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# Parameter Estimation of Hoek-Brown Rock Failure Criterion

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## Introduction

The Hoek-Brown rock failure criterion (Hoek and Brown, 1980) is widely used in rock mechanics. The parameters appearing there are determined from experimental data using statistical methods such as least square method (Shah and Hoek, 1992; Li et al., 2000) and formulating the problem as one of the constrained optimization. The parameters (termed as design or decision variables) are obtained by minimizing an objective function constructed with the purpose to find the error in estimation from the observed values. Shah and Hoek (1992) used simplex reflection method to obtain the optimal solution and observed that there is considerable improvement in the estimated statistical parameters in comparison to the same obtained by linear regression analysis. Li et al. (2000) have used generalized reduced gradient method, a direct constraint optimization technique suitable for linear constraints. Colmenares and Zoback (2002) used minimization of mean standard deviation misfit with grid search algorithm that is elementary and adds to more computational time. The performance of least square (LS) method is significantly affected by the presence of scattered data (outliers). In such situations it may be better to use a more robust and stable method like the least median square (LMS) method that is very efficient in noisy environment (Rousseeuw, 1998). Such an approach also helps in identifying the outliers that may be corrected or deleted from the data set for estimation of parameters.

With the above in view, an attempt has been made here to use LMS method for estimating parameters used in Hoek-Brown failure criterion. No

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single procedure or series of procedures can be the panacea to solve all problems to the last details. As such, application of any such technique to a new problem needs critical appraisal. Therefore, suitability of nonlinear programming (NLP) based optimization techniques to LMS method has been explored and presented here. A comparative study of the values of the parameters obtained by using LS and LMS schemes has also been undertaken. The method of identifying outliers with LMS method has been discussed and the technique for improving the efficiency in parameter estimation using reweighted least square (RLS) method as shown.

## Analysis

### *Failure Criteria*

The empirical Hoek–Brown rock failure criterion for intact and jointed rock mass is independent of intermediate principal stress and is expressed as:

$$\left(\sigma_1 - \sigma_3\right) - \left(m\sigma_3 C_0 + sC_0^2\right)^{1/2} = 0 \tag{1}$$

where $\sigma_1$ = major principal stress,

$\sigma_3$ = minor principal stress,

$C_0$ = uniaxial compressive strength of intact rock,

m and s = material parameters.

The value of s is to be taken as unity for intact rock.

and shown

### *Objective Function, Constraints*

The model parameters are estimated by minimizing the objective functions signifying the deviation (error) between a set of observed and estimated data points. The same for Hoek-Brown failure criterion using least square method is as follows.

$$\left[\mathrm{ERR}(f)\right] = \sum_{j=1}^{j=n}\left\{\sigma_{1\,\mathrm{Experimental}} - \sigma_{1\,\mathrm{Predicted}}\right\}^2 \tag{2}$$

subject to $m \geq 0$ and $C_0 \geq 0$

For LMS method as the objective is to minimize the median of square of errors (instead of the conventional sum of square of errors) the objective functions can be written as follows imposing the same constraints as described for LS method.

$$\left[\mathrm{ERR}(f)\right] = \text{Median}\left\{\sigma_{1\,\mathrm{Experimental}_i} - \sigma_{1\,\mathrm{Predicted}_i}\right\} \text{ for } i = 1 \text{ to } n \qquad (3)$$

subject to : $m \geq 0$ and $C_0 \geq 0$

The material parameters appearing in above mentioned models are to be found by minimizing their respective error function. The parameters (m and $C_0$) are decision variables and can be collectively called as design vector **D**. The error function and constraints are then functions of **D** and general formulation for the above problem can be represented as follows:

Find the decision vector **D** such that,

$$F = \min\left[\mathrm{ERR}(\mathbf{f})\right] = \min\left[f(\mathbf{D})\right]$$

subject to : $g_j(\mathbf{D}) < 0$ ; $j = 1, 2, \ldots \ldots, m$

where $g_j$ (**D**) are the inequality constraints and 'm' is their number.

There are a number of approaches for solving constraint optimization problems. In the present case as the constraints are in the form of bounds on decision vectors, the constraints can be eliminated by suitable transformation of the decision vectors (Rao, 1978). As the upper and lower bounds for a variable is of form $L_i \leq X_i \leq U_i$ the specified constraints are satisfied by transforming the variable $X_i$ to new variable $X_i^*$ as $X_i = L_i + (U_i - L_i)\sin^2 X_i^*$. With the elimination of the above constraints, the problem is now one of unconstrained optimization. Thus, there is a significant improvement in the numerical scheme in contrast to those by Shah and Hoek (1998) and Li et al. (2000), who formulated the problem as one of constrained optimization. The elements of the new design vectors for $X^*$ can take any value ($-\propto < X^* < \propto$). Thus, any unconstrained optimization technique can be applied and such method has been found to be successful in multivariable problems (Desai, 2001). The initial elements of the design vectors are chosen as,

$$X_0^* = \sin^{-1}\left(\sqrt{\frac{(X_0 - L_i)}{(U_i - L_i)}}\right) \qquad (4)$$

### *Adopted Optimization Methods*

Both for LS and LMS methods, the unconstrained minimization algorithms e.g. Hooke-Jeeve (HJ), steepest descent, (SD), Modified Newton-Raphson (NR), Fletcher and Reeve (FR), Polok-Ribiero, (PR), Davidon-Fletcher-Powel (DFP) and Broydon-Fletcher-Goldfarb-Shanno (BFGS) belonging to NLP were applied to obtain the minimum value of the objective function. All the above algorithms are gradient based except HJ which is a direct search method that does not need any gradient evaluation. Details of these methods with algorithms are available in standard textbooks on numerical methods (Rao, 1978; Press et al., 2000).

## Results and Discussion

Conventional Triaxial shear test data for different rock samples like Tennesse marble (Shah and Hoek, 1992), Pennant sandstone, Blackingstone quarry granite and Darley Dale sandstone (Franklin and Hoek, 1970) are considered here for analysis. As these test data are available in the cited literature, these are not presented here. Using some of the above data studies were reported on parameter estimation using LS methods (Shah and Hoek, 1992; Li et al., 2000).

Results of NLP methods are generally dependant on the choice of the initial design vector. Each of the NLP algorithms is tested by taking forty such randomly generated design vectors with its transformed values (Eqn. 4). The robustness (effectiveness) of the algorithm is judged by the number of failure (NF) out of the above initial design vectors, which resulted in failure (i.e. the obtained final result is not a global minimum) and is expressed in percentage. As such, the algorithms are also evaluated in terms of average of number of function evaluation (NFE) for successful initial points. The low values of NF and NFE indicate the corresponding algorithm to be effective and efficient.

For least square method all NLP based procedures gave identical results. The predicted values of m (5.482) and $C_0$ (135.03) for Tennessee marble with the present procedure are identical to those obtained by simplex reflection method (Shah and Hoek, 1992) and PREO-SOLVER method (Li et al., 2000). The values of these parameters for other rocks e.g. Pennant sandstone, Blackingstone quarry granite and Darley Dale sandstone were also estimated by using LMS method.

The robustness and efficiency of various algorithms in isolating the best solution using both LS and LMS are first evaluated and presented. These are respectively measured by NF and NFE as shown in Fig.1a and Fig.1b. It can be observed that for all the rock samples considered here,

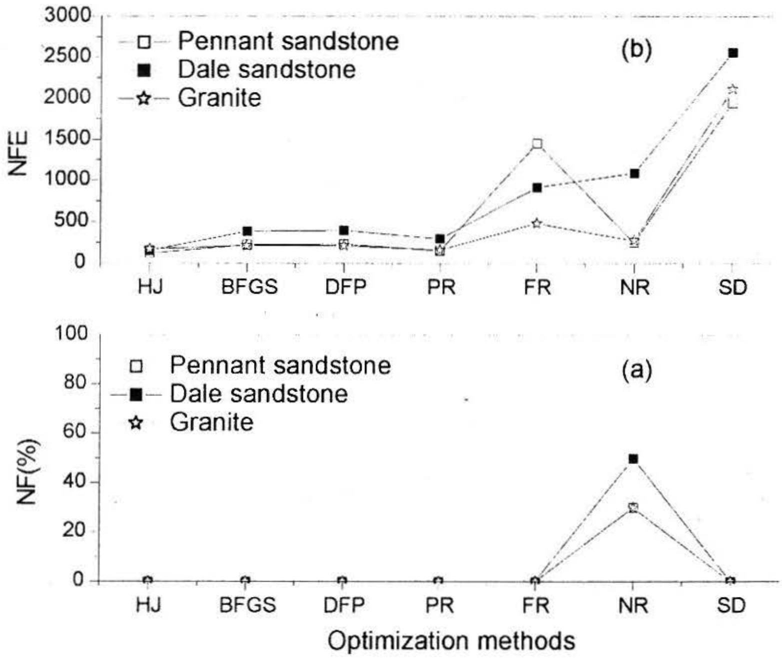



**FIGURE 1 : The Robustness (NF) and Efficiency (NFE) of Different Algorithm for Parameter Estimation using LS Method**

though all the NLP methods except NR are equally robust (as indicated by NF = 0 in Fig.1a), HJ and PR are more efficient than the other methods as the corresponding values of NFE for these methods are the least (Fig.1b). However, it may be highlighted that the other nonlinear optimization methods like FR, SD and NR are not efficient as signified by high NFE values.

The robustness and efficiency of the optimization methods for different rocks when applied to LMS method are shown in Figs.2a and 2b respectively. It is observed that in contrast to LS method, optimization techniques NR, DFP, BFGS and HJ could only find out the optimum parameter for granite and Pennant sandstone (signified by NF value less than 100%) but for Dale sandstone, NR method also failed to isolate the optimal point. It is also observed that the robustness of optimization methods (HJ, DFP, BFGS and NR) in finding out the optimum solution is reduced with LMS method (NF = 85-93%) compared to LS method (NF = 0%). Such difficulties in obtaining LMS parameter using NLP methods have also been reported in literature (Rousseeuw, 1998).

For other rocks as obtained from the present study using the data reported by Franklin and Hoek (1970), Table 1 shows the predicted values

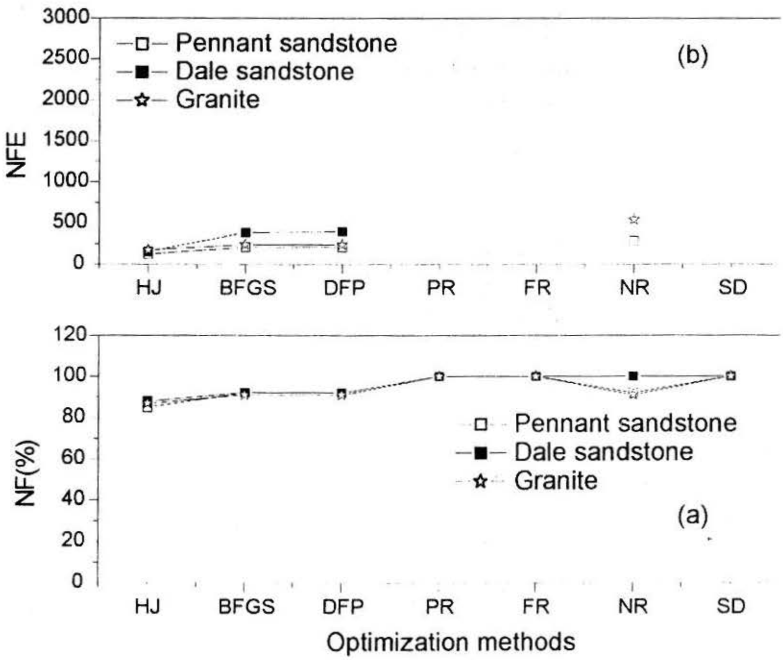



**FIGURE 2 : The Robustness (NF) and Efficiency (NFE) of Different Algorithm for Parameter Estimation using LMS Method**

of parameters (m and $C_0$) for different rocks. Comparison of LS and LMS methods are made in terms of least mean square error (MSE) and least median square error (LMSE) respectively. Absolute MSE using LS method ranges from 43.67 to 771.78, where as the LMSE for the LMS method is

**TABLE 1 : Comparisons Between LS and LMS Method for Hoek-Brown Failure Criteria**

| Methods | Optimized parameters | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Granite | | | Dale sandstone | | | Pennant sandstone | | |
| | m | $C_0$ | Error‡ | m | $C_0$ | Error‡ | m | $C_0$ | Error‡ |
| LS method | 19.8 | 216.6 | 771.78 | 16.96 | 76.75 | 43.67 | 12.3 | 206.47 | 116.85 |
| LMS method | 22.9 | 214.6 | 158.31 | 16.23 | 79.95 | 8.35 | 11.67 | 217.64 | 84.08 |

‡ MSE for LS and LMSE for LMS Method

8.35 to 158.31. For the same rock the ratio of the errors using LS and LMS method is of the order of 5 to 6. So LMS method is preferable over LS method.

Adverse effect of outliers on the parameter estimation is well recognized. To identify the outliers and improve upon the estimation of the model parameters, reweighted least squares (RLS) concept (Rousseeuw, 1998) is used here. First using LMS method the parameters are estimated and subsequently the error in each element (observation) of the data set from the predicted value corresponding to these parameters is estimated. The error is expressed in terms of standardized residuals as $|r_i/\hat{\sigma}|$ where $\hat{\sigma}$ is a consistent estimator of standard deviation for LMS method and defined as

$$\hat{\sigma} = 1.483\sqrt{\text{median } r_i^2}\ ,\ i = 1, \ldots, n$$

and $r_i$ are the residuals from LMS fit. Regression outliers, for which the standardized residuals are greater or equal to 2.5, are identified. LS regression is applied after deleting these outliers. This procedure is known as RLS method.

For both LS and LMS methods, Figs.3a, 3b and 3c show for different rocks the variation of the standardized residuals with principal stress values ($\sigma_1$). These figures reveal that there are several outliers in the data set when LMS method is used but in contrast when LS method is used there is no presence of such outliers. This indicates that LS method fails to handle the scattered data due to its failure in identifying the outliers, caused by the blowing up of the standard deviation with scattered data. The predicted strength envelopes for granite using LS and RLS methods are shown in Figs.4a and 4b respectively and compared with the observed data points. In Fig.4b the outliers (indicated by ☐) as obtained from LMS method are also shown along with other data points. While using RLS method these outliers are excluded from the data set but are made use of in LS method. It is observed that the MSE value obtained from RLS method (351.48) is less than half of the corresponding value predicted by LS method (771.8). The unconfined compressive strength computed by using RLS method (206.6 MPa) is closer to the observed value (179.6 MPa) in comparison to the same obtained from LS method (216.56 MPa). Similar studies are made for Dale sandstone and Pennant sandstone and the results are shown in Figs.5 and 6 respectively. In case of Dale sandstone also, the predicted unconfined compressive strength by RLS method (78.57 MPa) is closer to the observed value (80.1 MPa) than that of LS method (76.75 MPa). For both the sandstones as the data points are not very much scattered the decreases in MSE value are nominal (34% and 28% respectively) in comparison to those for granite, where the value is of the order of 200%.

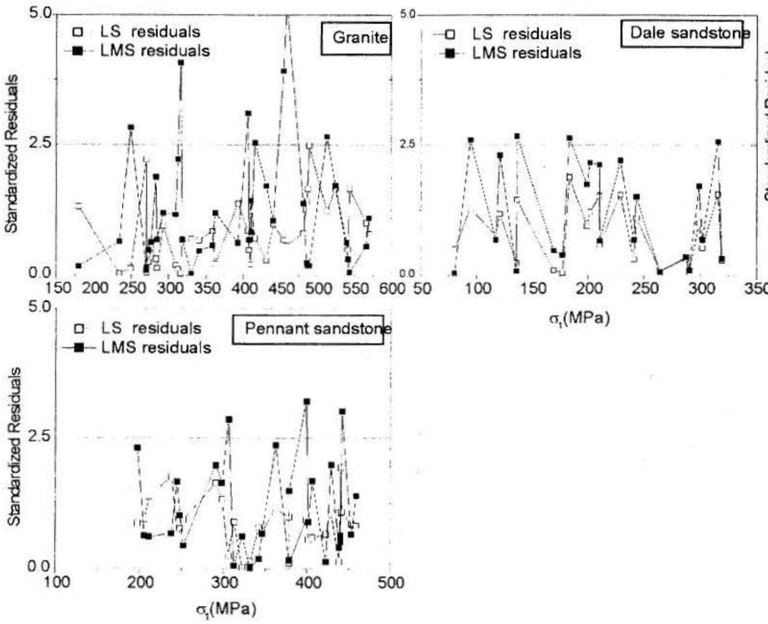



**FIGURE 3 : Standardized Residuals by LS and LMS Method for Different Rock Samples**

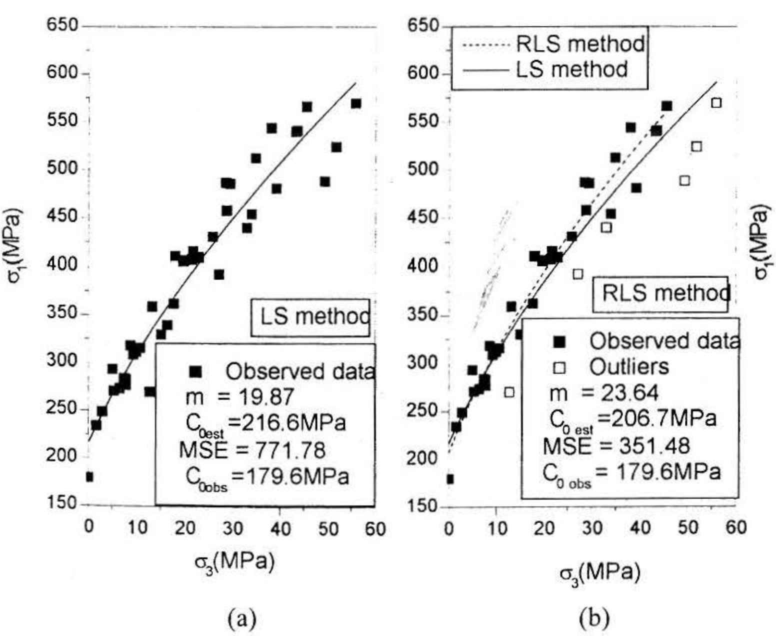
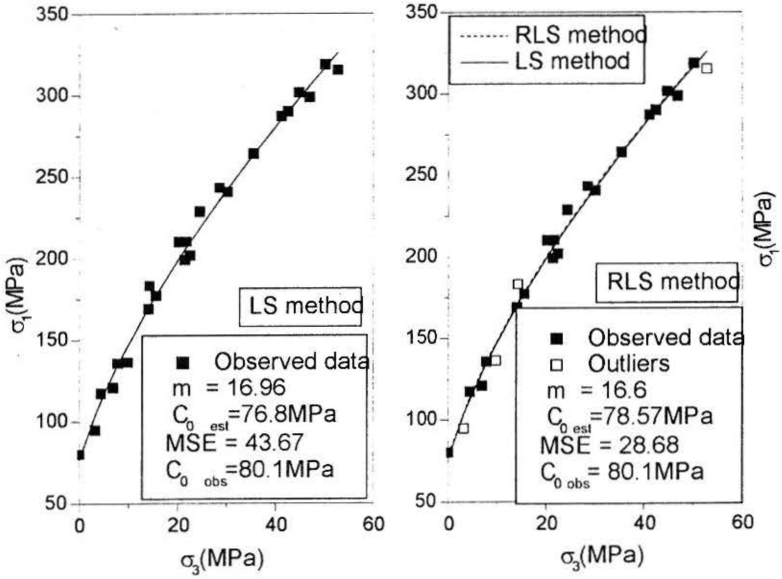
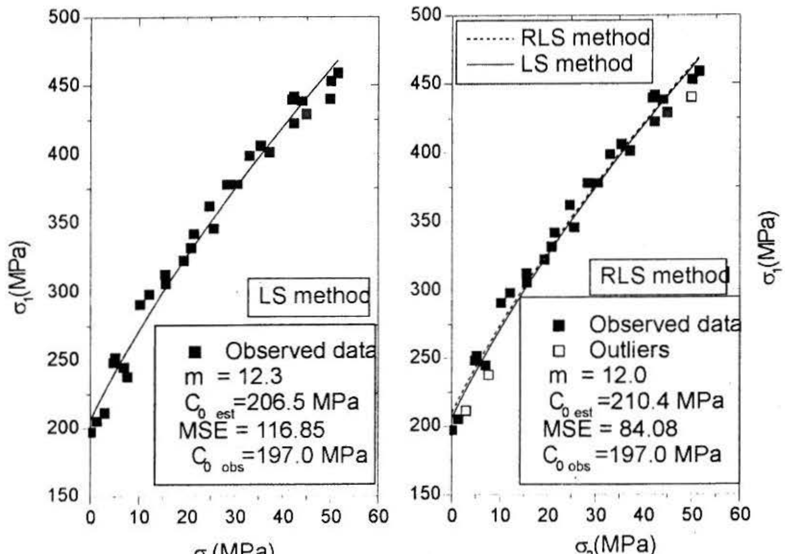



**FIGURE 4 : Triaxial Strength Envelope for Granite Corresponding to Hoek-Brown Failure Criterion using (a) LS Method and (b) RLS Method**




(a) (b)

**FIGURE 5 : Triaxial Strength Envelope for Dale Sandstone Corresponding to Hoek-Brown Failure Criterion using (a) LS Method and (b) RLS Method**




(a) (b)

**FIGURE 6 : Triaxial Strength Envelope for Pennant Sandstone Corresponding to Hoek-Brown Failure Criterion using (a) LS Method and (b) RLS Method**

Thus it is recommended that in case of scattered data RLS method should be preferred over conventional LS method for better estimation of the parameters appearing in the Hoek-Brown rock failure criterion.

## Conclusions

Based on the above studies following conclusions are drawn. Use of transformation technique to eliminate the side constraints on the variables simplifies the formulation. For isolating the optimal solution the nonlinear programming algorithms like Modified Newton-Raphson, Polok-Ribiero, Davidon-Fletcher-Powel, Broydon-Fletcher-Goldfarb-Shanno and Hooke-Jeeve could be used successfully in conjunction with LMS method; but, these methods are not robust. Identification of the outliers by using LMS method and their exclusion from the data set and subsequent use of RLS method resulted in considerable statistical improvement in the values of the estimated parameters. It is found that for granite and Dale sandstone, prediction of unconfined compressive strengths are better with RLS method compared to LS method

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