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Dynamic Response of Piles Under Vertical Loads

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Introduction

Based on the analytical solution of Baranov (1967), Novak (1974) proposed a method for evaluating the vertical response of piles under dynamic loading. Many researchers namely, Wolf and Von Arx (1978), Waas (1981), Kausel and Kaynia (1982), Banerjee and Sen (1987) have advanced their solutions to this problem, yet Novak's method is mostly preferred, particularly in design offices. Solution of pile and pile-group based on the method proposed by Banerjee and Sen (1985, 1987) is based on Boundary Element Method and has been found to give accurate results but it is computationally too exhaustive to find applications in a day-to-day design office work.

The Finite Element Method has been used where the pile is being modeled as beam elements and the soil being modeled as Winkler springs. Such models have, however, yielded good results either when the piles are single or the distance between piles is significant (≥5d, d being the diameter of the pile) where the pile-soil interaction can be neglected.

Novak's solution is mostly based on charts and it furnishes stiffness and damping of a pile and the solution is addressed to the fundamental degree of freedom.

However,

a) The solution does not take into account for the inertial effect of the pile;

b) Interpolation is required when the design data are out of range of the chart;

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c) Charts are available only for RCC or timber piles, whether these charts are applicable to the cases of steel piles[1], there is no clear-cut guideline;

e) The charts do not address to the issues where a pile is partially embedded;

f) It is also not clear if the charts are applicable for varying dynamic soil modulus.

In certain cases when the piles are supporting reciprocating compressors, it becomes essential to check the design for higher frequencies of the foundations to ensure that they are not matching with the second or third harmonics, when the machine at harmonic other than the first may induce the higher forces. In such cases one has no other options but to resort to an elaborate and expensive three-dimensional Finite Element based soil-pile foundation model to arrive at an answer to this problem and in number of cases uncertainties present in such results are many.

The results provided in the present paper are for the fundamental mode only and the solutions have been worked out for the first three modes.

Proposed Method

In the present paper an approach has been outlined where the pile stiffness and damping may be computed for any type of pile material like concrete, steel or timber and it takes into account for the variation of elastic modulus of soil, if any, and also the inertial and stiffness effect of pile cap have been incorporated.

Most of the work relating to dynamic stiffness of pile is based on Baranov's (1967) theory on the response of a soil embedded foundation. The present formulation is based on Novak and Beredugo's (1972) approach on embedded foundation.

Vibration of Piles in the Vertical Direction

Vibration of Friction Piles

Stiffness of the Pile: For this case the pile is assumed providing the resistance both through bearing as well as friction as shown in Fig.1. K_f represents the frictional stiffness of the pile and the pile tip bearing stiffness is taken as K_b. The longitudinal vibration of such beams having only the frictional stiffness may be represented by the expression.

$$EA\frac{\partial^2 u}{\partial z^2} + K_f u = m(z)\frac{\partial^2 u}{\partial t^2} \tag{1}$$

[1]This is an important issue for many real life projects specially in Arctic condition (like North Siberia) or very arid region (like Sudan, Algeria) due to extreme low temperature or absence of water makes concreting hazardous and almost all the structures and foundations are built on steel piles.

in which, E = Young's modulus of pile; A = area of pile; K_f = dynamic frictional stiffness of soil having dimension (F/L) and u(z,t) = dynamic amplitude of pile = $\phi(z)\,q(t)$; and m(z) = mass of element dz.

One of the solutions of eqn. (1) is given by

$$q(t) = C_3 \sin \omega t + C_4 \cos \omega t \tag{2}$$

With the definition of u and using eqn. (2), eqn. (1) may be written as

$$EA\frac{d^2\phi(z)}{dz^2} + K_f\phi(z) = m(z)\omega^2\phi(z) \tag{3}$$

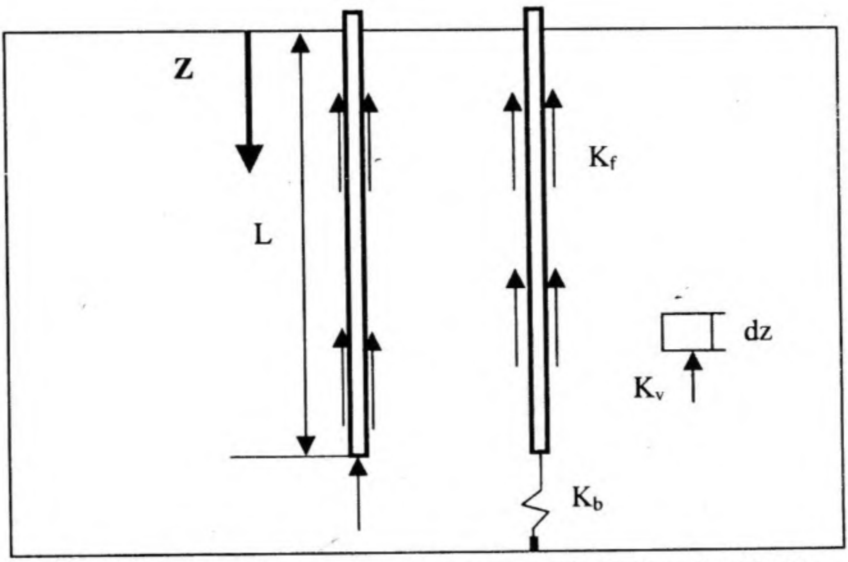

Fig.1 Pile Embedded in Ground up to a Depth L and its Mathematical Model

The above equation can further be simplified to

$$\frac{d^2\phi(z)}{dz^2} + p^2\phi(z) = 0 \tag{4}$$

where $p^2 = (m\omega^2 - K_f)/EA$.

Eqn. (4) suggests that the presence of frictional stiffness K_f does not affect the basic shape function of the pile and would remain same for the case had the pile would not have been embedded. However, the bearing stiffness K_b connected at the end of pile would affect the shape function depending on the appropriate boundary condition. For computing the correct shape function of the system, one has to start with the model as shown in Fig. 1. The general solution for the problem is given by (Humar,1990)

$$\phi(z) = (C_1 \cos pz + C_2 \sin pz)(C_3 \sin \omega t + C_4 \cos \omega t) \quad (5)$$

in which, C_1, C_2, C_3 and C_4 are the integration constants to be determined from the appropriate boundary conditions.

The pile has at the free head,

$z = 0$, $EA\dfrac{u}{dz} = 0$, which gives

$$EAp[-C_1 \sin pz + C_2 \cos pz][C_3 \sin \omega t + C_4 \cos \omega t] = 0 \rightarrow C_2 = 0. \quad (6)$$

and at the tip,

$z = L$, $EA\dfrac{du}{dz} = -K_b\, u(z)_{z=L}$, which gives

$$EAp[-C_1 \sin pL] = -K_b C_1 \cos pL \rightarrow pL \tan pL = K_b L/(EA) \quad (7)$$

in which $K_b = G_b r_0 C_b$ (8)

where, G_b = dynamic shear modulus of the soil at pile tip; r_0 = radius of the pile; C_b = a frequency independent dimensionless constant as suggested by Novak and Beredugo (1972) and is given in Table 2.

Combining eqns. (7) and (8), one can have

$$pL \tan pL = \frac{G_b C_b L}{E\pi\, r_0} \quad (9)$$

It will be observed that the right hand side of eqn. (9) is a dimensionless quantity.

If $\eta = \dfrac{G_b C_b L}{E\pi r_0} = \left(\dfrac{G_b}{E}\right)\left(\dfrac{C_b}{\pi}\right)\lambda$; where λ = slenderness ratio of the pile,

eqn. (9) can be represented as $pL \tan pL - \eta = 0$. (10)

Eqn. (10) is a transcendental equation in pL and can be solved numerically. The values of pL for various values of for the first mode are shown in Table 1.

TABLE 1: Roots of Equation pL tan(pL) - η= 0 for the First or Fundamental Mode

η	pL	η	pL	η	pL	η	pL
0	0.02	0.7	0.75	2	1.077	15	1.473
0 1	0 322	0 8	0 791	2 5	1 142	20	1 496
0 2	0 433	0 9	0 828	3	1 192	25	1 51
0.3	0.522	1	0.86	3.5	1.232	30	1.52
0.4	0.593	1.25	0.931	4	1.265	35	1.527
0 5	0 653	1 5	0 988	5	1 314	40	1 533
0.6	0.705	1.75	1.036	10	1.429	50	1.54

Writing, $pL = \beta$, the arbitrary shape function of the problem is given by

$$\phi(z) = \cos \beta \frac{z}{L} \tag{11}$$

The potential energy $d\Pi$ of an element of depth dz, shown in Fig.1, is given by (Shames and Dym, 1995)

$$d\Pi = \frac{EA}{2}\left[\frac{du}{dz}\right]^2 + \frac{K_v}{2}u^2 \tag{12}$$

where, E = Young's modulus of pile; A= area of pile; K_v = dynamic stiffness of soil having dimension kN/m; w = displacement of pile in the z direction and may be written as $\phi(z)\,q(t)$.

Eqn. (12), K_v consists of two parts, namely,

1) the bearing modulus at pile tip, and

2) the friction modulus along the shaft.

For a rigid circular embedded footing with embedment D_f, the stiffness of the footing may be expressed (Novak, 1974) as

$$K_v = G_b r_0 C_b + G D_f S_1 \tag{13}$$

where, K_v = foundation stiffness in the vertical direction; G = dynamic shear modulus of the soil along the pile length; G_b = dynamic shear modulus of the soil at the pile base; r_0 = radius of the foundation; C_b and S_1 = dimensionless constant which are basically frequency dependent.

However it has been shown by Novak and Beredugo (1972) that considering C_b and S_1 as frequency independent, no accuracy is lost for practical design problems and the analysis becomes quite simplified for rigid circular embedded footing. The frequency independent values of C_b and S_1 are as given below in Table 2.

TABLE 2: Suggested Frequency Independent Values Suggested by Novak and Beredugo (1972) for Embedded Footing

Poisson's ratio	C_b	S_1
0.0	3.9	2.7
0.25	5.2	2.7
0.5	7.5	2.7

However, it should be remembered that an embedded circular footing is usually considered to be rigid having infinite structural stiffness. On the contrary, a pile will be far more flexible member whose structural stiffness will be much lower, thus the above recommended value may be valid for certain pile geometry but may not be valid for others. Comparing the stiffness data of piles obtained by Novak (1974) and Dobry and Gazetas (1988), to make S_1 independent of frequency it is proposed that the following value of S_1 be used for dynamic analysis of piles in the vertical direction. This is similar to the technique used earlier by Lysmer and Richart (1966) for deriving equivalent stiffness and damping of circular footings for Lysmer's analog from the solutions of a similar solution based on elastic half space theory proposed by Bycroft (1956).

$$S_1 = \frac{9.553(1+\nu)}{\lambda^{0.333}} \tag{14}$$

where ν = Poisson's ratio of the soil; and λ = slenderness ratio of the pile.

The value C_b may be taken as suggested in Table 2 for it has no bearing on the flexibility of pile and is a function of the base area only. Considering pile base area is much smaller in comparison to a footing, its contribution is only marginal. Moreover in most of the practical cases its effect does not come into consideration (as will be shown subsequently) for analysis of such piles are either considered as bearing pile i.e. having infinite base stiffness or floating having no base effects.

The first term in eqn. (13) represents the contribution of base resistance, while the second term, the embedment effect of the foundation. Substituting eqn. (13) in eqn. (12) for an element dz, $d\Pi$ may be written as

$$d\Pi = \frac{EA}{2}\left[\frac{du}{dz}\right]^2 + \frac{G_b r_0 C_1}{2}u^2 + \frac{GS_1 dz}{2}u^2 \tag{15}$$

and the total potential energy over the total length of the pile (L) is given by

$$\Pi = \frac{EA}{2}\int_0^L \left[\frac{du}{dz}\right]^2 dz + \frac{GS_1}{2}\int_0^L u^2 dz + \frac{G_b r_0 C_b}{2}u^2 \tag{16}$$

Considering u(z, t) = ϕ (z) q(t), it can be proved (Hurty and Rubenstein,1967) that

$$K_{ij} = EA\int_0^L \phi_i'(z)\phi_j'(z)dz + GS_1\int_0^L \phi_i(z)\phi_j(z)dz + G_b r_0 C_b \phi_i(L)\phi_j(L) \tag{17}$$

where the shape function of the problem is given by eqn.(11).

The first derivative of the above with respect to z is given by

$$\phi'(z) = -\frac{\beta}{L}(\sin \beta \frac{z}{L}) \tag{18}$$

Using $z/L = \xi$ implying $dz = Ld\xi$, and converting the shape function as furnished in eqn. (11) from local to generalized co-ordinates, the limits of the problem get converted to 1 to zero.

$$F(\xi) = \cos \beta\xi \tag{19}$$

$$F_i'(\xi) = -\frac{\beta}{L}(\sin \beta\xi) \text{ , and} \tag{20}$$

$$K_{ij} = \frac{EA\beta_i\beta_j}{L}\int_0^{+1} F_i'(\xi)F_j'(\xi)d\xi + GS_1L\int_0^1 F_i(\xi)F_j(\xi)d\xi + G_br_0C_bF_i(1)F_j(1) . \tag{21}$$

For the fundamental mode i = j = 1 and eqn. (21) reduces to

$$K_1 = \frac{EA\beta^2}{L}\int_0^{+1} F_1'(\xi)^2 d\xi + GS_1L\int_0^1 F_1(\xi)^2 d\xi + G_br_0C_bF_1(1)^2 \tag{22}$$

Eqn. (22) can be rewritten as

$$K_1 = \frac{EA\beta^2}{L}\int_0^{+1} (\sin \beta\xi)^2 d\xi + GS_1L\int_0^1 (\cos \beta\xi)^2 d\xi + G_br_0C_b(\cos^2 \beta) \tag{23}$$

Eqn. (23) on integration and after some simplification may be expressed as

$$K_1 = I_1 + I_2 + I_3$$

in which $I_1 = \frac{EA\beta^2}{L}\left[\frac{1}{2} - \frac{\sin 2\beta}{4\beta}\right]; I_2 = GS_1L\left[\frac{1}{2} + \frac{\sin 2\beta}{4\beta}\right]; I_2 = \frac{G_br_0C_b}{2}(1 + \cos 2\beta)$ (24)

Finally, K_1 Can be written as

$$K_1 = \left(\frac{EA\beta^2}{2L} + \frac{GS_1L}{2} + \frac{G_br_0C_b}{2}\right) + \left(\frac{GS_1L}{4\beta} - \frac{EA\beta}{4L}\right)\sin 2\beta + \frac{G_br_0C_b}{2}\cos 2\beta \tag{25}$$

which may be further simplified to

$$K_1 = X_1 + X_2 \sin 2\beta + X_3 \cos 2\beta \tag{26}$$

in which $X_1 = \left(\frac{EA\beta^2}{2L} + \frac{GS_1L}{2} + \frac{G_br_0C_b}{2}\right); X_2 = \left(\frac{GS_1L}{4\beta} - \frac{EA\beta}{4L}\right); X_3 = \frac{G_br_0C_b}{2}$ (27)

Eqn. (27) gives a complete derivation of stiffness of the pile for the vertical mode, without any limitation to slenderness ratio, E/G or the material type.

***Mass of the Pile*:** For a conservative system, if T is the kinetic energy of the system then at any time t, the energy equations may be written as

$$T(t)=\frac{1}{2}\int_{0}^{H} m(z)\left[\frac{\partial u(z,t)}{\partial t}\right]^2 dz \tag{28}$$

Using, $u(z,t)=\sum_{i=1}^{n}\phi_i(z)q_i(t)$ (29)

where $u(z,t)$ = displacement function; $\phi_i(z)$ = shape function ; $q_i(t)$ = generalized co-ordinate; m(z) = mass of element dz and substituting eqn. (29) in eqn. (28), the energy equation may be written as

$$T(t)=\frac{1}{2}\int_{0}^{H} m(z)\left[\sum_{i=1}^{n}\phi_i(z)\dot{q}_i(t)\right]\left[\sum_{j=1}^{n}\phi_j(z)\dot{q}_j(t)\right]dz \tag{30}$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i(t)\dot{q}_j(t)\left[\int_{0}^{H} m(z)\phi_i(z)\phi_j(z)dz\right]$$

from which the mass matrix may be written as

$$m_{ij}=\left[\int_{0}^{H} m(z)\phi_i(z)\phi_j(z)dz\right] \text{ for } i, j = 1, 2, 3 \ldots\ldots n \tag{31}$$

Similarly the stiffness value with transformation from local to natural co-ordinate, the mass contribution of the pile may be obtained as

$$m_{ij}=\frac{\gamma_p AL}{g}\int_{0}^{1} F_i(\xi)F_j(\xi)d\xi \tag{32}$$

were γ_p=bulk density of pile material; A = area of pile cross section; L = pile length embedded in soil, and g = acceleration due to gravity.

For the fundamental mode, i = j = 1, and one can have

$$m_1=\frac{\gamma_p AL}{g}\int_{0}^{1} F_1(\xi)^2 d\xi \tag{33}$$

The above on expansion results in

$$m_1=\frac{\gamma_p AL}{g}\int_{0}^{1} (\cos\beta\xi)^2 d\xi \tag{34}$$

Integration of eqn. (34) gives

$$m_1 = \frac{\gamma_p AL}{2g}\left[1 + \frac{\sin 2\beta}{2\beta}\right] \tag{35}$$

which is the contributory mass of the pile for the fundamental mode in the vertical direction.

Damping of the Pile: The damping of the pile embedded in soil will constitute of two parts:

1) Material damping of the pile itself; and,

2) Radiation damping of the soil.

It is obvious that the material damping of the pile will be much lower than that of the soil radiation damping. As the first step for calculating the soil damping one may ignore the material damping of the pile for the time being. Material damping of soil also is part of the system vibration. However it has been found that for translational vibration their effect is insignificant and may be neglected without any significant effect. Else if one wishes, their values may be obtained from resonant column test from the laboratory.

For a rigid footing embedded in soil for a depth D_f, Novak (1974) has

$$C_z = r_0\sqrt{\rho_b G_b}\,\overline{C}_b + r_0\sqrt{\rho G}\,\overline{S}_2 D_f \tag{36}$$

where, r_0 = radius of the foundation; G_b= dynamic shear modulus at foundation base; G = dynamic shear modulus of soil in which the foundation is embedded; D_f = depth of embedment; $\overline{C}_b$ and S_2 = frequency independent constants as defined by Novak and furnished in Table 3.

With reference to Fig.1 for a pile element of length dz, embedded in the soil, the above equation may expressed as

$$C_z = r_0\sqrt{\rho_b G_b}\,\overline{C}_b + r_0\sqrt{\rho G}\,\overline{S}_2 dz \tag{37}$$

For systems having continuous function, the damping is usually expressed as (Pazz,1987)

$$C_z = c(z)\int \phi_i(z)\phi_j(z)dz \tag{38}$$

For the present case, eqn. (38) can be expressed as

$$C_z = r_0\sqrt{\rho G}\,\overline{S}_2\int_0^L \phi_i(z)\phi_j(z)dz + r_0\sqrt{\rho_b G_b}\,\overline{C}_b\phi_i(L)\phi_j(L) \tag{39}$$

Considering $\phi(z) = \cos\beta\frac{z}{L}$, for the fundamental mode, one can have

$$C_z = r_0\sqrt{\rho G}\bar{S}_2\int_0^L \cos^2\beta \frac{z}{L}dz + r_0\sqrt{\rho_b G_b}\bar{C}_b\cos^2\beta \tag{40}$$

and hence,

$$C_z = r_0\sqrt{\rho G}\ \bar{S}_2 L\int_0^1 \cos^2\beta\xi d\xi + r_0\sqrt{\rho_b G_b}\ \bar{C}_b\cos^2\beta \tag{41}$$

Eqn. (41) on integration simplifies to

$$C_z = \frac{1}{2}r_0\sqrt{\rho G}\ \bar{S}_2 L + \frac{1}{2}r_0\sqrt{\rho_b G_b}\ \bar{C}_b + \frac{r_0\sqrt{\rho G}\ S_2 L}{4\ \ \beta}\sin 2\beta + \frac{r_0\sqrt{\rho_b G_b}\ \bar{C}_b}{2}\cos 2\beta \tag{42}$$

Eqn. (42) expresses the soil damping for a single pile under vertical mode of vibration. Here the Factor $\bar{S}_2$ and $\bar{C}_b$ are damping coefficients which are frequency dependent. Fortunately the damping factor is required for calculation of the amplitude when the eigen solution of the problem is already done *vis-a-vis*, the dimensionless frequency factor $a_0 = \omega r / V_s$ is known. Polynomial fit curve for $\bar{S}_2$ and $\bar{C}_b$ are available in terms of a_0 which can be used to arrive at these parameters. The damping constants are as given in Table 3.

TABLE 3: Values of Damping Coefficients Based on Novak and Beredugo (1972)

Poisson's ratio	$\bar{C}_b$	$\bar{S}_2$
0.0	$3.438a_0+0.5742a_0^2-1.154a_0^3+0.7433a_0^4$	$6.059a_0 + \frac{0.7022a_0}{a_0+0.01616}$
0.25	$5.06a_0$	Do
0.5	$7.414a_0-2.986a_0^2+4.324a_0^3-1.782a_0^4$	Do

where $a_0 = \omega.r/v_s$ in which ω = natural frequency of the system in rad/sec; r = radius of the pile; v_s = shear wave velocity of the soil.

Consideration of Material Damping of Pile: The structural stiffness contribution of the pile is given by eqn. (25), while that of the mass is given in eqn. (35). Thus, if C_c be the critical damping of the pile then it can be expressed as $C_c = 2\sqrt{Km_p}$, where K and m_p are the stiffness and mass matrices of the pile.

Depending on the material used for pile like (RCC, steel etc) a suitable damping ratio (ζ) can be assumed. The damping (C) for the pile can be expressed as

$$C_p = \zeta\, C_c \tag{43}$$

This, when added to the radiation damping, calculated earlier, gives the complete damping quantity for the soil-pile system. It should be noted that for perfectly floating piles structural contribution of pile vanishes, and the material damping of the pile mentioned in the preceding need not be considered.

Vibration Of Bearing Piles

The expressions derived so far give a general case when the load is transferred from the pile to the soil both through friction and bearing. There will be cases when the pile is pre-dominantly bearing in load transfer.

Using the above formulation when $\lim \eta \to \infty$ (i.e. G_b is very large compared to E),

pL(tanpL) $= \infty$,when $\beta \to \frac{\pi}{2}$, the pile reduces to a perfectly bearing pile (i.e. fixed at the base), however for practical case when $\eta \to 50$, it will not be too erroneous to assume , $\beta \to \frac{\pi}{2}$ when the stiffness of the pile reduces to

$$K_1 = \left(\frac{EA\pi^2}{8L} + \frac{GS_1L}{2}\right) \quad \text{and} \tag{44}$$

the damping may be expressed as

$$C_1 = \frac{1}{2} r_0 \sqrt{\rho G} S_2 L \tag{45}$$

and the mass is

$$m_1 = \frac{\gamma_p AL}{2g}. \tag{46}$$

Vibration of Friction Piles

When G_b is very small the load is transferred mainly through pile friction. In the above formulation when $\lim \eta \to 0$, pL tanpL = 0, when $\beta \to 0$, the pile becomes a perfectly friction pile.

Thus, for $\beta \to 0$, the stiffness of pile is given by

$$K_1 = \left(\frac{GS_1L}{2}\right) \tag{47}$$

The damping matrix may be expressed as

$$C_1 = \frac{1}{2} r_0 \sqrt{\rho G}\, S_2 L + r_0 \sqrt{\rho_b G_b}\, \bar{C}_b \tag{48}$$

From eqn.(48) it should be noted that for a friction pile, the damping factor increases, while the stiffness term in eqn.(47) is less than the bearing case in eqn. (44). A similar observation also has been made by Novak (1974).

For very poor soil, the term G_b in eqn.(48) may be ignored. However for cases when piles located in medium to stiff homogenous clayey soil where $G = G_b$ and yet the load is basically transferred through friction, the last term cannot be ignored and would further enhance the radiation damping. The mass matrix shall be same as stated in eqn. (35).

Vertical Vibration of Partially Embedded Piles

In many instances, especially in the arctic condition, due to environmental reasons, the steel piles are driven into the ground when they protrude about 2-3m above the ground over which the pile cap and vibrating equipments as placed. In such cases Novak's (1974, 1983) chart cannot be used readily.

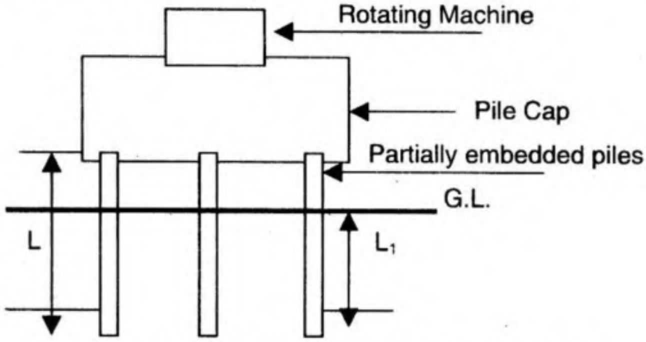

Fig. 2 Schematic Diagram of Partially Embedded Piles

To evaluate the pile stiffness for such cases, the stiffness eqn. (21) is to be modified as

$$K_{ij} = EA\int_0^L \phi_i'(z)\phi_j'(z)dz + GS_1\int_0^{L_1}\phi_i(z)\phi_j(z)dz + G_b r_0 C_b \phi_i(L)\phi_j(L) \qquad (49)$$

where L_1 = partial depth of embedment of pile and L = total length of pile.

It is apparent from eqn. (49) that the first and last term remains unchanged and the second term based on depth of embedment gets modified, where the integration limits changes to (L_1- 0) and the stiffness expression for the fundamental mode reduces to

$$K_1 = \left(\frac{EA\beta^2}{2L} + \frac{GS_1L_1}{2} + \frac{G_b r_0 C_b}{2}\right) + \left(\frac{GS_1L_1}{4\beta} - \frac{EA\beta}{4L}\right)\sin 2\beta + \frac{G_b r_0 C_b}{2}\cos 2\beta \qquad (50)$$

The damping of the pile-soil system is given by

$$C_z = \frac{1}{2}r_0\sqrt{\rho G}\bar{S}_2L_1 + \frac{1}{2}r_0\sqrt{\rho_b G_b}\bar{C}_b + \frac{r_0\sqrt{\rho G}S_2L_1}{4\beta}\sin 2\beta + \frac{r_0\sqrt{\rho_b G_b}\bar{C}_b}{2}\cos 2\beta \qquad (51)$$

The mass matrix remains the same as stated in eqn. (35).

It should be noted that for this case while calculating the value of S_1 [eqn.(14)], the slenderness ratio is to be calculated based on *the embedded length* of the pile.

Stiffness of the Pile in Soil with Varying Elastic Property

In the previous section, the calculation of stiffness as well as the damping of soil was based on the dynamic shear modulus of soil invariant with depth. While this could be possible for clayey soils, there are many cases when the dynamic shear modulus of the soil has been found to vary with depth. Generally this can be expressed as

$$G' = G\left(z/H\right)^{\alpha} \tag{52}$$

where α = a number varying from 0-2 [considered 0 when G is constant with depth , assumed 1 for linear variation and 2 for parabolic distribution].

For instance for the soil with variable elastic property, eqn. (52) may be modified to

$$G' = G\xi^{\alpha} \tag{53}$$

where $\xi = z/H$.

For the cases mentioned above, Novak's chart (1976) is possibly not valid. To accommodate the above variation, the stiffness equation can be modified to

$$K_{ij} = \frac{EA\beta^2}{L^2}\int_0^{+1} F_i'(\xi)F_j'(\xi)d\xi + GS_1L\int_0^1 \xi^{\alpha}F_i(\xi)F_j(\xi)d\xi + G_br_0C_bF_i(L)F_j(L) \tag{54}$$

Shear Modulus Having a Linear Variation

When the soil has linear distribution with depth, the stiffness eqn. (54) may be expressed as

$$K_1 = \frac{EA\beta^2}{L}\int_0^{+1}(\sin\beta\xi)^2 d\xi + GS_1L\int_0^1 \xi(\cos\beta\xi)^2 d\xi + G_br_0C_b(\cos\beta)^2 \tag{55}$$

which on integration and subsequent simplification, gives rise to

$$K_1 = \frac{1}{2}\left[\frac{EA\beta^2}{L} + \frac{GS_1L}{4}\left(1-\frac{1}{\beta^2}\right) + \frac{G_br_0C_b}{2}\right] + \frac{1}{2}\left[\frac{GS_1L}{\beta} - \frac{EA\beta}{L}\right]\sin 2\beta + \left[\frac{GS_1L}{4\beta^2} + \frac{G_br_0C_b}{2}\right]\cos 2\beta \tag{56}$$

It may be noted that while for bearing pile $\beta = \pi/2$, for friction pile (unlike constant G case), $\beta = 0$ is an admissible function in this case. For the fundamental mode the admissible function is $\beta = \pi$, which is the next higher mode. This is logical also for the soil having stiffness increasing with depth and the pile will have a natural tendency to wobble about its centre rather than moving en-mass.

The damping matrix in this case can be expressed as

$$C_z = r_0\sqrt{\rho G}\,\overline{S}_2L\int_0^1\sqrt{\xi}\cos^2\beta\xi d\xi + r_0\sqrt{\rho_bG_b}\,\overline{C}_b\cos^2\beta \tag{57}$$

The integration of the first term in eqn. (57) being cyclic in nature and can be solved approximately by expanding the cosine function in series. On integration, eqn. (57) reduces to

$$C_z = r_0\sqrt{\rho G}\,\overline{S}_2 L\left[\frac{2}{3} - 2\beta^2\left(\frac{1}{7} - \frac{\beta^2}{33} + \frac{2}{675}\beta^4\right)\right] + r_0\sqrt{\rho_b G_b}\,\overline{C}_b \cos^2\beta \qquad (58)$$

Shear Modulus having a Parabolic Variation

When the soil modulus has a parabolic distribution with depth, the stiffness equation may be expressed as

$$K_1 = \frac{EA\beta^2}{L}\int_0^{+1}(\sin\beta\xi)^2\,d\xi + GS_1 L\int_0^1 \xi^2(\cos\beta\xi)^2\,d\xi + G_b r_0 C_b(\cos\beta)^2 \qquad (59)$$

which on integration and subsequent simplification reduces to

$$K1 = \left[\frac{EA\beta^2}{2L} + \frac{GS_1 L}{6} + \frac{G_b r_0 C_b}{2}\right] + \left[\frac{GS_1 L}{2}\left(\frac{1}{2\beta} + \frac{1}{3\beta^3}\right) - \frac{EA\beta}{4L}\right]\sin 2\beta + \left[\frac{GS_1 L}{\beta} + \frac{G_b r_0 C_b}{2}\right]\cos 2\beta \qquad (60)$$

In this case, the first admissible function will be $\beta = \pi$ for a friction pile and $\beta = \pi/2$ for a bearing pile.

The mass matrix for both the cases remains same as stated in eqn. (35) while the damping matrix can be obtained from the expression

$$C_z = r_0\sqrt{\rho G}\,\overline{S}_2 L\int_0^1 \xi\cos^2\beta\xi\,d\xi + r_0\sqrt{\rho_b G_b}\,\overline{C}_b \cos^2\beta \qquad (61)$$

which on integration and simplification reduces to

$$C_z = \left[\frac{r_0\sqrt{\rho_b G_b}\,C_b}{2} - \frac{r_0\sqrt{\rho G}\,S_2 L}{4}\right] + \frac{r_0\sqrt{\rho G}\,S_2 L}{8\beta}\sin 2\beta + \left[\frac{r_0\sqrt{\rho G}\,S_2 L}{4\beta} + \frac{r_0\sqrt{\rho_b G_b}\,C_b}{2}\right]\cos 2\beta. \qquad (62)$$

Group Effect of Pile

The formulation given in the preceding is valid for single piles which needs to be modified to consider the group effect when K_{group} is not necessarily $\sum_{i=1}^{n} K_{zi}$ where n = number of piles in a group. For such cases, the method proposed by Poulos (1968) is possibly the best-suited technique and can well be used to modify the total stiffness of a pile group having n number of piles. Accordingly

$$K_{group} = \sum_{i=1}^{n} K_{zi} \Big/ \sum_{i=1}^{n} \alpha_{zi} \qquad (63)$$

where α_{zi} are the interaction factors provided by Poulos (1968).

Similarly the damping of the pile group may be obtained from the expression

$$C_{group} = \sum_{i=1}^{n} C_{zi} \Big/ \sum_{i=1}^{n} \alpha_{zi} \tag{64}$$

Effect of Pile Cap on Pile Stiffness

The pile cap has been found to affect the response of footing significantly. Before considering its effect within the proposed framework, it would be worthwhile to recapitulate the practice in vogue.

The sketch given in Fig. 3 can represent a pile group with cap.

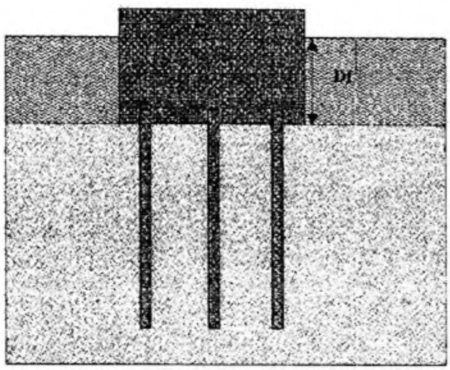

Fig. 3. Schematic Diagram of Pile and Pile-cap with Embedment.

In such case usually the embedment stiffness GS_fD_f is added to the pile group stiffness and the system is considered as a lumped mass single degree freedom system where

$$\omega = \sqrt{\frac{K_{group} + G_f S_f D_f}{M}} \tag{65}$$

where G_f = dynamic shear modulus of the soil surrounding the pile cap; D_f = depth of embedment; S_f = constant as suggested by Novak has been furnished in Table 2; M = mass of pile cap and machine placed on it.

It may be noted that contributing effect of the pile mass is ignored in the above which could be significant for a pile group having large number of piles. To overcome the above limitation and also to derive a better response, a two-mass lumped model has been proposed and shown in Fig. 4.

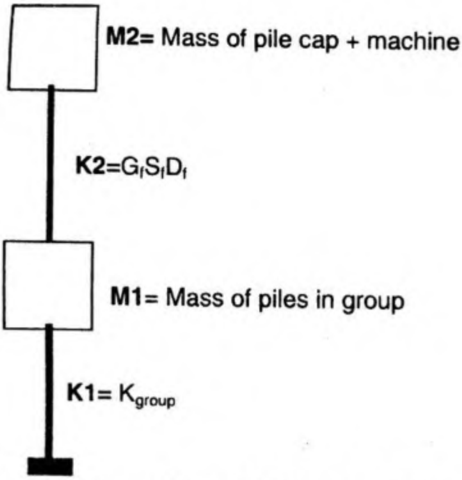

Fig. 4 Proposed Two-mass Lumped Model for the Pile and Pile Cap

The mass and stiffness matrices for the above model may be written as

$$K = \begin{bmatrix} K_{group} + G_f S_f D_f & -G_f S_f D_f \\ -G_f S_f D_f & G_f S_f D_f \end{bmatrix} \quad (66)$$

and
$$M = \begin{bmatrix} M & 0 \\ 0 & \dfrac{n\gamma_p AL}{2g}\left(1+\dfrac{\sin 2\beta}{2\beta}\right) \end{bmatrix} \quad (67)$$

where n = number of piles in the pile group.

Since eqn. (66) is statically coupled, the damping matrix is given by

$$\mathbf{C} = \begin{bmatrix} C_{group} + C_f & -C_f \\ -C_f & C_f \end{bmatrix} \quad (68)$$

where $C_z = r_0\sqrt{\rho G}\bar{S}_2 D_f$ and D_f is the embedment depth of pile cap. (69)

Once the stiffness, mass and damping matrices are established, the natural frequency of the system may be obtained from the standard expression

$$[\mathbf{K}] - [\mathbf{M}]\,\omega^2 = 0 \quad (70)$$

leading to

$$\lambda_{1,2} = \frac{(m_p B + MA) \pm \sqrt{\left[(m_p B + MA)^2 - 4m_p M\left(AB - B^2\right)\right]}}{2m_p M} \quad (71)$$

in which $m_p = \frac{n\gamma_p AL}{2g}\left[1+\frac{\sin 2\beta}{2\beta}\right]$ and A = Kgroup + $G_t S_t D_t$; B = $G_t S_t D_t$

where, $\omega_1 = \sqrt{\lambda_1}$ and $\omega_2 = \sqrt{\lambda_2}$ here ω_1 and ω_2 are the natural frequency of the structure.

The damping matrix generated here is non-classical in nature and will not be de-coupled on orthogonal transformation. However, since the degrees of freedom considered here is two, the same can also be converted into an equivalent Rayleigh damping (Chowdhury and Dasgupta, 2002) when the matrix will decouple and standard modal solution can be applied.

Solutions for Higher Modes

This case is usually not considered in design office practices and neither any guidelines presently exists for the same except treating the pile as a beam and the soil as Winkler springs and solving the same based on finite element method. Using the proposed methodology, the stiffness, damping and mass matrices can be computed for the higher modes.

Referring to eqn.(21), the stiffness matrix can be stated as

$$[Kij] = \frac{EA}{L}\begin{bmatrix} \beta_1^2\int_0^1 F_1(\xi)^2 & \beta_1\beta_2\int_0^1 F_1(\xi)F_2(\xi) & \beta_1\beta_3\int_0^1 F_1(\xi)F_3(\xi) & \cdots & \beta_1\beta_n\int_0^1 F_1(\xi)F_n(\xi) \\ \beta_2\beta_1\int_0^1 F_2(\xi)F_1(\xi) & \beta_2^2\int_0^1 F_2(\xi)^2 & \cdots & \cdots & \beta_2\beta_n\int_0^1 F_2(\xi)F_n(\xi) \\ \beta_3\beta_1\int_0^1 F_3(\xi)F_1(\xi) & \beta_3\beta_2\int_0^1 F_3(\xi)F_2(\xi) & \beta_3^2\int_0^1 F_3(\xi)^2 & \cdots & \beta_3\beta_n\int_0^1 F_3(\xi)F_n(\xi) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n\beta_1\int_0^1 F_n(\xi)F_1(\xi) & & & & \beta_n^2\int_0^1 F_n(\xi)^2 \end{bmatrix} d\xi$$

$$+\frac{GS1L}{2}\begin{bmatrix} F_1(\xi)^2 & F_1(\xi)F_2(\xi) & F_1(\xi)F_3(\xi) & \cdots & F_1(\xi)F_n(\xi) \\ F_2(\xi)F_1(\xi) & F_2(\xi)^2 & \cdots & \cdots & F_2(\xi)F_n(\xi) \\ F_3(\xi)F_1(\xi) & F_3(\xi)F_2(\xi) & F_3(\xi)^2 & \cdots & F_3(\xi)F_n(\xi) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_n(\xi)F_1(\xi) & & & & \int_0^1 F_n(\xi)^2 \end{bmatrix} +$$

$$Gbr_0C_bF_i(0)F_j(0) \quad \text{etc.} \tag{72}$$

For first three modes this can simply be presented as

$$[K]_{i=1,3\, j=1,3} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \tag{73}$$

where for i = j

$$K_{ii} = \left[\frac{EA\beta_i^2}{2L} + \frac{GS_1L}{2} + G_br_0Cb\right] + \left[\frac{GS_1L}{4\beta_i} - \frac{EA\beta_i}{4L}\right]\sin 2\beta_i + \frac{G_br_0C_b}{2}\cos 2\beta_i \quad (74)$$

For i≠j we have

$$K_{ij} = \left[\frac{EA\beta i\beta j}{2L} - \frac{GS_1L}{2}\right]\frac{\sin(\beta_i - \beta_j)}{\beta_i - \beta_j} +$$

$$\left[\frac{EA\beta_i\beta_j}{2L} - \frac{GS_1L}{2}\right]\frac{\sin(\beta_i + \beta_j)}{\beta_i + \beta_j} + G_br_0C_b\cos\beta_i\beta_j \quad (75)$$

It should be noted at this point that there are no suggestive values available for S_1 and C_b for higher modes either by Novak or any other research. However it may be reasonably stated that for higher modes the dimensionless frequency a_0 would be ≥ 1.0(or near 1.0 at worse) when the curve for S_1 becomes almost constant (Novak,1974) and the values furnished in Table 2 may be used without much error.

The value of β for the fundamental mode is already furnished in Table 1 for the next two modes the values of beta are furnished in Table 3 and Table 4.

TABLE 3: Roots of Equation Pl Tan(pl) - η = 0 For Second Mode

η	pl	η	pl	η	pl	η	pl
0	3 141	0 7	3 348	2	3 644	10	4 425
0.1	3.173	0.8	3.374	2.25	3.689	20	4.491
0 2	3 204	0 9	3 4	2 5	3 732	25	4 533
0.3	3.234	1	3.426	3.0	3.809	30	4.561
0 4	3 264	1 25	3 486	3 5	3 876	35	4 582
0.5	3.292	1.5	3.542	4	3.935	40	4.598
0.6	3.320	1.75	3.595	5	4.034	50	$\cong 3\pi/2$

TABLE 4: Roots of Equation PL Tan(pl) - η = 0 for Third Mode

ηη	pl	η	pl	η	pl	η	pl
0	6.28	0.7	6.392	2	6.578	15	7.316
0.1	6.299	0.8	6.407	2.25	6.611	20	7.495
0.2	6.315	0.9	6.422	2.5	6.643	25	7.56
0.3	6.331	1	6.437	3.0	6.704	30	7.606
0.4	6.346	1.25	6.474	3.5	6.761	35	7.639
0.5	6.362	1.5	6.510	4	6.814	40	7.665
0.6	6.377	1.75	6.544	5	6.910	50	$\cong 5\pi/2$

Mass matrix is similarly given by

For i = j

$$m_{ii} = \frac{\gamma_p AL}{2g}\left[1 + \frac{\sin 2\beta_i}{2\beta_i}\right] + M\cos^2\beta_i \quad (76)$$

where M= Mass of machine plus pile cap

For i ≠ j

$$m_{ij} = \frac{\gamma_p AL}{2g}\left[\frac{\sin(\beta_i+\beta_j)}{\beta_i+\beta_j} - \frac{\sin(\beta_i-\beta_j)}{\beta_i-\beta_j}\right] + \frac{M}{2}\left[\cos(\beta_i+\beta_j) - \cos(\beta_i-\beta_j)\right] \quad (77)$$

The damping matrix can be obtained as

For i = j

$$C_{ii} = \frac{1}{2} r_0 \sqrt{\rho G}\,\overline{S}_2 L + \frac{1}{2} r_0 \sqrt{\rho_b G_b}\,\overline{C}_b + \frac{r_0\sqrt{\rho G} S_2 L}{4\beta}\sin 2\beta_i + \frac{r_0 \sqrt{\rho_b G_b}\,\overline{C}_b}{2}\cos 2\beta_i \quad (78)$$

For i ≠ j

$$C_{ij} = r_0 \sqrt{\rho G}\,\overline{S}_2 L\left[\frac{\sin(\beta_i+\beta_j)}{\beta_i+\beta_j} - \frac{\sin(\beta_i-\beta_j)}{\beta_i-\beta_j}\right] + \frac{r_0^2 \sqrt{\rho G_b}\,\overline{C}_b}{2}\left[\cos(\beta_i+\beta_j) - \cos(\beta_i-\beta_j)\right]. \quad (79)$$

Comparison of Results

It is apparent that the dynamic analyses of piles with pile cap are standard and the validity of the same would depend on how correctly the pile stiffness values have been obtained. To this end, the pile stiffness as obtained by eqn. (44) and (47) has been compared with Novak's chart (1974) and equation based on rigorous analysis as proposed by Dobry and Gazetas (1988). It should be noted that the analysis is valid *for floating piles* of length, say L, and embedded in an elastic half space of length 2L. The results have been compared for a single pile of various slenderness ratio(λ varying from 20 to 100) and Ep/Gs value of soil varying from 250 to 10,000 for an RCC pile of diameter of 600 mm and having E_p = 30 GPa. Poisson's ratio value for soil considered is 0.4.

Here Ep = Young's modulus of pile material; Gs = dynamic shear modulus of soil. The results for $K_{pile(bearing)}$ and $K_{pile(friction)}$ are shown in Figs.5 through 14 for various slenderness ratios.

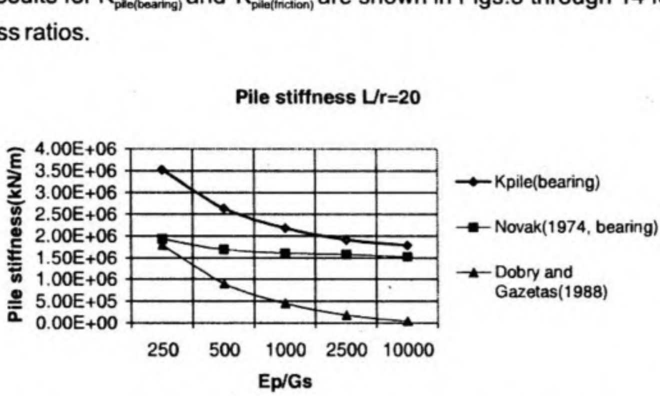

Fig.5. Comparison of Bearing Pile Stiffness for Slenderness Ratio =20.

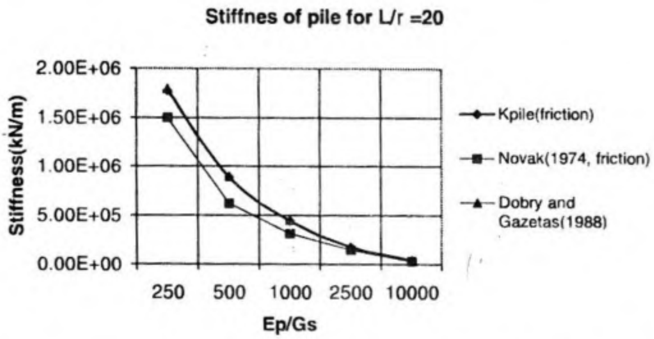

Fig. 6. Comparison of Friction Pile Stiffness for Slenderness Ratio =20.

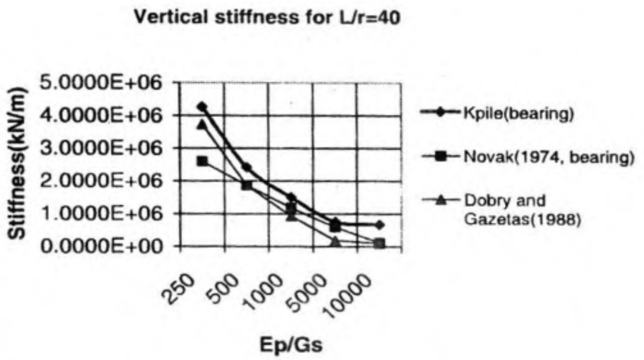

Fig. 7. Comparison of Bearing Pile Stiffness for Slenderness Ratio =40.

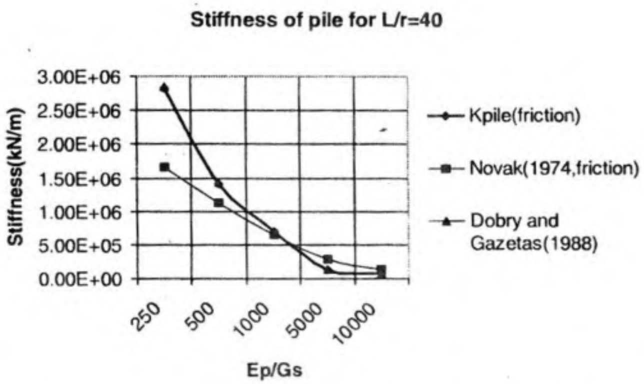

Fig. 8. Comparison of Friction Pile Stiffness for Slenderness Ratio =40.

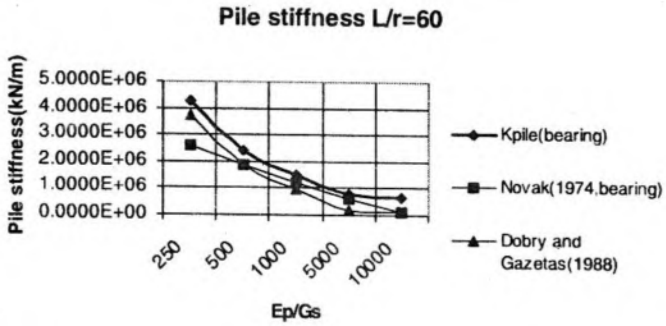

Fig. 9. Comparison of Bearing Pile Stiffness for Slenderness Ratio =60.

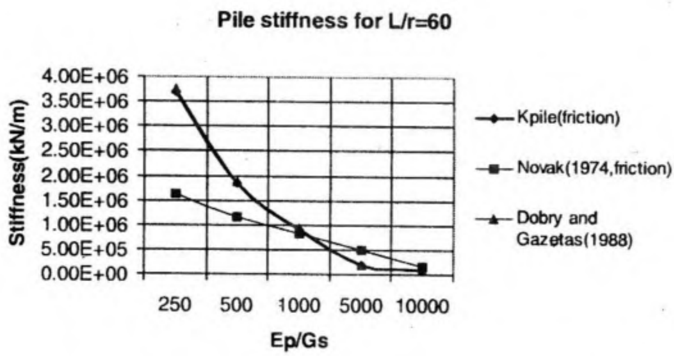

Fig. 10. Comparison of Friction Pile Stiffness for Slenderness Ratio =60.

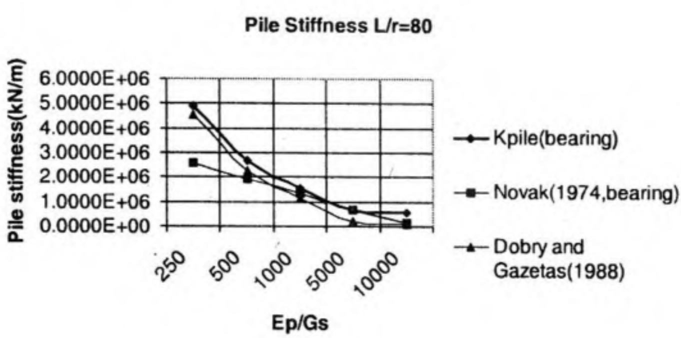

Fig. 11. Comparison of Bearing Pile Stiffness for Slenderness Ratio =80.

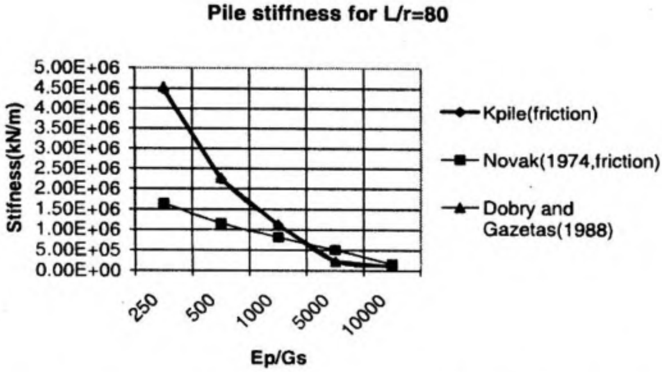

Fig. 12. Comparison of Friction Pile Stiffness for Slenderness Ratio =80.

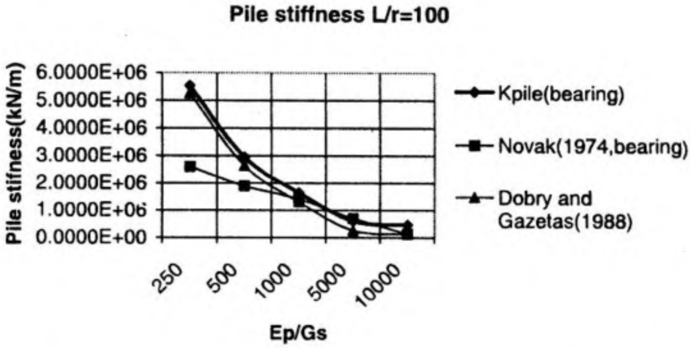

Fig. 13. Comparison of Bearing Pile Stiffness for Slenderness Ratio =100.

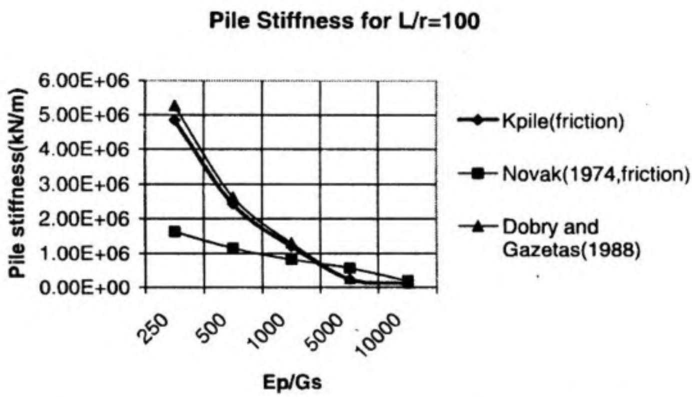

Fig. 14. Comparison of Friction Pile Stiffness for Slenderness Ratio =100.

Finally, the natural frequency of a real life centrifugal compressor foundation supported on 9 RCC piles, 45 meter long having diameter of 950 mm, have been compared. The piles are spaced at 3.0 m c/c. The size of pile cap is 7mX5mX2.0 m, embedded to depth of 1.4 meter. The weight of the generator supported on it weighs 400 kN. The frequencies are again compared for a range of Ep/Gs varying from 250 to 10000.

The results based on K_p(bearing) and K_p(friction) has been compared to Dobry and Gazetas'(1988) results and presented in Table-4. The results have not been compared with Novak(1974) in this case for the charts are too crude especially in the range when the ratio of Ep/Gs= 2500-10000 and significant variation can occur based on eye estimate of stiffness function. Results have been found to be excellently matching particularly for friction piles.

TABLE 4: Variation of Vertical Frequency for Compressor Foundation

Sl No	Ep/Gs	Freq(rad/sec) for K_{pile} (**bearing**)	Freq(rad/sec) forK_{pile} (**friction**)	As per Dobry and Gazetas (rad/sec)
1	250	196	195	197
2	500	139	138	139
3	1000	99	98	98
4	2500	64	62	62
5	5000	47	44	44
6	7500	39	36	36
7	10000	35	31	31

Experiments conducted by Novak and El-Sharnouby (1983) and by Prakash and Jadi (2001) in Laboratory and field (Bellie river USA) respectively shows good agreement in sandy type soil but theoretical values (as derived by Novak) over estimate by 15-20 for clayey soil was reported by Prakash and Jadi. This is attributed to the thixotropic change in the soil which looses a part of its shear strength during driving or boring of the piles.

However Novak and Gazetas' pile stiffness has also been successfully used in Industry for design of machine foundations.

So long as the stiffness values derived is in close proximity to the above should be acceptable for practical design.

Results and Discussions

As stated earlier, the results from eqn. (25)(with appropriate boundary condition for bearing and friction) have been compared with Novak's chart and Dobry and Gazetas' expression. The results have been studied against both the bearing and friction pile coefficients as suggested by Novak and El-Sharnouby (1983).

It will be observed in Figs. 5 through 14 that the frictional stiffness values obtained are very close to the reported results in all the cases for various L/r and

E/Gs values. For the bearing piles, the values obtained are slightly higher than the reported values but the present solution matches very closely to the Novak's data from Ep/Gs = 500 onwards. This is expected. It was pointed out by Novak and others that the bearing stiffness of a pile is slightly more than that of friction stiffness.

At L/r =20 the bearing values obtained are higher than that of the reported values of Dobry and Gazetas (which is logical as the case considered is that of a floating pile) but the difference reduces considerably from Ep/Gs =1000 onwards, and this is the range in which piles are commonly used in practice. The values, where Ep/Gs is ≤ 1000 are actually far too stiff for any piles to be bored or driven.

Moreover, a pile with L/r = 20 is actually a fictitious values. For instance a standard pile of length 30 meter, the diameter becomes 3.0 m, which in effect is actually a cassion and not a pile. It is possibly in such cases, the axial stiffness is far too high and this shows a significant higher stiffness in bearing compared to friction piles for such an unrealistic L/r ratio. For real life problems, the values of L/r is around 50-100 and Ep/Gs >1000. It will be observed that the values obtained by the proposed method are quite close to the reported results useful for practical ranges of application. As for the frequencies obtained for various Ep/Gs values the results in Table 4 are extremely encouraging.

The major advantage with the proposed method is that instead of solving the differential equation (especially when the boundary condition gets complicated with cases like partial embedment or variable soil) the stiffness, damping and mass matrices are directly derived from energy principles and the subsequent derivation gets quite simplified.

Finally, the formulation have been derived for a general case when pile can act both as bearing and friction pile for which no direct solutions are available-and this could be the reality in many cases when the pile is neither in full bearing or full floating. Comparing the results it can be well inferred that the method can be used for practical design office work without the limitations as stated at the outset.

Design Steps

Based on the derivations presented, the design steps may be summarized as follows:

- Determine the soil properties like G, G_b, G_f and ν (Poisson's ratio of the soil); The parameters G, G_b, G_f and ν for sandy and clayey soil can be obtained as mentioned hereunder:

Sandy soils: Though the processes of evaluation of dynamic shear modulus of both loose and dense sandy soil are outlined, it should be pertinent to point out that for dense sand (with Dr = 35-65 and N = 35-50), generally, piles will not be required.

For cohesion-less soil, the dynamic shear modulus can be obtained in the field employing seismic cross borehole test. This is a very standard procedure used in the industry for determining the in-situ dynamic shear modulus of soil.

This test can very well be made an integral part of the Standard Penetration Test SPT) which is usually carried out for sandy soils where, the G value for various layers can be effectively evaluated.

Seismic cross borehole test can be effectively used for both loose and dense sand in field.

For details of the test, reference may be made to Prakash (1981) or Kramer (1996).

Clayey Soil: For stiff clay which does not collapse on boring, the dynamic shear modulus can be obtained in field by seismic borehole test (Prakash, 1981). However, for very long piles some portion at the top may have to be provided with casing for the prevention of collapse of the soil mass.

For soft clay susceptible to collapse one can either carry out seismic cross borehole test or can collect undisturbed sample from the site at different depth and evaluate the dynamic shear modulus of the soil in the laboratory by resonant column test.

Details of this test is available in the text book by Richart et al. (1970)

Alternatively theoretical correlation exists (Hardin and Drnevich, 1972) and are appended below:

$$G_{max}=1230\frac{(2.973-e)^2}{(1+e)}(OCR)^k(\sigma_0)^{0.5}\ \text{in psi}$$

Where, e =void ratio; OCR = over consolidation ratio; σ_0 = mean effective stress in psi; = $0.333(\sigma_v+2\sigma_h)$; σ_v = vertical effective stress in psi; σ_h = horizontal effective in psi = $K_0\sigma_v$. K_0 = earth pressure at rest ,and is a function of the plasticity index and the over-consolidation ratio; k = is a function of the plasticity index of the soil and is given as follows

Plasticity Index	k
0	0
20	0.18
40	0.30
60	0.41
80	0.48
>=100	0.51

The Poisson's ratio (ν) for the soil usually varies from 0.25(Rock) to 0.45 (soft organic clay). For all practical purpose a value of ν= 0.4 will suffice (Dowrick, 1988).

- Determine the pile properties like Length of pile L and diameter of pile ($2r_0$) and also the Young's Modulus E of the pile material;

- Determine the pile cap property like its mass and depth of embedment D_f;
- Determine the weight of machine supported on the pile cap;
- Obtain Novak's stiffness and damping coefficients C_b, S_1, $\bar{C}_b$, S_2 from Table 2 and Table 3, eqn.(14) etc.;
- Establish the dimensionless parameter $\eta = \left(\frac{G_b}{E}\right)\left(\frac{C_b}{\pi}\right)\lambda$;
- For the given value η determine the value of pL from Table 1;
- If the pile is bearing (known priori) $\beta = \pi/2$;
- Consider $\beta = pL$;
- Determine K_1 and m_p from eqns. (25) and (35) respectively;

  Determine the embedment stiffness from the eqn. (56);
- Form the mass, stiffness;
- Perform eigen solution.
- Find the value of the frequency and obtain the dimensionless frequency number a_0.
- Find the value of S_2 from Novak's expression as given in Table 3;
- Determine the damping of the system based on eqn. (42, 45, 48);
- Perform Modal analysis to obtain the amplitude of vibration.

Conclusion

A comprehensive analytical method has been outlined for the design of piles under vertical vibration, which is chart independent and not restricted to the type of material to be used. It also takes into account for the variation of shear modulus with depth as well as the partial embedment of piles for which no standard method is available. The method can be easily programmed to standard design office software using simple spreadsheets.

References

Bannerjee, P.K and Sen, R. (1987): *Dynamic Behaviour of Axially and Laterally Loaded Piles and Pile Groups*, Chapter 3, Developments in Soil Mechanics and Foundation Engineering, Vol 3, Ed. Bannerjee, P.K. and Butterfield, R., Elsivier Applied Science, London

Baranov, V.A. (1967): *On the Calculation of Excited Vibrations of an Embedded Foundation*, Voprosy Dinamiki Prochnocti, No 14, Polytech. Ins. Riga, pp. 195-209.

Bycroft, G.N. (1956) : "Forced Vibrations of a Rigid Circular Plate on a Semi-Infinite Elastic Half Space And on an Elastic Stratum", *Philosophical Transactions*, Royal Society London Series, 248(1956), pp.327-368.

Chowdhury, I., Dasgupta, S.P. and Ghosh, B. (2002): "Analysis of Hammer Foundation Considering Soil Damping", *Int. conf. Advancement in Civil Engng.*, Indian Institute of Technology, Kharagpur, India, 2, pp.1019-1028.

Dobry, R. and Gazetas, G.(1988): "Simple Method for Dynamic Stiffness and Damping of Floating Piles Groups", *Geotechnique,* 38(4), pp.557-574.

Dowrick, D.J. (1988): *Earthquake Resistant Design for Engineers and Architects*, 2nd Edition, John Wiley, NY.

Hardin, B.O. and Drnevich, V.P. (1972): "Shear Modulus and Damping in Soils: Design Equations and Curves", *J. Soil Mech. and Found. Division*, ASCE,98(SM7).

Hurty, W.C. and Rubenstein, M.F. (1967): *Dynamics of Structures*, Prentice-Hall of India, New Delhi.

Humar, J.L. (1990): *Dynamics of Structure*, Prentice-Hall of India, New Delhi.

Kaynia, A.M. and Kausel, E. (1982): "Dynamic Behavior of Pile Groups", *2nd Int. Conf. on Numer. Meth. in Offshore Piling*, Austin, Texas.

Kramer, S. L. (1996): *Geotechnical Earthquake Engineering*, Prentice Hall, Upper Saddle River, NJ.

Lysmer J. and Richart, F.E.(1966): "Dynamic Respose of Footings to Vertical Loading", *J. Soil Mech. Found. Division*, ASCE, 92(SM7),pp. 65-91.

Mario, P. (1987): *Structural Dynamics*, CBS Publishers Ltd., New Delhi.

Novak M. and Beredugo Y.O. (1972): "Vertical Vibration of Embedded Footings", *J. Soil Mech. Found. Division*, ASCE, 12, pp.1291-1310.

Novak, M. (1974): "Dynamic Stiffness and Damping of Piles", *Can. Geotech. J.*, Vol.11, pp.574-598.

Novak, .M. and El Sharnouby. B. (1983): "Stiffness and Damping Constants for Single Piles", *J. Geotech. Engng. Division*, ASCE, 109, pp. 961 974.

Poulos, H.G.(1968): "Analysis and Settlement of Pile Group", *Geotechnique*, 18, pp. 449-471.

Prakash, S. (1981): *Soil Dynamics*, McGraw Hill, NY.

Prakash, S. and Jadi, H. (2001): "Prediction of Lateral Dynamics Response of Single Piles Embedded in Fine Soils," Paper No. 6.50, *Proc. Fourth Intern. Conf. on Recent Advances in Geot. EQ. Eng. and Soil Dyn. San Diego*, CA, Paper No. 7.05.

Shames, I.H. and Dym, C.L. (1995): *Energy and Finite Element Method in Structural Mechanics*, New Age International Publishers Ltd., New Delhi.

Waas, G. and Hartmann H.G. (1981): "Piles Subjected to Dynamic Horizontal Loads-European Simulation Meeting on Modeling and Simulation of Large Scale Structural Systems", Capri, p. 17.

Wolf, J.P. and VonArx, G.A. (1978): "Impedance Function of a Group of Vertical Piles", *Proc. ASCE Speciality Conf. on Earthquake Engng. and Soil Dynamics*, Pasadena ,California, pp. 1024-1041.

Richart, F. E.,Jr., Woods, R.D. and Hall, J. R., Jr. (1970): *Vibration of Soils and Foundation*, Prentice-Hall, Inc., Englewood Cliffs, NJ.

Notations

A	=	area of pile ;
C_1 through C_4	=	integration constants to be determined from the boundary condition;
C_b	=	a frequency independent constant;
C_b and S_1	=	constants which are basically frequency dependent;
C_c	=	critical damping of the pile, $C_c = 2\sqrt{Km_p}$ where m_p and K are the mass and stiffness of the pile;
C_{group}	=	damping of the pile group $= \sum_{i=1}^{n} C_{zi} \Big/ \sum_{i=1}^{n} \alpha_{zi}$;
$\overline{C}_b$ and S_2	=	frequency independent constants;
C_z	=	soil damping;
D_f	=	depth of embedment for a rigid circular footing;
Ep	=	Young's modulus of pile material;
F (ξ)	=	Shape function in generalized co-ordinate;
G	=	dynamic shear modulus of soil in which the foundation is embedded;
G_b	=	dynamic shear modulus at foundation base;
G'	=	variation of G as per $G' = G\xi^{\alpha}$, where α = a number which varies from 0-2;
G_f	=	dynamic shear modulus of the soil surrounding the pile cap;

g	=	Acceleration due to gravity;
G_b	=	dynamic shear modulus of the soil at pile base;
G	=	dynamic shear modulus of the soil along pile length;
G_b	=	dynamic shear modulus of the soil at pile tip;
K_1	=	the stiffness matrix for the fundamental mode;
K_b	=	end bearing stiffness of the pile;
K_f	=	dynamic frictional stiffness of soil having dimension (F/L);
K_{ij}	=	the stiffness matrix;
K_v	=	foundation stiffness in the vertical direction;
K_{group}	=	stiffness of the pile group $=\sum_{i=1}^{n} K_{zi} \Big/ \sum_{i=1}^{n} \alpha_{zi}$, where α_{zi} are the interaction factors;
L	=	length of the pile;
L_1	=	pile length embedded in soil;
M	=	mass of pile cap and machine placed on it;
m1	=	the mass matrix for the fundamental mode;
m_{ij}	=	the mass matrix;
$q_i(t)$	=	generalized co-ordinate;
r_0	=	radius of the pile; r_0 = radius of the foundation;
S_f	=	constant as suggested by Novak;
T	=	kinetic energy of the system;
$u(z,t)$	=	displacement function;
u(z,t)	=	dynamic amplitude of pile $=\phi(z)\,q(t)$;
w	=	displacement of pile in the z direction = $\varphi(z)\ q(t)$;
γ_p	=	bulk density of pile material;
$\phi_i(z)$	=	shape function;
Π	=	total potential energy of the system;
ω_1, ω_2	=	natural frequencies of the system.