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Sliding Compliance Functions for Embedded Circular Footings in Horizontal Shearing Interaction

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Introduction

Solutions to the problem of foundation response to horizontal loads can be based either on the elastic half space theory or on the lumped parameter approach. Though the former is more rigorous and exact, its application is limited to a few cases where the boundary conditions are simple. Even such common practical issues as embedment of foundation into the soil and layered nature of soil deposit make the method highly unwieldy and complicated. On the other hand, the latter method can rather easily incorporate such features, and hence is more versatile for general applications. This point is further substantiated considering the fact that almost all practical cases involve such complications, and there are only a few cases with simple boundary conditions. However, the accuracy of the results obtained from the latter method is highly dependent on the exactitude of the parameters involved, viz. stiffness and damping coefficients. Since the stiffness of a foundation-soil system is the major parameter influencing its response under load, evaluating the same correctly is essential for reliable solutions. This paper describes an analytical method to formulate closed-form expressions for compliance functions in horizontal translation at any depth of embedment for a circular footing embedded into a homogeneous, isotropic and elastic half space. The corresponding stiffness can then be evaluated from the compliance function. The effects of the commonly encountered types of

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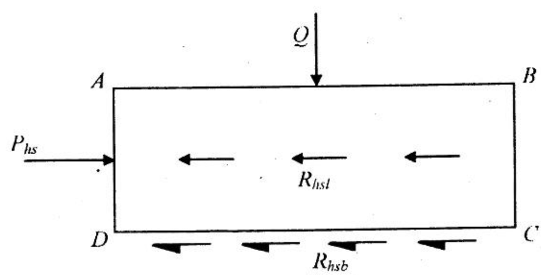
contact pressure distributions under the footing, namely rigid-base, uniform and parabolic pressures are also considered in the analysis.

Numerous solutions to the problem of horizontal vibration of non-embedded footings are reported in the literature. Analytical solutions were worked out by Arnold et al. (1955) for surface foundations subjected to pure horizontal vibrations, using elastic half space theory. Hall (1967) and Ratay (1971) used these results for the analysis of coupled sliding and rocking of footings. Elastic half space theory being cumbersome, many investigators like Hsieh (1962), Lysmer and Richart (1966), etc. have used the concept of mass-spring-dashpot model with a single degree of freedom to represent the foundation-soil system. In this method, an equivalent stiffness and a damping coefficient were used to simplify the problem. Bycroft (1956) has given the expression for equivalent horizontal stiffness of a surface foundation resting on a semi-infinite elastic medium with rigid base pressure distribution underneath. Nagendra et al. (1981) have similarly derived the same for uniform and parabolic pressure distributions. Investigators like Anandakrishnan and Krishnaswamy (1973), Novak and Beredugo (1972) and Ramiah et al. (1977) have dealt with embedment and its effects in vertical vibrations. Lysmer and Kuhlemeyer (1969) and Kaldjian (1969) used finite element method for solution to vertical vibration problems. Veletsos and Younan (1995) analyzed the dynamic response of deeply embedded cylindrical foundations to horizontal loads, modeling the soil medium by a series of elastically constrained thin horizontal layers with a circular hole at the centre. Chandrakaran (2001) evaluated the horizontal static stiffness of embedded circular foundations, assuming the problem as axi-symmetric for easy evaluation of the deformations. This assumption leads to erroneous results as the problem as a whole cannot be considered axi-symmetric. This is because neither the load nor the horizontal deformation function is axi-symmetric, though the foundation geometry, being circular, is so.

A careful study of the literature reveals that there are only very few investigations related to the effect of embedment on horizontal sliding stiffness of footings, and none reported so far which presents closed-form mathematical expressions for the same under horizontal static loads. This paper addresses the problem of evaluating horizontal sliding compliance function and hence the sliding stiffness of embedded footings under static loads using Mindlin's (1936) theory. It makes no simplifying assumption of axi-symmetry of the problem, and evaluates the integrals involved exactly, paying due regard to the radial and circumferential variation of the deformation function. Closed-form solutions are obtained for the static compliance functions in sliding mode. The proposed sliding compliance functions can be easily used for evaluating horizontal stiffness at all depths of embedment due to shearing interaction involving load transfer through base friction. Thus, the enhancement in horizontal stiffness can be predicted corresponding to any depth of embedment. The expressions for stiffness of non-embedded circular footings evaluated from the present method are compared with those available in the literature. The agreement in the results is brought out, indicating that the present method can be used for quantifying the sliding stiffness at any depth of embedment.

Analysis

Of the various possible modes of horizontal load transfer between a horizontally loaded embedded footing and the supporting soil underneath, we consider the shear load transfer through frictional interaction alone for deriving the sliding compliance functions (Fig.1). The horizontal load, P_{hs} on the footing is transmitted to the underlying medium through friction on the bottom face, R_{hsb} and that on the lateral faces, R_{hsl}. This load transfer induces elastic deformations in the soil in the direction of the applied load. As long as there is no slippage between the soil and the footing, the sliding deformation in the soil is proportional to the applied load on it. This condition of no slippage is generally realized in any footing, which is safe against sliding failure. Therefore, the deformation being proportional to the load, the system is linearly elastic and hence can be represented by an equivalent stiffness, K_{hs} mobilized due to shear deformations in the soil. As already pointed out, the total horizontal force P_{hs} is transmitted to the soil as a frictional reaction, R_{hs}, which can be considered as made up of the components transferred through bottom horizontal face R_{hsb}, and lateral vertical faces R_{hsl} such that



$$P_{hs} = R_{hsl} + R_{hsb}$$

ABCD : Side View of the Footing
Q : Vertical Load
P_{hs} : Horizontal Load
R_{hsl} : Frictional Reaction on Lateral Vertical Sides
R_{hsb} : Frictional Reaction on Bottom Face

FIGURE 1 : Horizontal Load Transfer in Friction / Shearing Interaction between the Footing and the Soil

$$P_{hs} = R_{hs} = R_{hsb} + R_{hsl} \tag{1}$$

Since the footing can be considered as rigid in its own plane, the sliding deformation at all points on the same can be assumed to be equal, say δ. Let the horizontal stiffness mobilized under R_{hsb} be K_{hsb} and that under R_{hsl} be K_{hsl}. Taking the horizontal stiffness of the embedded foundation as K_{hemb}, for a given horizontal displacement δ of the foundation,

$$\begin{aligned} R_{hs} &= K_{hemb}\delta \\ R_{hsb} &= K_{hsb}\delta \\ R_{hsl} &= K_{hsl}\delta \end{aligned} \tag{2}$$

Substituting Eqn.2 in Eqn.1,

$$K_{hemb}\delta = K_{hsb}\delta + K_{hsl}\delta \tag{3}$$

From which,

$$K_{hemb} = K_{hsb} + K_{hsl} \tag{4}$$

If we consider the load transfer to the soil through the bottom sides alone, Eqn.4 becomes

$$K_{hemb} = K_{hsb} \tag{5}$$

The term K_{hsb} is now referred to as the horizontal sliding stiffness. This stiffness is mobilized in shearing of soil underneath through frictional load transfer between the same and bottom face of footing. Dropping the suffix b, as the term with suffix l is no longer considered, K_{hsb} may be denoted as K_{hs}. Hence Eqn.5 is rewritten as

$$K_{hemb} = K_{hs} \tag{6}$$

It is reported in the literature that this type of horizontal load transfer as considered above is governed by a horizontal stiffness, which increases as the depth of embedment increases. This increase in horizontal stiffness is termed as the "trench effect" (Gazetas and Tassoulas, 1987). Experimental work on dry sand by Erden (1974) reveals that "even with no side contact of the footing along the embedment depth, the effect of embedment to reduce the displacements is not caused solely by an increase in shear modulus beneath the footing", indicating the presence of the trench effect. So the

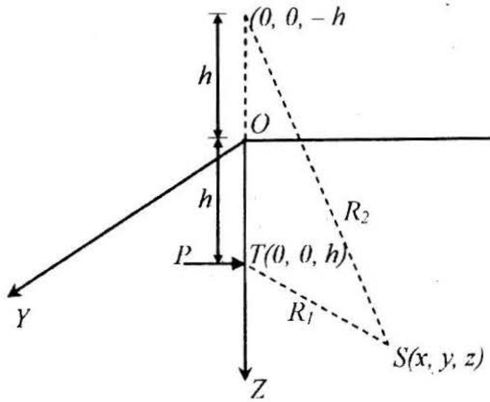


FIGURE 2 : Horizontal Displacement at Point S due to Point Load at T (Mindlin, 1936)

present analysis can provide a theoretical interpretation to the trench effect by formulating closed-form mathematical expressions for the same.

Evaluation of K_{hs}

To evaluate K_{hs}, we use of Mindlin's (1936) theory for horizontal deformation at any point S(x, y, z) within a semi-infinite elastic medium due to a horizontal point load P at another point T(0, 0, h), i.e. at a depth of h from the free surface (Fig.2). The horizontal displacement Δ at the point S(x, y, z) is given by

$$\Delta = \frac{P}{16\pi G(1-\upsilon)}\left[\begin{array}{l}\dfrac{3-4\upsilon}{R_1}+\dfrac{1}{R_2}+\dfrac{x^2}{R_1^3}+\dfrac{(3-4\upsilon)x^2}{R_2^3}+\dfrac{2hz}{R_2^3}\left\{1-\dfrac{3x^2}{R_2^2}\right\}\\ \qquad +\dfrac{4(1-\upsilon)(1-2\upsilon)}{R_2+z+h}\left\{1-\dfrac{x^2}{R_2(R_2+z+h)}\right\}\end{array}\right] \quad (7)$$

where $R_1 = \sqrt{r^2+(z-h)^2}$

$R_2 = \sqrt{r^2+(z+h)^2}$

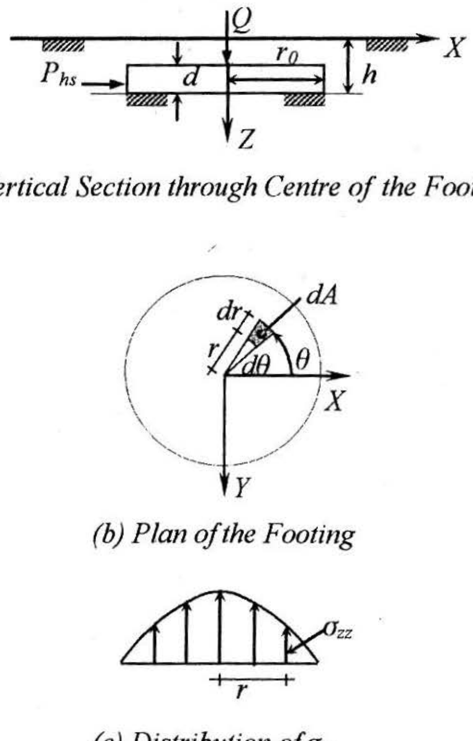
$r^2 = x^2 + y^2$

G = modulus of rigidity, and

υ = Poisson's ratio of the medium.

Equation 7 can be used for finding the horizontal deformation immediately under the centre of the footing consequent on the horizontal load transfer to the soil from the footing. Accordingly, consider a circular footing of radius r_0, thickness d, embedded into a semi-infinite soil mass up to a depth of h and with an applied horizontal load P_{hs} as shown in Fig.3a. Generally, P_{hs} is less than the maximum sliding resistance $\mu_m.Q$, where μ_m is the maximum coefficient of friction between footing base and soil, and Q is the vertical load on the footing. This is so because in all safe footings, the factor of safety against sliding is more than 1.5. Therefore, there is no slippage between the soil and the footing, and hence a linear relation exists between the load and the corresponding deformation.

Let σ_{zz} be the vertical stress under the footing at any point located at a radial distance r from the centre of the footing as shown in Fig.3c. σ_{zz} depends on the type of pressure distribution under the footing, viz. rigid-base, uniform or parabolic. The mathematical expression for σ_{zz} under various pressure distributions are as follows (Baidya and Sridharan, 1994)



(a) Vertical Section through Centre of the Footing

(b) Plan of the Footing

(c) Distribution of σ_{zz}

FIGURE 3 : Embedded Circular Footing under Horizontal Load

$$\sigma_{zz} = \frac{Q}{2\pi r_0 \sqrt{r_0^2 - r^2}} \text{, for rigid-base contact pressure} \tag{8a}$$

$$\sigma_{zz} = \frac{Q}{\pi r_0^2} \text{, for uniform contact pressure and} \tag{8b}$$

$$\sigma_{zz} = \frac{2Q\left(r_0^2 - r^2\right)}{\pi r_0^4} \text{, for parabolic contact pressure.} \tag{8c}$$

On an elemental area dA shown in Fig.3b, the vertical reaction, dQ is given by

$$dQ = \sigma_{zz} dA \tag{9}$$

If μ is the coefficient of friction mobilized for the frictional load transfer ($\mu < \mu_m$, as the footing is safe in sliding), the elemental horizontal reaction, dR_{hs} on the elemental area dA is

$$dR_{hs} = \mu dQ = \mu \sigma_{zz} dA \tag{10}$$

This load dR_{hs} acts inside the soil as a horizontal point load. Therefore, substituting dR_{hs} for P in Eqn.7, the elemental horizontal deformation dU_{hs} at the bottom centre of the footing due to dR_{hs} is given by

$$dU_{hs} = \frac{dR_{hs}}{16\pi G(1-\upsilon)} X = \frac{\mu \sigma_{zz} dA}{16\pi G(1-\upsilon)} X \tag{11}$$

where X is the term within square brackets in Eqn.7. G is the modulus of rigidity and μ is the Poisson's ratio of the soil.

Hence the total horizontal deformation at the centre of the footing due to shear load transfer to the soil over the entire base area A is obtained by integrating Eqn.11 over A. So we get

$$U_{hs} = \frac{\mu}{16\pi G(1-\upsilon)} \int_A \sigma_{zz} X \, dA \tag{12}$$

On conversion to polar co-ordinates as shown in Fig.3b,

$$U_{hs} = \frac{\mu}{16\pi G(1-\upsilon)} \int_{r=0}^{r_0} \int_{\theta=0}^{2\pi} X \sigma_{zz} r \, dr \, d\theta \tag{13}$$

In the problem being discussed, both the load and deformation are considered at points having the same z coordinate, i.e $z = h$. So, observing that $z - h = 0$ and $z + h = 2h$, R_1, R_2 and x in X of Eqn.7 are

$$R_1 = \sqrt{r^2 + (z-h)^2} = r \tag{14}$$

$$R_2 = \sqrt{r^2 + (z+h)^2} = \sqrt{r^2 + 4h^2} \tag{15}$$

$$x = r\cos\theta \tag{16}$$

Substituting for the six terms of X as given in Eqn.7 using the values of R_1, R_2 and x, the integral in Eqn.13 can be represented as the sum of six integrals $I_1, I_2, \ldots, I_6$ as

$$I_1 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{3-4\upsilon}{r} r\sigma_{zz}\, dr\, d\theta \tag{17a}$$

$$I_2 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{1}{\sqrt{r^2+4h^2}} r\sigma_{zz}\, dr\, d\theta \tag{17b}$$

$$I_3 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{r^2\cos^2\theta}{r^3} r\sigma_{zz}\, dr\, d\theta \tag{17c}$$

$$I_4 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{(3-4\upsilon)r^2\cos^2\theta}{\left(r^2+4h^2\right)^{3/2}} r\sigma_{zz}\, dr\, d\theta \tag{17d}$$

$$I_5 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{2h^2}{\left(r^2+4h^2\right)^{3/2}}\left[1-\left\{\frac{3r^2\cos^2\theta}{\left(r^2+4h^2\right)}\right\}\right] r\sigma_{zz}\, dr\, d\theta \tag{17e}$$

$$I_6 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{4(1-\upsilon)(1-2\upsilon)}{\left(2h+\sqrt{r^2+4h^2}\right)}\left[1-\left\{\frac{r^2\cos^2\theta}{\sqrt{r^2+4h^2}\left(2h+\sqrt{r^2+4h^2}\right)}\right\}\right] r\sigma_{zz}\, dr\, d\theta$$

... (17f)

In other words,

$$U_{hs} = \frac{\mu}{16\pi G(1-v)}\left\{I_1 + I_2 + I_3 + I_4 + I_5 + I_6\right\} \tag{18}$$

The integrals I_1 to I_6 can be evaluated in closed-form by exact integration after a series of substitution and change of variables. Since the steps involved in their closed-form evaluation are very lengthy, not all the steps are presented here. However, in order to bring out the general nature of the result, the evaluation of one of the integrals is presented as an example. We choose the integral I_2 as the typical example for the sake of elucidation. Substituting for σ_{zz} from Eqn.8a to Eqn.8c, I_2 is evaluated as given below for the three cases of base pressure distributions.

In the case of rigid-base pressure distribution,

$$I_2 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{1}{\sqrt{r^2+4h^2}}\, r \frac{Q}{2\pi r_0\sqrt{r_0^2-r^2}}\, dr\, d\theta \tag{19a}$$

or

$$I_2 = \frac{Q}{2\pi r_0}\int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{1}{\sqrt{r^2+4h^2}} \frac{r}{\sqrt{r_0^2-r^2}}\, dr\, d\theta \tag{19b}$$

Simplifying,

$$I_2 = \frac{Q}{2\pi r_0}\int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{r\, dr\, d\theta}{\sqrt{\left(r^2+4h^2\right)\left(r_0^2-r^2\right)}}$$

$$= \frac{Q}{r_0}\sin^{-1}\left\{\frac{1}{\sqrt{1+4\left(r_0/h\right)^2}}\right\} = \frac{Q}{r_0}\sin^{-1}\left\{\frac{1}{\sqrt{1+4e^2}}\right\} \tag{19c}$$

In the above expression, term e is called as the embedment ratio of the footing, and is equal to h/r_0, the depth of embedment in a non-dimensional form.

For uniform pressure distribution,

$$I_2 = \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} \frac{1}{\sqrt{r^2+4h^2}}\, r \frac{Q}{\pi r_0^2}\, dr\, d\theta \tag{20a}$$

or

$$I_2 = \frac{Q}{\pi r_0^2} \int_{r=0}^{r_0} \int_{\theta=0}^{2\pi} \frac{r}{\sqrt{r^2+4h^2}} \, dr \, d\theta \tag{20b}$$

Simplifying,

$$\begin{aligned} I_2 &= \frac{Q}{r_0} \times 2 \left\{ \sqrt{1+4(r_0/h)^2} - 2(r_0/h) \right\} \\ &= \frac{Q}{r_0} \times 2 \left\{ \sqrt{1+4e^2} - 2e \right\} \end{aligned} \tag{20c}$$

Similarly for parabolic pressure distribution,

$$I_2 = \int_{r=0}^{r_0} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{r^2+4h^2}} \, r \, \frac{Q\left(r_0^2 - r^2\right)}{\pi r_0^4} \, dr \, d\theta \tag{21a}$$

or

$$I_2 = \frac{Q}{\pi r_0^4} \int_{r=0}^{r_0} \int_{\theta=0}^{2\pi} \frac{r\left(r_0^2 - r^2\right)}{\sqrt{r^2+4h^2}} \, dr \, d\theta \tag{21b}$$

Simplifying,

$$\begin{aligned} I_2 &= \frac{Q}{r_0} \times \frac{8}{3} \left[\left\{ 1+4(r_0/h)^2 \right\}^{3/2} - 8(r_0/h)^3 - 3(r_0/h) \right] \\ &= \frac{Q}{r_0} \times \frac{8}{3} \left[\left\{ 1+4e^2 \right\}^{3/2} - 8e^3 - 3e \right] \end{aligned} \tag{21c}$$

Equations 19c, 20c and 21c may conveniently be rewritten as

$$I_2 = \frac{Q}{r_0} F_{2hs}(e) \tag{22}$$

In other words, the integral I_2 is a function of the embedment ratio, e. In a similar way, all the other integrals can also be simplified into the above form. So, in general we can write

$$I_j = \frac{Q}{r_0} F_{jhs}(e), \text{ where } j = 1 \text{ to } 6 \quad (23)$$

So Eqn.18 becomes

$$U_{hs} = \frac{\mu}{16\pi G(1-\upsilon)} \frac{Q}{r_0} \left\{ \begin{array}{l} F_{1hs}(e) + F_{2hs}(e) + F_{3hs}(e) \\ \quad + F_{4hs}(e) + F_{5hs}(e) + F_{6hs}(e) \end{array} \right\} \quad (24)$$

or

$$U_{hs} = \frac{\mu Q}{16\pi G r_0 (1-\upsilon)} F_{hs}(e) \quad (25)$$

where $F_{hs}(e)$ is given by

$$F_{hs}(e) = F_{1hs}(e) + F_{2hs}(e) + F_{3hs}(e) + F_{4hs}(e) + F_{5hs}(e) + F_{6hs}(e) \quad (26)$$

From horizontal equilibrium of the footing, vide Fig.3a,

$$P_{hs} = R_{hs} = \mu Q \quad (27)$$

Hence Eqn.25 is rewritten as

$$U_{hs} = \frac{P_{hs}}{16\pi G r_0 (1-\upsilon)} F_{hs}(e) \quad (28)$$

Here U_{hs} is the central deformation of the footing. Therefore, the horizontal stiffness, K_{hs} in shearing interaction through base friction, as per central deformation criterion is given by

$$K_{hs}(e) = \frac{P_{hs}}{U_{hs}} = \frac{R_{hs}}{U_{hs}} = \frac{16\pi G r_0 (1-\upsilon)}{F_{hs}(e)} \quad (29)$$

By definition, the reciprocal of stiffness function (or impedance function), $K_{hs}(e)$ is called as the compliance function, $C_{hs}(e)$. Therefore

$$C_{hs}(e) = \{K_{hs}(e)\}^{-1} = \frac{F_{hs}(e)}{16\pi G r_0 (1-\upsilon)} \quad (30)$$

Equation 30 can be written as

$$16\pi G r_0 (1-\upsilon) \cdot C_{hs}(e) = F_{hs}(e) \tag{31}$$

Since the left hand side of Eqn.31 represents the sliding compliance function in a dimensionless form after multiplying with a constant $16\pi G r_0 (1-\upsilon)$, the right hand side function $F_{hs}(e)$, henceforth, will be called as the non-dimensional sliding compliance function in shear load transfer. It is representative of the sliding flexibility of the footing, and comprises of six terms as given in Eqn.26. Each of the six terms appearing in Eqn.26 is a result of closed-form integration of I_1 to I_6 as explained in Eqn.19 to Eqn.22 for the typical case of I_2.

The constituent terms of the dimensionless sliding compliance function $F_{hs}(e)$ are presented in Table 1 for the three common types of contact pressure distributions, namely rigid-base, uniform and parabolic distributions under the footing. It may be noted that each of the terms appearing in Column 2 of Table 1 is obtained by exactly integrating Eqns.17a to 17f for the case of rigid base pressure distribution. Similarly, the terms in Column 3 and Column 4 of the same table can be obtained by exact integration of the above equations for uniform and parabolic pressure distributions, respectively.

Since the above compliance functions are closed-form expressions, horizontal stiffness can readily be evaluated for any embedment ratio e from Eqn.28. Two extreme cases of e for stiffness calculation correspond to a surface (e = 0) and a very deeply embedded (e = ∞) footings, so that the results can be compared with those available in the literature. At e = 0, $F_{hs}(e)$ can be evaluated by direct substitution. However at e = ∞, $F_{hs}(e)$ is indeterminate. Hence it has to be evaluated using L'Hospital's Rule. These extreme stiffness values so evaluated for all the three types of pressure distributions are shown in Table 2.

The variation of horizontal sliding stiffness $K_{hs}(e)$ with embedment ratio e can now be studied. This variation can be best illustrated with the help of stiffness increase factor $I_{hs}(e)$, defined as the ratio of the stiffness at any depth of embedment of the footing to the stiffness at soil surface of the same footing. It is nothing but the trench effect factor applicable to horizontal stiffness. Therefore, the stiffness increase factor is given by

$$I_{hs}(e) = \frac{K_{hs}(e)}{K_{hs}(0)} \tag{32}$$

Accordingly, introducing the suffixes r for rigid, u for uniform and

TABLE 1 : Terms of Non-dimensional Sliding Compliance Function $F_{hs}(e)$ in Horizontal Shearing Interaction

Term	Contact Pressure		
	Rigid-base	Uniform	Parabolic
F_{1hs}	$\frac{\pi\beta_1}{2}$	$2\beta_1$	$\frac{8\beta_1}{3}$
F_{2hs}	$\sin^{-1}\left(\frac{1}{\sqrt{1+4e^2}}\right)$	$2\left(\sqrt{1+4e^2}-2e\right)$	$\frac{8}{3}\left\{\left(1+4e^2\right)^{3/2}-8e^3-3e\right\}$
F_{3hs}	$\frac{\pi}{4}$	1	$\frac{4}{3}$
F_{4hs}	$\frac{\beta_1}{2}\left\{\sin^{-1}\left(\frac{1}{\sqrt{1+4e^2}}\right)-\frac{2e}{1+4e^2}\right\}$	$\beta_1\left\{\frac{8e^2+1}{\sqrt{1+4e^2}}-4e\right\}$	$\frac{2\beta_1}{3}\left\{\frac{128e^4+40e^2+2}{\sqrt{1+4e^2}}-64e^3-12e\right\}$
F_{5hs}	$\frac{4e^3}{\left(1+4e^2\right)^2}$	$\frac{2e^2}{\left(1+4e^2\right)^{3/2}}$	$4e^2\left\{\frac{64e^4+24e^2+2}{(1+4e^2)^{3/2}}-8e\right\}$
F_{6hs}	$\beta_2\tan^{-1}\left(\sqrt{1+4e^2}-2e\right)$	$\beta_2\left(\sqrt{1+4e^2}-2e\right)$	$\frac{2}{3}\beta_2\left\{3\left(\sqrt{1+4e^2}-2e\right)-8e^3+\left(\sqrt{1+4e^2}-2e\right)^3-2\left(1+4e^2\right)\left(\sqrt{1+4e^2}-3e\right)\right\}$

where $\beta_1 = 3-4\upsilon$ and $\beta_2 = 4(1-\upsilon)(1-2\upsilon)$

TABLE 2 : Values of $K_{hs}(e)$ at $e = 0$ and $e = \infty$.

Horizontal Stiffness of Circular Footings at						Proposed By
Soil Surface, $K_{hs}(0)$			Infinite Embedment, $K_{hs}(\infty)$			
Rigid-base	Uniform	Parabolic	Rigid-base	Uniform	Parabolic	
$\frac{8}{2-v}Gr_0$	$\frac{2\pi}{2-v}Gr_0$	$\frac{1.5\pi}{2-v}Gr_0$	$\frac{64(1-v)}{7-8v}Gr_0$	$\frac{16\pi(1-v)}{7-8v}Gr_0$	$\frac{12\pi(1-v)}{7-8v}Gr_0$	Authors (Present Study)
$\frac{8}{2-v}Gr_0$	–	–	–	–	–	Bowles (1996), Wolf (1988), Sankaran et al. (1977)
$\frac{32(1-v)}{7-8v}Gr_0$	$\frac{8\pi(1-v)}{7-8v}Gr_0$	$\frac{6\pi(1-v)}{7-8v}Gr_0$	–	–	–	Nagendra et al. (1981)
$\frac{32(1-v)}{7-8v}Gr_0$	–	–	–	–	–	Bycroft (1956)

p for parabolic contact pressures, the stiffness increase factors can be evaluated as given below. We make use of Eqn.29 and and the values of $K_{hs}(0)$ given in Table 2 for this purpose.

$$I_{hsr}(e) = \frac{K_{hsr}(e)}{K_{hsr}(0)} = \frac{16\pi G r_0 (1-v)}{F_{hsr}(e)} \div \frac{8Gr_0}{2-v} = \frac{2\pi(1-v)(2-v)}{F_{hsr}(e)} \quad (33)$$

$$I_{hsu}(e) = \frac{K_{hsu}(e)}{K_{hsu}(0)} = \frac{16\pi G r_0 (1-v)}{F_{hsu}(e)} \div \frac{2\pi G r_0}{2-v} = \frac{8(1-v)(2-v)}{F_{hsu}(e)} \quad (34)$$

$$I_{hsp}(e) = \frac{K_{hsp}(e)}{K_{hsp}(0)} = \frac{16\pi G r_0 (1-v)}{F_{hsp}(e)} \div \frac{1.5\pi G r_0}{2-v} = \frac{32(1-v)(2-v)}{3F_{hsp}(e)} \quad \ldots (35)$$

By definition, i.e. by Eqn.32, the stiffness increase factor at infinite embedment is given by

$$I_{hs}(\infty) = \frac{K_{hs}(\infty)}{K_{hs}(0)} \quad (36)$$

Using the values of $K_{hs}(\infty)$ and $K_{hs}(0)$ from Table 2, it is seen that $I_{hs}(\infty)$works out to be the same for all contact pressures, and its value is given by

$$I_{hsr}(\infty) = I_{hsu}(\infty) = I_{hsp}(\infty) = \frac{8(1-v)(2-v)}{7-8v} \quad (37)$$

Discussion

The values of horizontal stiffness of circular surface footings as reported by previous investigators are presented in Table 2, along with those deduced by the authors based on the present method. On comparison of these values, it can be seen for the case of a rigid base pressure that there is a perfect agreement between the horizontal stiffness obtained by the present method and that available in Wolf (1988) and Bowles (1996), and used by Sankaran et al. (1977). For other cases of pressure distribution and embedment, nothing has been reported in the literature. On the other hand, the formulae for stiffness at very deep embedment, $K_{hs}(\infty)$ as per the present method is twice that reported for ground surface, $K_{hs}(0)$ by Nagendra et al. (1981) and Bycroft (1956). In other words, the difference is in terms of a multiplying factor 2. This means that the sliding stiffness as per the present method for a very deeply embedded circular footing, $K_{hs}(\infty)$ is twice that reported previously for the surface footing, $K_{hs}(0)$. However, the ratio of $K_{hs}(\infty)$ to $K_{hs}(0)$ when calculated using the present theory is as given in Eqn.37, which is a function of Poisson's ratio, υ.

The variation of horizontal sliding stiffness with embedment ratio can be studied indirectly from the graphs between stiffness increase factor $I_{hs}(e)$ and embedment ratio e, making use of the expressions in Eqns.33, 34 and 35, after substituting for $F_{hsr}(e)$, etc. from Table 1. These graphs are presented for the following two sets of conditions

(a) for a constant Poisson's ratio, varying the type of contact pressure (Figs.4, 5 and 6)

(b) for a fixed contact pressure distribution, varying the Poisson's ratio (Figs.7, 8 and 9)

Figure 4 represents the variation of sliding stiffness increase factor with embedment ratio for the three types of contact pressure distributions at a fixed value of Poisson's ratio, $\upsilon = 0.0$. The same kind of graph is plotted in Figs.5 and 6 for $\upsilon = 0.3$ and $\upsilon = 0.5$ respectively. It can be seen from these graphs that for a given Poisson's ratio, the stiffness increase factor is the highest in the case of parabolic pressure distribution and the least for the

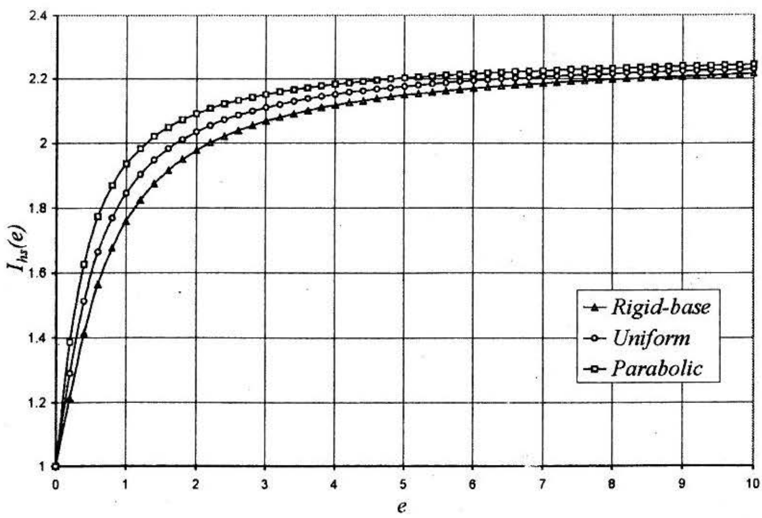


FIGURE 4 : Sliding Stiffness Increase Factor vs. Embedment Ratio (υ = 0.0)

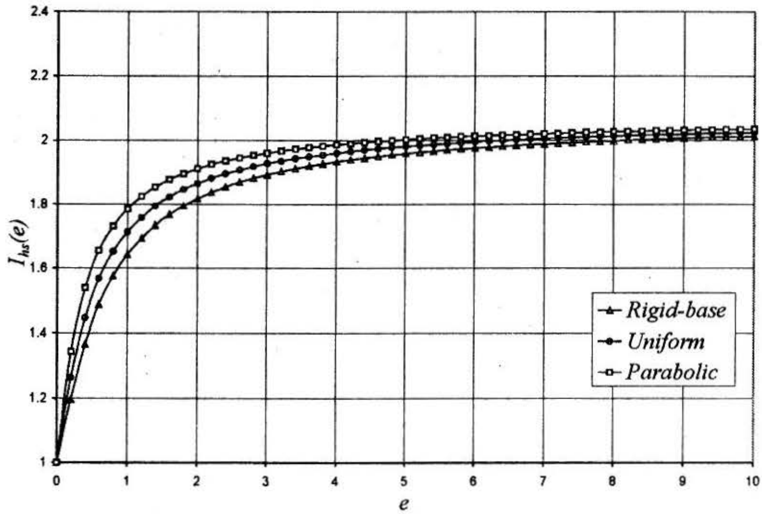


FIGURE 5 : Sliding Stiffness Increase Factor vs. Embedment Ratio (υ = 0.3)

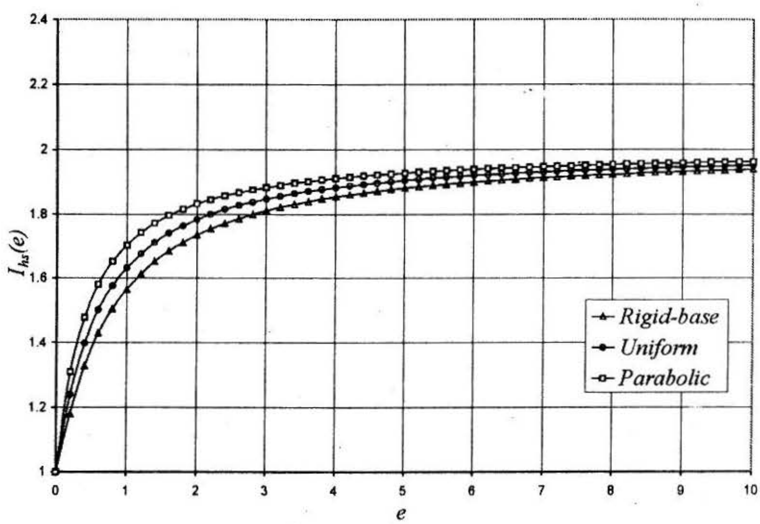


FIGURE 6 : Sliding Stiffness Increase Factor vs. Embedment Ratio (υ = 0.5)

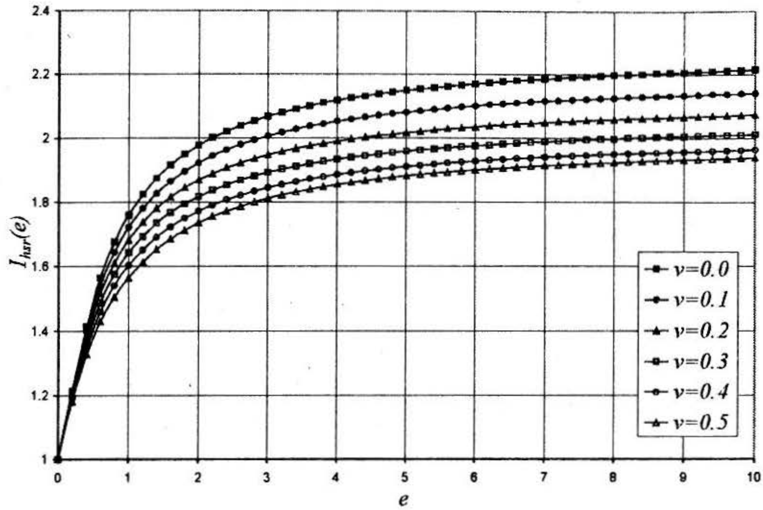


FIGURE 7 : Sliding Stiffness Increase Factor vs. Embedment Ratio (Rigid-base Contact Pressure Distribution)

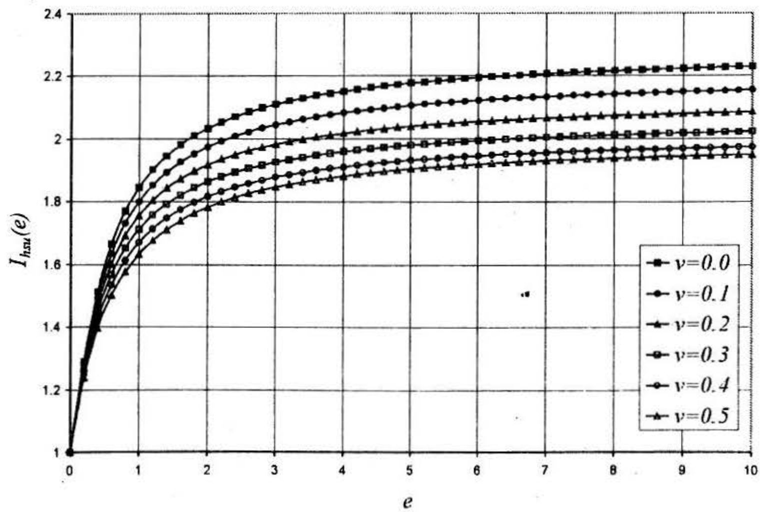


FIGURE 8 : Sliding Stiffness Increase Factor vs. Embedment Ratio (Uniform Contact Pressure Distribution)

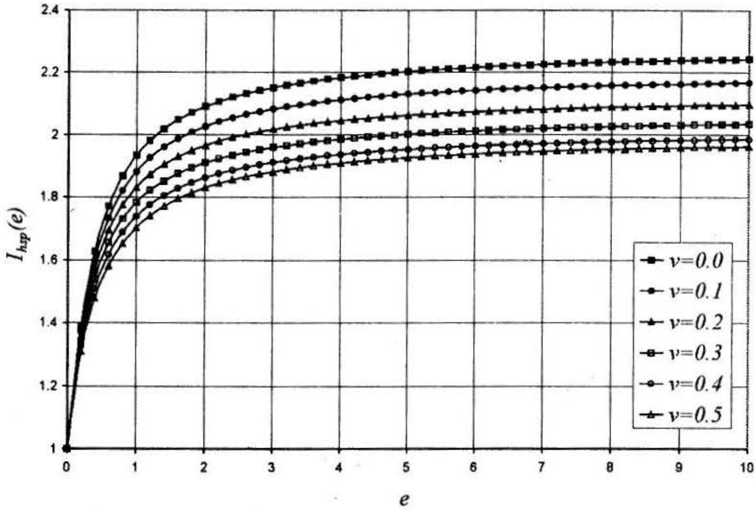


FIGURE 9 : Sliding Stiffness Increase Factor vs. Embedment Ratio (Parabolic Contact Pressure Distribution)

rigid one, for normal range of embedment. However, for very large embedment ratios, stiffness increase factors have identical value for all types of contact pressures, as indicated in Eqn.37. The same type of variation prevails in the case of all the values of Poisson's ratios.

It is also noticed that for a given type of contact pressure distribution, the highest value of stiffness increase factor corresponds to the lowest value of Poisson's ratio, viz $\upsilon = 0.0$. The lowest stiffness increase factor corresponds to the case of highest value of $\upsilon = 0.5$. These are elucidated in Figs.7, 8 and 9. These figures show the variation of stiffness increase factor with embedment ratio for rigid, uniform and parabolic pressure distributions respectively. The effect of Poisson's ratio as discussed above is clear from this graph. The trends observed in the case of all the three types of pressure distributions are identical.

Conclusion

Closed-form sliding compliance functions are greatly useful in solution of problems involving displacement of foundations in static or dynamic conditions, because the stiffness coefficient can easily be determined from the same. In the case of embedded footings, stiffness increases with embedment for obvious reasons. Though this increase has been recognized in the past, as evident from the various related literature, a closed-form representation of the same using exact mathematical expressions for horizontal sliding motion is non-existent hitherto. This paper presents closed-form functions for the sliding compliance and thereby for stiffness of an embedded circular footing under shearing interaction with the soil. The variation of sliding stiffness with respect to depth of embedment can be easily obtained using these functions under the above mode of interaction between the footing and the soil. Any depth of embedment up to infinity can be considered. It is assumed that the soil is a homogeneous, isotropic and elastic half-space. Elasticity equations are used of for the purpose of analysis. The results obtained are compared with those presented in the literature, for non-embedded circular footings. It has been noticed that there is a good agreement between these results.

References

ANANDAKRISHNAN, M. and KRISHNASWAMY, N.R. (1973) : "Response of Embedded Footings to Vertical Vibrations", *Journal of Soil Mechanics & Foundation Engineering Division*, ASCE, Vol.99, No.SM 10, pp.863-883.

ARNOLD, R.N., BYCROFT, G.N. and WARBURTON, G.B. (1955) : "Forced Vibrations of a Body on an Infinite Elastic Solid", *Journal of Applied Mechanics*, Transactions of ASME, September 1955, pp.391-400.

BAIDYA, D.K. and SRIDHARAN, A. (1994) : "Stiffness of the Foundations Embedded into Elastic Stratum", *Indian Geotechnical Journal*, Vol.24, No.4, pp.353-367.

BOWLES, J.E. (1996) : *Foundation Analysis and Design*, Fifth Edition, McGraw Hill International Editions, Civil Engineering Series, pp.1101.

BYCROFT, G.N. (1956) : "Forced Vibrations of a Rigid Circular Plate on a Semi-infinite Elastic Space and on an Elastic Stratum", *Philosophical Transactions of Royal Society of London*, Vol.248, No.A948, pp.327-368.

CHANDRAKARAN, S. (2001) : "A Simplified Method for Design of Embedded Foundations Subjected to Horizontal Vibrations", *Proceedings IGC 2001*, December 2001, pp.540-543.

ERDEN, S.M. (1974) : "Influence of Shape and Embedment on Dynamic Foundation Response", *Ph.D Thesis*, University of Massachusetts, at Amherst, Mass., USA.

GAZETAS, G. and TASSOULAS, J.L. (1987) : "Horizontal Stiffness of Arbitrarily Shaped Embedded Foundations", *Journal of Geotechnical Engineering*, ASCE, Vol.113, No.5, pp.440-457.

HALL, J.R., Jr. (1967) : "Coupled Rocking and Sliding Oscillations of Rigid Circular Footings", *Proceedings of International Symposium on Wave Propagation and Dynamic Properties of Earth Materials*, pp.139-148.

HSIEH, T.K. (1962) : "Foundation Vibration", *Proceedings of Institution of Civil Engineers*, London, Vol.22, pp.221-226.

KALDJIAN, M.J. (1969) : "Discussion on Design Procedures for Dynamically Loaded Foundations", *Journal of Soil Mechanics and Foundations Division*, ASCE, Vol.95, No.SM1, pp.364-366.

LYSMER, J. and KUHLEMEYER, R.L. (1969) : "Finite Dynamic Model for Infinite Media", *Journal of Engineering Mechanics Division*, ASCE, Vol.95, No.EM4, pp.859-877.

LYSMER, J. and RICHART, F.E. (1966) : "Dynamic Response of Footings to Vertical Loading", *Journal of Soil Mechanics and Foundation Engineering Division*, ASCE, No.SM1, pp.65-91.

MINDLIN, R.D. (1936) : "Force at a Point in the Interior of a Semi-infinite Solid", *Journal of Applied Physics*, Vol.7, pp.195-202.

NAGENDRA, M.V., SRIDHARAN, A. and SREENIVASAN, M. (1981) : "Foundation Response to Horizontal Vibrations", *Indian Geotechnical Journal*, Vol.16, pp.132-151.

NOVAK, M. and BEREDUGO, Y.O. (1972) : "Vertical Vibration of Embedded Footings", *Journal of Soil Mechanics and Foundation Engineering Division*, ASCE, No.SM12, pp.1291-1309.

RAMIAH, B.K., CHIKANAGAPPA, L.S and RAMAMURTHI, T.N. (1977) : "Vertical Vibration of Embedded Footings", *Proceedings of IX International Conference on Soil Mechanics and Foundation Engineering*, Vol.2, pp.343-346.

RATAY, R.T. (1971) : "Sliding-Rocking Vibration of Body on Elastic Medium", *Journal of Soil Mechanics and Foundations Division*, Proceedings of ASCE, Vol.97, No.SM1, pp.177-192.

SANKARAN, K.S., SUBRAMANIAM, M.S. and SASTRI, K.K. (1977) : "Horizontal Vibrations – New Lumped Parameter Model", *Proceedings of IX International Conference on Soil Mechanics and Foundation Engineering*, Vol.2, pp.365-368.

VELETSOS, A.S. and YOUNAN, A.H. (1995) : "Dynamic Modeling and Response of Rigid Embedded Cylinders", *Journal of Engineering Mechanics*, ASCE, No.9, Vol.121, pp.1026-1035.

WOLF, J.P. (1988) : *Soil-Structure Interaction Analysis in Time Domain*, Prentice Hall, Englewood Cliffs, New Jersey, pp.16.

Notations

P_{hs}	=	Horizontal force on footing.
Q	=	Vertical load on footing.
R_{hs}, R_{hsb}, R_{hsl}	=	Horizontal reaction in sliding shear mode.
K_{hs}, K_{hsb}	=	Horizontal sliding stiffness in shearing interaction at the base of the footing.
K_{hsl}	=	Horizontal sliding stiffness in shearing interaction on the lateral vertical sides of the footing
K_{hemb}	=	Horizontal sliding stiffness of embedded footing as a whole.

δ, Δ	=	Horizontal deformation in soil due to a point load.
U_{hs}	=	Horizontal deformation at the centre of the footing.
d	=	Thickness of footing.
h	=	Depth of embedment of footing.
x, y, z	=	Co-ordinates of a point within the soil.
R_1	=	$\sqrt{r^2+(z-h)^2}$
R_2	=	$\sqrt{r^2+(z+h)^2}$
r	=	$\sqrt{x^2+y^2}$
dA	=	Elemental area of the base of footing.
r_0	=	Radius of footing.
e	=	Embedment ratio, (h/r_0).
μ	=	Coefficient of friction between the footing and soil.
μ_m	=	Maximum coefficient of friction between the footing and soil.
σ_{zz}	=	Vertical pressure under footing.
G	=	Modulus of rigidity of soil.
υ	=	Poisson's ratio of soil.
$I_1, I_2, \ldots, I_6$	=	Values of integrals.
$F_{1hs}(e), \ldots, F_{6hs}(e)$	=	Terms of dimensionless sliding compliance function.
$F_{hs}(e)$	=	Dimensionless sliding compliance function.
$C_{hs}(e)$	=	Sliding compliance function.
$K_{hs}(e)$	=	Sliding impedance function.
$I_{hsr}(e)$	=	Stiffness increase factor for the footing with rigid-base pressure distributions underneath.
$I_{hsu}(e)$	=	Stiffness increase factor for the footing with uniform pressure distributions underneath.
$I_{hsp}(e)$	=	Stiffness increase factors for the footing with parabolic pressure distributions underneath.
β_1	=	A constant = $3-4\upsilon$
β_2	=	A constant = $(1-\upsilon)(2-\upsilon)$