gamma}\left[\frac{\sin\gamma c}{\gamma}-c\cos\gamma c\right]\cos\beta x\,d\beta\,d\gamma \tag{39}$$

Making similar substitutions as in the vertical case to make the infinite integral to finite one, the simplified expression for vertical ground compliance at any point in the rocking mode of vibrations is given by

$$\frac{w\mu c^2}{M_R}=\frac{3}{\pi^2 a_0}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\sqrt{r^2-n^2}}{F(r)}\left[\frac{\sin\left(\frac{c}{b}a_0 r\sin\theta\right)}{\frac{c}{b}a_0 r\sin\theta}-\cos\left(\frac{c}{b}a_0 r\sin\theta\right)\right]$$
$$\left[\frac{\sin(a_0 r\cos\theta)\sin\left(\frac{c}{b}\frac{y}{c}a_0 r\sin\theta\right)\cos\left(\frac{x}{b}a_0 r\cos\theta\right)}{r\sin\theta\cos\theta}\right]dr\,d\theta \tag{40}$$

and rotation ϕ is obtained as

$$\phi=\frac{w\big]_{z=0,\,y=c,\,x=0}}{c}$$

$$\frac{\phi\mu c^3}{M_R}=\frac{3}{\pi^2 a_0}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\sqrt{r^2-n^2}}{F(r)}\left[\frac{\sin\left(\frac{c}{b}a_0 r\sin\theta\right)}{\frac{c}{b}a_0 r\sin\theta}-\cos\left(\frac{c}{b}a_0 r\sin\theta\right)\right]$$
$$\left[\frac{\sin(a_0 r\cos\theta)\sin\left(\frac{c}{b}a_0 r\sin\theta\right)}{r\sin\theta\cos\theta}\right]dr\,d\theta. \tag{41}$$

While evaluating the numerical values singularities will occur due to the nature of the Rayleigh function at the Rayleigh pole. Following the similar arguments given for the vertical case, to avoid error due to these singularities, the above equation should be added to the following quantity

$$\frac{\phi\mu c^3}{M_R}=-\frac{3i}{\pi a_0}\int_0^{\frac{\pi}{2}}\frac{\sqrt{z_0^2-n^2}}{F(z_0)}\left[\frac{\sin\left(\frac{c}{b}a_0 z_0\sin\theta\right)}{\frac{c}{b}a_0 z_0\sin\theta}-\cos\left(\frac{c}{b}a_0 z_0\sin\theta\right)\right]$$
$$\left[\frac{\sin(a_0 z_0\cos\theta)\sin\left(\frac{c}{b}a_0 z_0\sin\theta\right)}{z_0\sin\theta\cos\theta}\right]d\theta. \tag{42}$$

Zero Frequency (Static) Displacement

The above-derived equations are not suitable for zero frequency case $a_0=0$, thus limiting case $\lim_{\omega \to 0}$ is solved and a closed form solution is presented hereunder.

Vertical loading

As $\omega/c_2 = k$, when $\omega \to 0$; $k \to 0$ Hence in eqn. (32) for $k \to 0$

$$\lim_{k \to 0} \left[\frac{k^2 \sqrt{r'^2 - n^2 k^2}}{F(r',k)} \right] = \frac{1}{2r'(n^2 - 1)}$$

Eqn. (32) will become

$$\left. \frac{wc\mu}{P_v} \right]_{\omega=0} = \frac{1}{\pi^2 b} \int_0^{\infty} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2(n^2-1)} \right] \left[\frac{\sin(r'b\cos\theta)\ \sin(r'c\sin\theta)}{r'^2 \sin\theta\cos\theta} \right] \left[\cos(r'x\cos\theta)\cos(r'y\sin\theta) \right] d\theta dr' \quad (43)$$

A simplified form may be written as

$$\left. \frac{wc\mu}{P_v} \right]_{\omega=0} = \frac{1}{16\pi(n^2-1)} \left[I_1 + I_2 + I_3 + I_4 \right]$$

in which

$$I_{1,4} = \frac{c}{b}\left(1 \pm \frac{y}{c}\right) \int_0^{\theta} \frac{d\theta}{\cos\theta} + \left(1 \pm \frac{x}{b}\right) \int_0^{\frac{\pi}{2}-\theta} \frac{d\theta}{\cos\theta} \qquad \theta = \tan^{-1}\left(\frac{1 \pm \frac{x}{b}}{1 \pm \frac{y}{c}} \right)$$

$$I_{2,3} = \frac{c}{b}\left(1 \pm \frac{y}{c}\right) \int_0^{\theta} \frac{d\theta}{\cos\theta} + \left(1 \mp \frac{x}{b}\right) \int_0^{\frac{\pi}{2}-\theta} \frac{d\theta}{\cos\theta} \qquad \theta = \tan^{-1}\left(\frac{1 \mp \frac{x}{b}}{1 \pm \frac{y}{c}} \right).$$

Horizontal Loading

The limiting value for the static case will be

$$\lim_{k \to 0} \left[\frac{F(r',k) - r'^2 \cos^2\theta \left[4r'^2 - 3k^2 - 4\sqrt{r'^2 - k^2}\sqrt{r'^2 - n^2 k} \right]}{F(r',k)\sqrt{r'^2 - k^2}} \right]$$

$$= \frac{1}{r'} \left[1 - \frac{\cos^2\theta(2n^2 - 1)}{2(n^2 - 1)} \right]$$

The expression for zero frequency displacement then reduces to

$$\left.\frac{uc\mu}{P_H}\right]_{\omega=0} = \frac{1}{\pi^2 b}\int_0^{\infty}\int_0^{\frac{\pi}{2}}\left[1-\frac{\cos^2\theta\left(2n^2-1\right)}{2\left(n^2-1\right)}\right]\left[\frac{\sin\left(r'b\cos\theta\right)\sin\left(r'c\sin\theta\right)}{r'^2\sin\theta\cos\theta}\right] \left[\cos\left(r'x\cos\theta\right)\cos\left(r'y\sin\theta\right)\right]dr'd\theta \tag{44}$$

On simplification as in vertical case the above equation can be written as

$$\left.\frac{wc\mu}{P_H}\right]_{\omega=0} = \frac{I_1+I_2}{4\pi}-\left(\frac{2n^2-1}{8\pi\left(n^2-1\right)}\right)I_3-\left(\frac{2n^2-1}{8\pi\left(n^2-1\right)}\right)I_2 \tag{45}$$

in which

$$I_1=\frac{c}{b}\left(1\pm\frac{y}{c}\right)\int_0^{\theta}\frac{d\theta}{\cos\theta};\quad \theta=\tan^{-1}\left(\frac{1\pm\frac{x}{b}}{1\pm\frac{y}{c}}\right)$$

$$I_2=\left(1\pm\frac{x}{b}\right)\int_0^{\frac{\pi}{2}-\theta}\frac{d\theta}{\cos\theta};\quad \theta=\tan^{-1}\left(\frac{1\pm\frac{x}{b}}{1\pm\frac{y}{c}}\right)$$

$$I_3=\frac{c}{b}\left(1\pm\frac{y}{c}\right)\sin\theta+\left(1\pm\frac{x}{b}\right)\cos\theta;\quad \theta=\tan^{-1}\left(\frac{1\pm\frac{x}{b}}{1\pm\frac{y}{c}}\right).$$

Rocking Loading

Form eqn. (42), by applying limits the final equation for rocking will be

$$\frac{\phi\mu c^3}{M_R}=\frac{3}{2\pi^2\frac{c}{b}\left(n^2-1\right)}\int_0^{\infty}\int_0^{\frac{\pi}{2}}\left[\frac{\sin\left(\frac{c}{b}r\sin\theta\right)\sin\left(r\cos\theta\right)\sin\left(\frac{c}{b}r\sin\theta\right)}{r^3\sin^2\theta\cos\theta}-\frac{\cos\left(\frac{c}{b}r\sin\theta\right)\sin\left(r\cos\theta\right)\sin\left(\frac{c}{b}r\sin\theta\right)}{r^2\sin\theta\cos\theta}\right]drd\theta. \tag{46}$$

Computation of Response

The procedure outlined by Elorduy et al. (1967) has been followed for the evaluation of response. A rigid rectangular footing with sides 2b (width) and 2c (length) resting on the elastic halfspace (Fig. 1) is assumed. Solutions given by eqns. (33)+(34), (36)+(37) and (41)+(42) were taken as the response of the footing under vertical, horizontal and rocking modes of vibration, respectively.

Vertical and Horizontal Modes:

The following scheme has been adopted for the computation:

1. Let the force at any instant t is given by

   $P = P_0 \cos(\omega t + \varphi_{P-R})$

   Where P_0 is the amplitude, ω the circular frequency of the exciting force and φ_{P-R} is the phase between the force and the reaction R of the halfspace.

2. Dynamic response(displacement) of the base at any instant t is

$$w = \frac{R}{\mu c}(f_1 \cos \omega t - f_2 \sin \omega t)$$

3. Amplitude of displacement, A and reaction, R of the halfspace respectively are:

$$A = \frac{P_0}{\mu c}\sqrt{\frac{f_1^2 + f_2^2}{\left(1 + \chi b_1 a_0^2 f_1\right)^2 + \left(b_1 a_0^2 f_2\right)^2}} = \frac{P_0}{\mu c} M \, , \tag{47}$$

where M is called the response factor.

$$R = \frac{P_0}{\sqrt{\left(1 + \chi b_1 a_0^2 f_1\right)^2 + \left(b_1 a_0^2 f_2\right)^2}} \tag{48}$$

Phase angles

Between R and w: $\varphi_{R-w} = \tan^{-1}\left(-\frac{f_2}{f_1}\right)$

Between P and R: $\varphi_{P-R} = \tan^{-1}\left(\frac{b_1 a_0^2 f_2}{1 + \chi b_1 a_0^2 f_1}\right)$

Between P and w: $\varphi_{P-w} = \tan^{-1}\left(\frac{-f_2}{\chi f_1 + b_1 a_0^2\left(f_1^2 + f_2^2\right)}\right)$

4. Input power:

$$L = \frac{-P_0^2}{2bc\sqrt{\mu\rho}}\left[\frac{a_0 f_2}{\left(1 + \chi b_1 a_0^2 f_1\right)^2 + \left(b_1 a_0^2 f_2\right)^2}\right]$$

In the above

b_1 = Mass ratio = $\frac{m_0}{\rho c b^2}$; a_0 = Frequency factor = $\frac{\omega b}{c_2}$ where c_2 is the shear wave velocity of the halfspace and χ = -1 for the vertical and = 1 for the horizontal response.

Rocking Mode

The following scheme has been adopted for the computation:

1. Let the force at any instant t is given by

   $M_R = M_{R0} \cos(\omega t + \varphi_{T\text{-}R})$

Where M_{R0} is the amplitude, ω the circular frequency of the exciting force and $\varphi_{P\text{-}R}$ is the phase between the force and the reaction R of the halfspace.

2. Dynamic response (rotation with respect to the x-axis, Fig. 1) of the base at any instant t is

$$\phi = \frac{R}{\mu c^3}(f_1 \cos \omega t - f_2 \sin \omega t)$$

3. Amplitude of displacement, A_θ and reaction, R of the halfspace respectively are:

$$A_\theta = \frac{M_{R0}}{\mu c^3}\sqrt{\frac{f_1^2 + f_2^2}{\left(1 - b_1 a_0^2 f_1\right)^2 + \left(b_1 a_0^2 f_2\right)^2}} = \frac{M_{R0}}{\mu c^3} M, \tag{49}$$

where M is called the response factor.

$$R = \frac{M_{R0}}{\sqrt{\left(1 - b_1 a_0^2 f_1\right)^2 + \left(b_1 a_0^2 f_2\right)^2}} \tag{50}$$

Phase angles

Between R and ϕ: $\varphi_{R-\phi} = \tan^{-1}\left(-\frac{f_2}{f_1}\right)$

Between M_R and R: $\varphi_{M_R - R} = \tan^{-1}\left(\frac{b_1 a_0^2 f_2}{1 - b_1 a_0^2 f_1}\right)$

Between M_R and ϕ: $\varphi_{M_R - \phi} = \tan^{-1}\left(\frac{-f_2}{\chi f_1 + b_1 a_0^2\left(f_1^2 + f_2^2\right)}\right)$

4. Input power:

$$L = \frac{-M_{R0}^2}{2c^3\sqrt{\mu\rho}}\left[\frac{a_0 f_2}{\left(1-b_1 a_0^2 f_1\right)^2+\left(b_1 a_0^2 f_2\right)^2}\right]$$

In the above

b_1 = Mass ratio = $\frac{I_0}{\rho c^3 b^2}$; a_0 = Frequency factor= $\frac{\omega b}{c_2}$; I_0 is the mass moment of inertia about the base with respect to the rocking axis(x-axis) and c_2 is the shear wave velocity of the halfspace.

Resonant values

If it is assumed that a_0^* is the frequency factor at which the response is the maximum, the relation for resonant condition (Dasgupta and Kameswara Rao, 1978) may be written as

$$f_1 + b_1 a_0^{*2}(f_1^2 + f_2^2) = 0 \tag{51}$$

__ an approximate relation, but the accuracy is within 2%.

Computational Scheme

General

The equations to be evaluated are complex functions and require special consideration. In evaluating the integral, it is necessary to use 96 points Gaussian quadrature because of the complexity of the sine and cosine fluctuations. The integration has been tested for other possible Gauss points and conclusion is that a 96-point Gauss quadrature leads to a stable converging solution since a closer interval is needed to account for the sharp variation of sine and cosine functions. In addition, the entire interval is divided into five parts to account for the nature of function F(r), namely

(0 to 0.5) → real;

(0.5 to pn) → real;

(pn to 1) → complex;

[1 to Z_0(root of equation F(r)] → positive real;

and (Z_0 to ∞) → negative real.

The Dynamic Case

1. Suitable values of Poisson's ratio, frequency ratio and also x/b, y/c, c/b values are selected.
2. Numerical values of the function F(r) its derivative and F´(r) are computed.
3. The compliance functions have been evaluated by Gaussian double integration scheme.
4. A 96-Gauss-point Gauss quadrature has been used for evaluating the integral.
5. While evaluating the integral, the interval should be divided into parts to account for the complexities in the evaluation of function F(r).
6. The intervals are (0 to 0.5), (0.5 to pn), (pn to 1), [1 to Z_0 (root of equation F(r) = 0] and (Z_0 to ∞).
7. The interval (Z_0 to ∞) can be evaluated by substituting instead $1/r$ instead of r in the integral.
8. The value thus obtained should be subtracted to account for singularities /standing waves occur due to the nature of the Rayleigh function F(r).

The static case

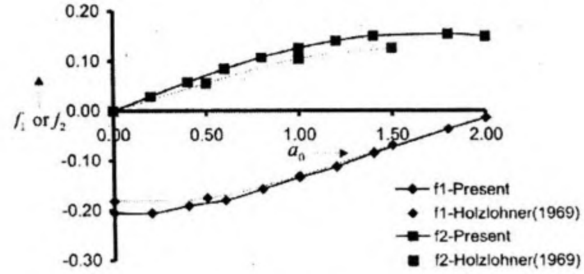
1. Solutions are available in the literature, for vertical and horizontal cases. However, these are now simplified to a single integral form where numerical values have been found by integrating the terms and applying the limits.
2. To find numerical values in rotational case Gaussian double integration method

   was used with a substitution $r = \frac{2\alpha}{(1+x_i)} - \alpha$, where α =1

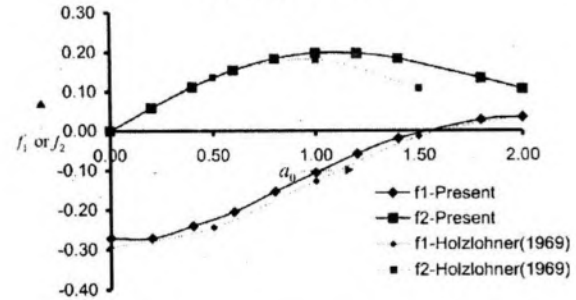
Results and Discussion

Equations have been derived for all three modes of vibrations (vertical, horizontal, rotational) at any point of a rectangular footing as described in the previous chapter. Numerical evaluation of the equations was performed using standard techniques discussed in the preceding.

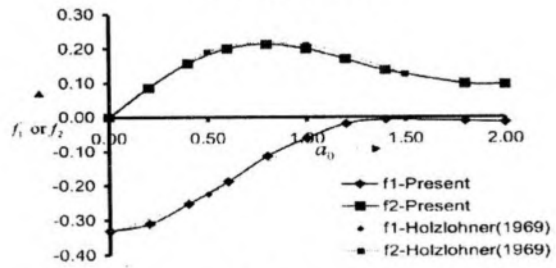
Comparison of results with the previous calculations indicates the results obtained from our study are in good agreement with the other. Present results have been compared with Holzlhner (1969) solutions for different length to width ratio. These comparisons are as shown in Fig. 2(a) to 2(c). The values are slightly greater than the Holzlhner (1969) results, which are expected according to the clarifications given by him.



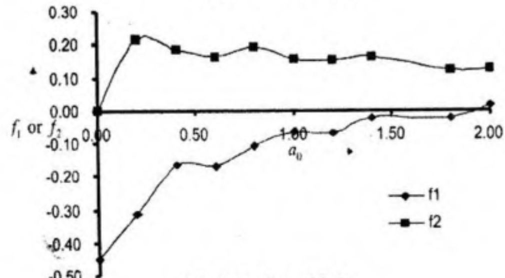
(a) For c/b= 1.0



(b) For c/b = 2.0



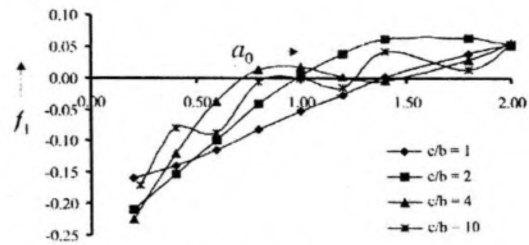
(c) For c/b = 3.0



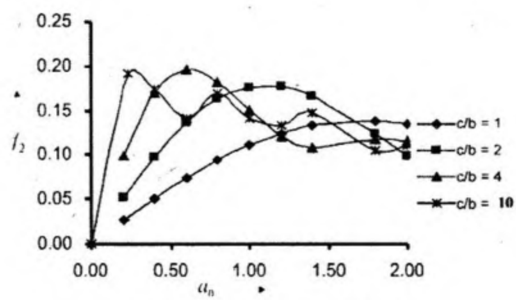
(d) For c/b = 10.0

Fig. 2.Comparison of Non-dimensional Vertical Compliance Vs. Frequency Factors for Poisson's Ratio 0.25 at Center of the Footing.

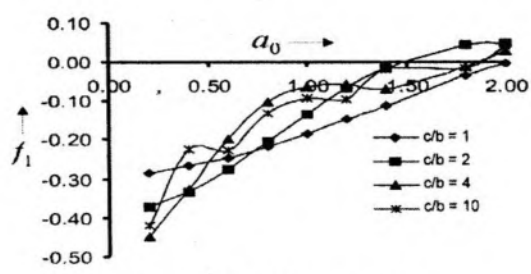
Compliance functions at centre of the rectangular footing are obtained for vertical mode of vibration and are plotted against frequency ratio for different length to width ratio and Poisson's ratio. These results are shown in Figs. 2(d) to 3(f). As expected, the real part of the compliance is negative and is decreases with a_0. However, the imaginary part of the compliance is positive and increases up to certain value of a_0 and then becomes almost constant. These compliance values are considerably changing with length to width ratio of the footing and Poisson's ratio. As expected, for constant frequency ratio the compliance functions are getting decreased as the Poisson's ratio increases. This is because as the soil stiffens the compliance values are becoming lesser and lesser.



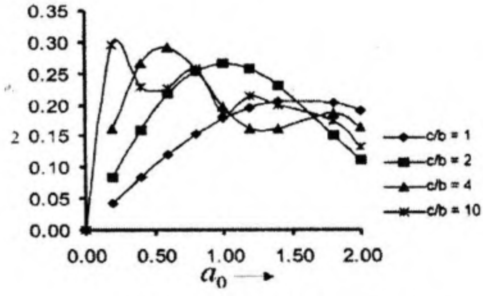
(a) Variation of f_1 for ν = 0.33



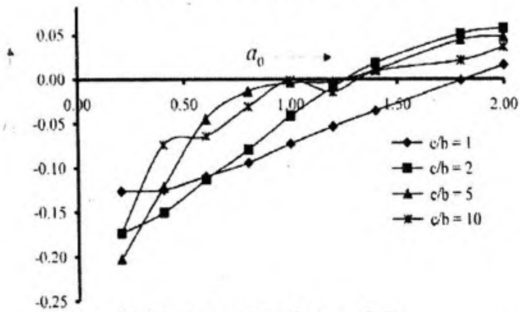
(b) Variation of f_2 for ν = 0.33



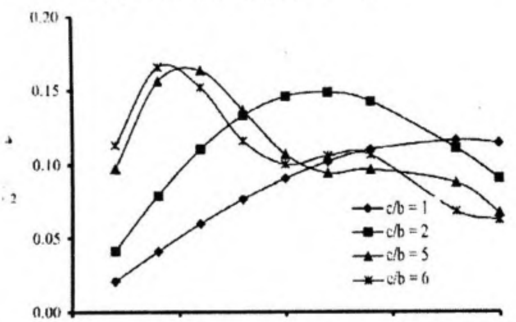
(c) Variation of f_1 for ν = 0.00



(d) Variation of f_2 for $\nu = 0.00$



(e) Variation of f_1 for $\nu = 0.5$

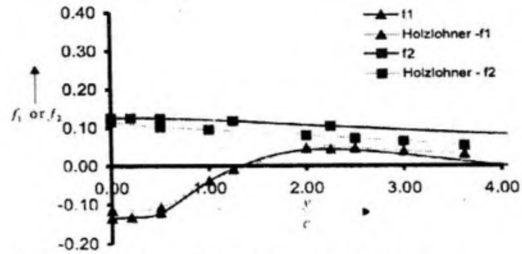


(f) Variation of f_2 for $\nu = 0.5$

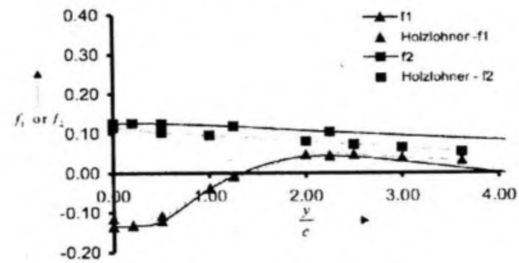
Fig. 3 Non Dimensional Vertical Compliance Vs. Frequency Factors for Different Poisson's Ratio at Center of the Footing

Comparisons have been made for vertical mode of vibrations and values away from the centre of the footing as shown in Fig. 3(a) and 3(b).

Plots are also made, as shown in Fig. 4(a) and 4(b), for compliance functions against the distance away from the centre of the footing. As expected these values are decreasing as the distance from the centre is increasing.



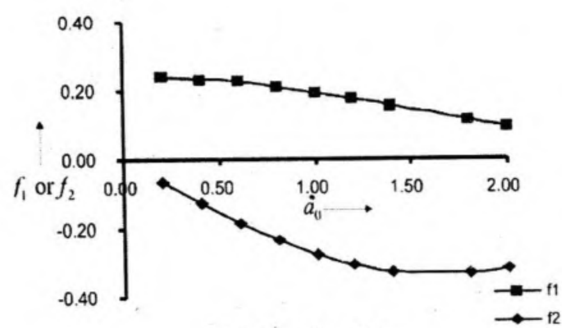
(a) For frequency factor of 1.0



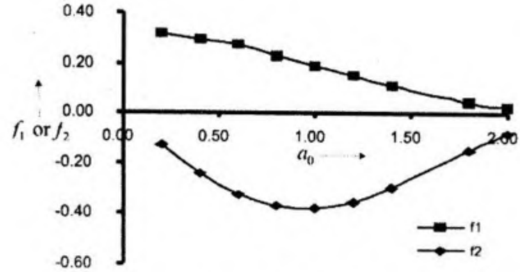
(b) For frequency factor of 1.5

Fig. 4 Comparison of Vertical Compliance Away
from Center of Footing for Poisson's Ratio ν = 0.25.

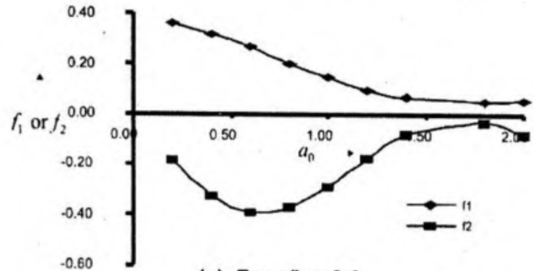
Compliance functions at the centre of the rectangular footing for horizontal mode of vibrations are plotted against frequency ratio for different length to width ratio and Poisson's ratios. These are as shown in Fig. 5(a) to 5(d). Real part of the compliance is positive and is decreasing with the frequency ratio. And also, imaginary part of the compliance is decreasing with increase in frequency ratio. Plots are also made as shown in Fig. 6(a) and 6(d) for compliance functions against the distance away from centre of the footing, as expected these values are decreasing as the distance from the center is increasing.



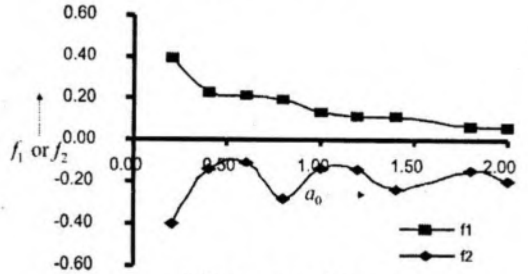
(a) For c/b = 1.0



(b) For c/b = 2.0

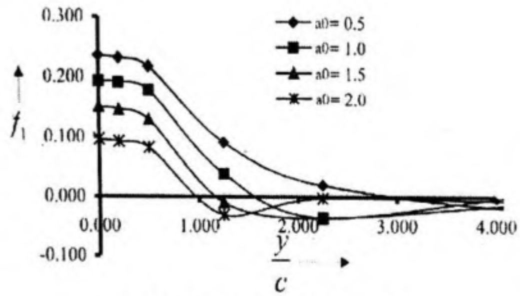


(c) For c/b = 3.0

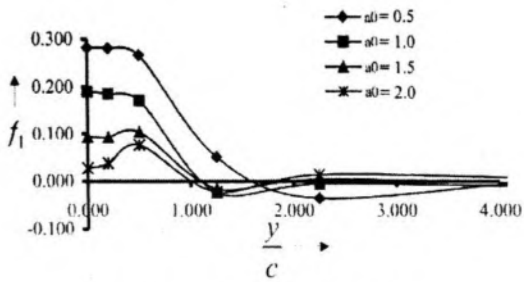


(d) For c/b = 10.0

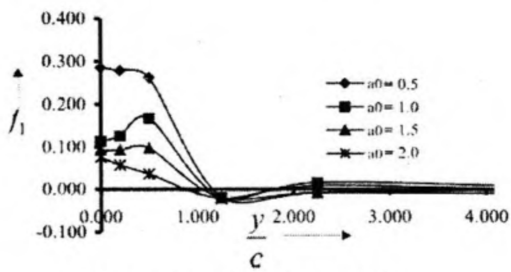
Fig. 5 Non-dimensional Horizontal Compliance vs. Frequency Factors for Poisson's Ratio ν = 0.25 **at** Center **of the** Footing.



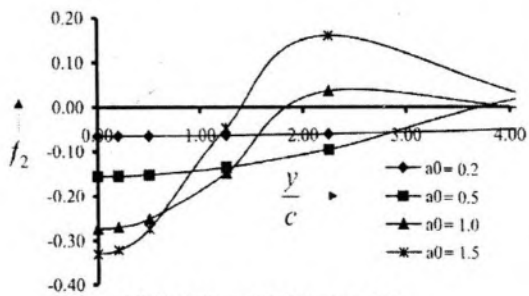
(a) Variation of f_1 for c/b = 1.0



(b) Variation of f_1 for c/b = 2.0



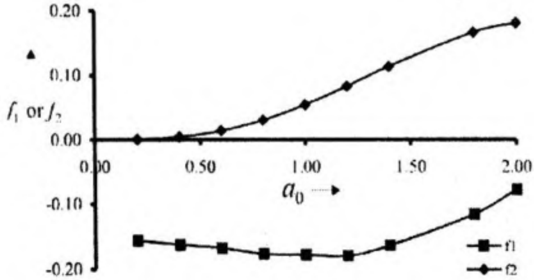
(c) Variation of f_1 for c/b = 4.0



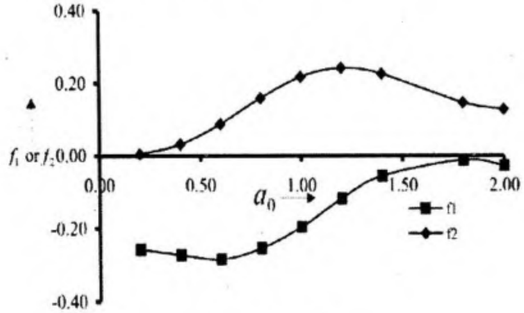
(d) Variation of f_2 for c/b = 4.0

Fig. 6 Horizontal Compliance Away from Center of Footing forpoisson's Ratio $\nu = 0.25$.

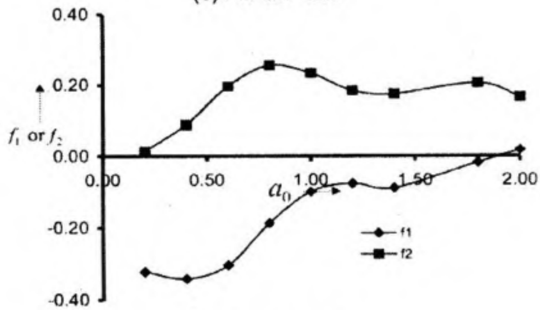
Compliance functions for rotation mode of vibrations are plotted as in Fig. 7(a) to 7(d) real part of the compliance is negative and is decreasing with frequency ratio. The imaginary part of the compliance is increasing with increase in frequency ratio. Non-dimensional static displacement factors are also obtained and are presented Fig. 8 to 10(b).



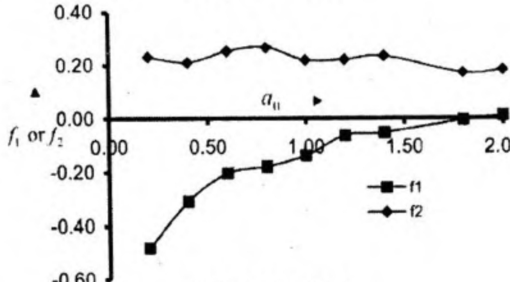
(a) For c/b = 1.0



(b) For c/b =2.0



(c) For c/b = 3.0



(d) For c/b =10.0

Fig. 7 Non-dimensional Rotational Compliance vs. Frequency Factors for Poisson's Ratio ν = 0.25.

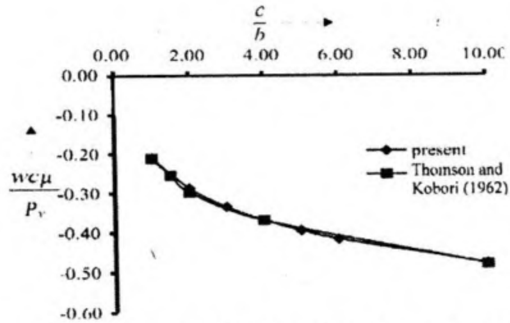
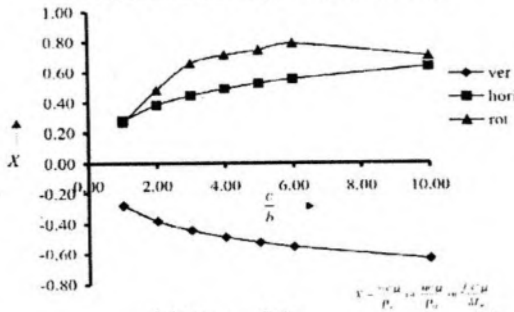
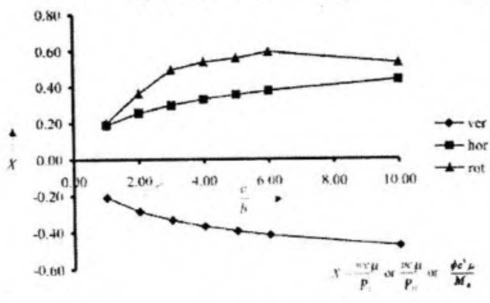


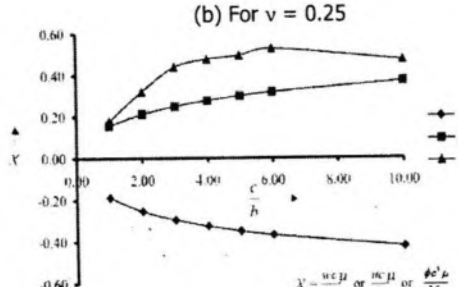
Fig. 8 Comparison Non Dimensional Static Vertical Displacement vs. **c**/b Value for ν = 0.25



(a) For ν = 0.0

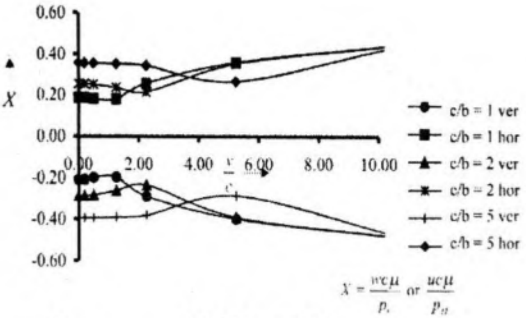


(b) For ν = 0.25

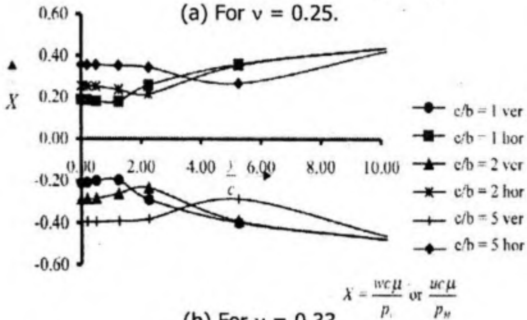


(c) For ν = 0.33

Fig. 9 Non Dimensional Static Displacement vs. c/b Value for Various of Poisson's Ratios.



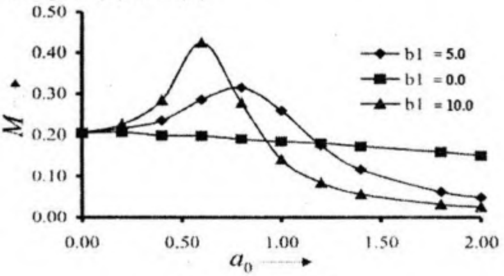
(a) For ν = 0.25.



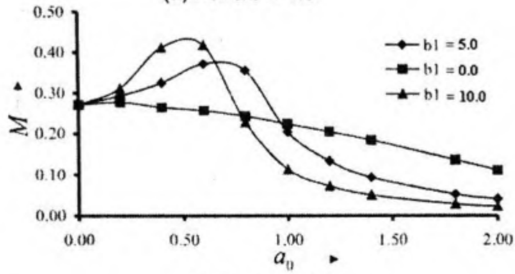
(b) For ν = 0.33.

Fig. 10 Non Dimensional Static Displacements Away from Center of Footing

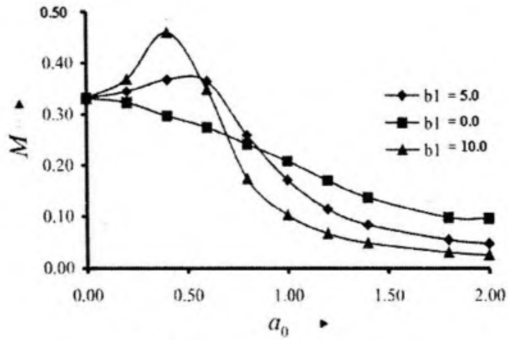
Response factor (M) has been calculated for different frequency ratios and for the different mass ratio of 0, 5 and 10 varying with length to width ratio of the footing. These are shown in Fig. 11(a) to 11(c).



(a) For c/b = 1.0



(b) For c/b = 2.0



(c) For c/b = 3.0

Fig. 11. Response Factor vs. Frequency Factors At Center of a Rectangular Footing for Vertical Mode of Vibrations for ν **= 0.25**

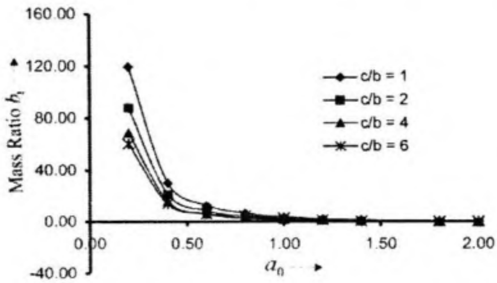


Fig. 12 Mass ratio Vs. frequency ratio at resonance frequency at center of a rectangular footing for vertical mode of vibrations for ν = 0.25.

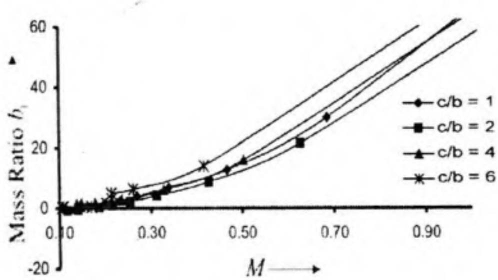


Fig. 13 Mass Ratio Vs. Response Factor at Resonance Frequency at Center of a Rectangular Footing for Vertical Mode of Vibrations for ν = 0.25.

In Figs. 12 and 13 plots have been made between mass ratio against frequency ratio and mass ratio against the response factor at resonant frequency from where resonance amplitude can be found out.

Conclusions

A numerical solution has been presented for three modes of vibrations of a rectangular footing resting on semi-infinite, homogeneous and isotropic elastic medium. Solutions were obtained for vertical, horizontal, and rotational modes of vibrations. The expressions for zero frequency displacements i.e. static non-dimensional displacements are also obtained by applying limits to the above derived equations. Fourier triple integration technique was used to solve the above problem.

Compliance functions and hence the displacements, at any point on the soil surface have been obtained for three modes of vibrations by solving the integral expressions using Gaussian qudrature.

For demonstration purpose, as is the standard procedure, compliance functions f_1 *and* f_2 at centre of the footing were drawn against frequency ratio for all three modes of vibrations. Plots are also made for compliance functions against the distance away from centre of the footing. Non-dimensional static displacement factors are also been obtained and are presented against various c/b ratios.

Response factor (M) has been calculated for different frequency factors, for the mass ratios of 0, 5 and 10 and for various lengths to with ratio of the footing (c/b). Plots are also made between mass ratio against frequency ratio and mass ratio against response factor at resonant frequency. Comparisons are also made with the existing solutions. These comparisons shows that the results obtained from the present study are in good agreement with the previous studies.

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Notations

A	=	Amplitude of dynamic displacement
A_θ	=	Amplitude of dynamic rotation
a_0	=	Dimensionless frequency factor
$2b,\ 2c$	=	Width and length of the rectangular footing
b_1	=	Mass ratio
c_1	=	Dilatational wave velocity
c_2	=	Shear wave velocity
E	=	Modulus of elasticity of soil
f_1 and f_2	=	Compliance functions
G	=	Shear modulus of soil
h	=	A parameter which is the ratio of
k	=	A parameter which is the ratio of
M	=	Response factor
M_R	=	Moment produced due to rocking
M_{R0}	=	Amplitude of moment due to rocking
n	=	Velocity ratio which is the ratio of
P_H	=	Dynamic horizontal load applied on the footing

P_H	=	Dynamic horizontal load applied on the footing
P_V	=	Dynamic vertical load applied on the footing
Q(X, Y, Z, t)	=	Surface traction
q_o	=	Uniform stress produced due to the application of dynamic loads
t	=	Time
$w,\ u$	=	Vertical and horizontal displacement amplitudes respectively
z_0	=	Rayleigh pole
ν	=	Poisson's ratio
ρ	=	Mass density of soil
λ	=	Lame's constant
μ	=	Shear modulus of elasticity
σ_z	=	Vertical stress
τ_{xz}, τ_{yz}	=	Shear stresses acting perpendicular to z- axis along x and y directions respectively
ω	=	Circular frequency of vibrations
θ	=	Rotation
φ	=	Phase angle
ϕ	=	Rotation due to rocking.