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Coupled Vibrations of Machine Foundations Subjected to General Loads

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Introduction

Machine installations are generally supported on rigid concrete foundation blocks. Depending upon the strength and other characteristics of the soil, these foundation blocks can directly rest on soil or can be supported through piles.

Analysis of such foundations involves solution of coupled, linear, second order differential equations to get the response of machine foundation to loads and moments coming from various units of the machine assembly (Kameswara Rao, 1980; Kameswara Rao, 1985). Mode-Superposition method (Craig, 1981) is the most commonly used method for solving such equations. But this method is very inefficient for systems, which have higher degrees of freedom. Also it cannot be used if damping matrix of the equations does not satisfy orthogonality conditions. In such cases methods suggested by Hurty and Rubinstein (1964), Warburton and Soni (1977) can be tried. However, in case of most general problems, it is considered best to use step by step numerical integration methods, such as average-acceleration method, Wilson's θ method (Craig, 1981) and Runge-Kutta method (James, 1989; Venugopal Rao, 1994), which are highly computationally-intensive. Moreover, all the above methods do not take advantage of the periodic nature of the forcing functions, which is the case for almost all general problems.

In this investigation a general method based on Fourier analysis is developed, which is capable of solving any number of coupled, linear differential equations to obtain the steady state response for any kind of

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periodic forcing functions (i.e., net loads and moments coming on foundation may comprise of different components of different periods). The method is illustrated by applying it to a few practical problems and comparing the results with the existing ones from other numerical methods.

Derivation of General Equations of Motion

Foundation for machine assemblies can be directly supported by soil medium if the soil is (a) of medium strength (b) of low compressibility (c) is not made up or reclaimed and is a natural formation. Otherwise such foundations have to be supported by piles taken deeper into the soil. The governing equations of motion for both the cases can be obtained as follows (Kameswara Rao, 1977, 1980).

Machine-Foundation-Soil System

A typical assembly of a simple machine foundation, and the convention of co-ordinates, displacements, forces and moments are shown in Figs.1 and 2 respectively. Point O is the combined centre of gravity (C.G.) of the machine and foundation (assumed as rigid body) and is chosen as origin of the co-ordinate system. If A is any point of the system with co-ordinates x_A, y_A, z_A, then displacements and rotations at A can be expressed as

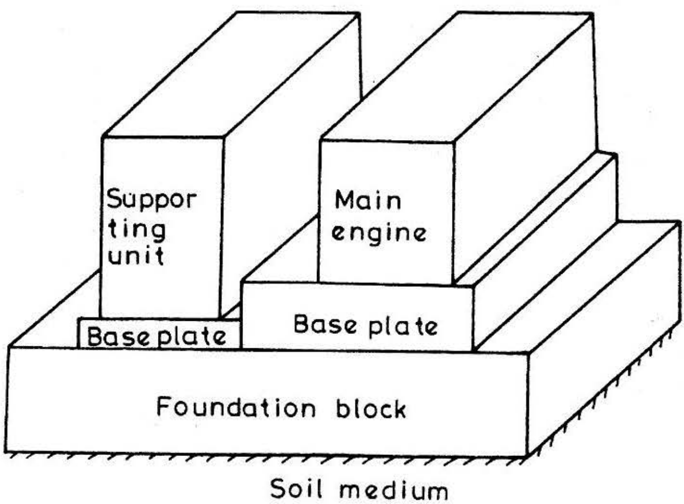

FIGURE 1 : A Simple Machine Foundation

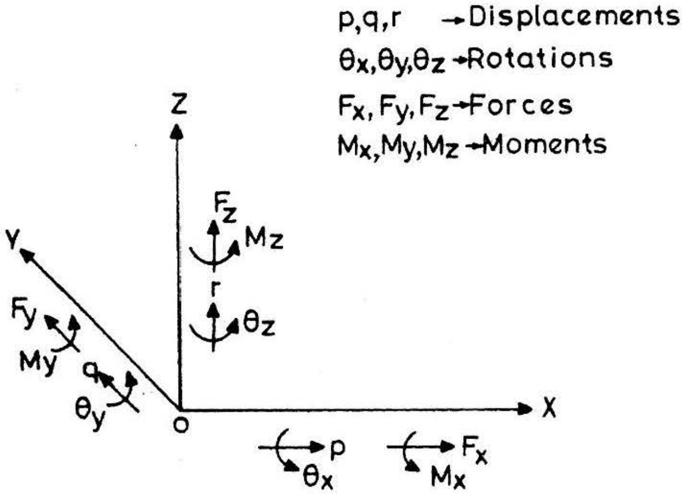

FIGURE 2 : Sign Convention for Displacements, Rotations, Forces and Moments (Positive Quantities Shown)

$$
\begin{aligned}
p_A &= p_o - y_A\theta_z + z_A\theta_y \\
q_A &= q_o + x_A\theta_z - z_A\theta_x \\
r_A &= r_o - x_A\theta_y + y_A\theta_x \\
\theta_{xA} &= \theta_x \\
\theta_{yA} &= \theta_y \\
\theta_{zA} &= \theta_z
\end{aligned} \tag{1}
$$

where p, q, r are displacement and θ_x, θ_y, θ_z are rotations along and about x, y, z axes respectively and subscripts refer to the point.

Taking point A at the centre of contact area, the net soil reaction components acting at this point can be written as (Kameswara Rao, 1977).

$$
\begin{aligned}
F_x &= -c_x\dot{p}_A - k_x p_A \\
F_y &= -c_y\dot{q}_A - k_y q_A \\
F_z &= -c_z\dot{r}_A - k_z r_A
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
M_x &= -h_x\dot{\theta}_A - s_x\theta_x \\
M_y &= -h_y\dot{\theta}_y - s_y\theta_y \qquad \text{(2 cont.)} \\
M_z &= -h_z\dot{\theta}_z - s_z\theta_z
\end{aligned}
$$

where c, k, h, s are damping and spring constants of the equivalent single degree of freedom analogues, which can be evaluated using expressions given by Kameswara Rao (1977), Richart et al. (1970). These expressions are given in Appendix 1. Dots denote derivative with respect to time.

Substituting the values of p_A, q_A, r_A from Eqn.1, these reactions can be written in matrix form as

$$
\begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} -c_x & 0 & 0 & 0 & -c_x z_A & c_x y_A \\ 0 & -c_y & 0 & c_y z_A & 0 & -c_y x_A \\ 0 & 0 & -c_z & -c_z y_A & c_z x_A & 0 \\ 0 & 0 & 0 & -h_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -h_y & 0 \\ 0 & 0 & 0 & 0 & 0 & -h_z \end{bmatrix} \begin{Bmatrix} \dot{p}_o \\ \dot{q}_o \\ \dot{r}_o \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix}
$$

$$
+ \begin{bmatrix} -k_x & 0 & 0 & 0 & -k_x z_A & k_x y_A \\ 0 & -k_y & 0 & k_y z_A & 0 & -k_y x_A \\ 0 & 0 & -k_z & -k_z y_A & k_z x_A & 0 \\ 0 & 0 & 0 & -s_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_y & 0 \\ 0 & 0 & 0 & 0 & 0 & -s_z \end{bmatrix} \begin{Bmatrix} p_o \\ q_o \\ r_o \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \qquad (3)
$$

The above soil reactions as well as forces and moments from various units of the assembly can all be reduced into generalized forces and moments acting at the combined C.G. of machine foundation assembly.

From the dynamic equilibrium of the rigid body, the general equations of motion of the rigid body can now be written as

$$
\begin{aligned}
m\ddot{p}_o &= \bar{F}_x + F_x \\
m\ddot{q}_o &= \bar{F}_y + F_y \\
m\ddot{r}_o &= \bar{F}_z + F_z \\
I_x\ddot{\theta}_x &= \bar{M}_x + \left(M_x - F_y z_A + F_z y_A\right) \\
I_y\ddot{\theta}_y &= \bar{M}_y + \left(M_y + F_x z_A - F_z x_A\right) \\
I_z\ddot{\theta}_z &= \bar{M}_z + \left(M_z + F_y x_A - F_x y_A\right)
\end{aligned}
\tag{4}
$$

where m is the total mass of the machine and foundation block, I_x, I_y, I_z are mass moments of inertia about x, y, z axes respectively and $\bar{F}_x$, $\bar{F}_y$, $\bar{F}_z$, $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ are net forces and moments coming from different units, acting at combined C.G.

Equation 4 with the help of Eqn.3 can be written in matrix form (Kumar, 1994) as

$$
\begin{bmatrix}
m & 0 & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 & 0 \\
0 & 0 & m & 0 & 0 & 0 \\
0 & 0 & 0 & I_y & 0 & 0 \\
0 & 0 & 0 & 0 & I_y & 0 \\
0 & 0 & 0 & 0 & 0 & I_z
\end{bmatrix}
\begin{Bmatrix}
\ddot{p}_o \\ \ddot{q}_o \\ \ddot{r}_o \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z
\end{Bmatrix}
$$

$$
+\begin{bmatrix}
c_x & 0 & 0 & 0 & c_x z_A & -c_x y_A \\
0 & c_y & 0 & -c_y z_A & 0 & c_y x_A \\
0 & 0 & c_z & c_z y_A & -c_z x_A & 0 \\
0 & -c_y x_A & c_z y_A & \begin{pmatrix} h_x + c_y z_A^2 \\ + c_z y_A^2 \end{pmatrix} & -c_z x_A y_A & -c_y x_A z_A \\
c_x z_A & 0 & -c_z x_A & -c_z x_A y_A & \begin{pmatrix} h_y + c_z x_A^2 \\ + c_x z_A^2 \end{pmatrix} & -c_x y_A z_A \\
-c_x y_A & c_y x_A & 0 & -c_y x_A z_A & -c_x y_A z_A & \begin{pmatrix} h_z + c_x y_A^2 \\ + c_y x_A^2 \end{pmatrix}
\end{bmatrix}
\begin{Bmatrix}
\dot{p}_o \\ \dot{q}_o \\ \dot{r}_o \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z
\end{Bmatrix}
$$

$$
+\begin{bmatrix}
k_x & 0 & 0 & 0 & k_x z_A & -k_x y_A \\
0 & k_y & 0 & -k_y z_A & 0 & k_y x_A \\
0 & 0 & k_z & k_z y_A & -k_z x_A & 0 \\
0 & -k_y z_A & k_z y_A & \begin{pmatrix} s_x + k_y z_A^2 \\ + k_z y_A^2 \end{pmatrix} & -k_z x_A y_A & -k_y x_A z_A \\
k_x z_A & 0 & -k_z x_A & -k_z x_A y_A & \begin{pmatrix} s_y + k_z x_A^2 \\ + k_x z_A^2 \end{pmatrix} & -k_x y_A z_A \\
-k_x y_A & k_y x_A & 0 & -k_y x_A z_A & -k_x y_A z_A & \begin{pmatrix} s_z + k_x y_A^2 \\ + k_y x_A^2 \end{pmatrix}
\end{bmatrix}
\begin{Bmatrix} p_o \\ q_o \\ r_o \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}
$$

$$
= \begin{Bmatrix} \bar{F}_x \\ \bar{F}_y \\ \bar{F}_z \\ \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{Bmatrix} \tag{5}
$$

Machine-Pile-Foundation-Soil System

Here dynamic reaction from each pile is to be considered and summation is carried over all the piles to obtain net forces and moments acting at the combined C.G. due to soil reaction.

The governing equations of motion in this case are obtained in the following form (Kumar, 1994).

$$
\begin{bmatrix}
m & 0 & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 & 0 \\
0 & 0 & m & 0 & 0 & 0 \\
0 & 0 & 0 & I_y & 0 & 0 \\
0 & 0 & 0 & 0 & I_y & 0 \\
0 & 0 & 0 & 0 & 0 & I_z
\end{bmatrix}
\begin{Bmatrix} \ddot{p}_o \\ \ddot{q}_o \\ \ddot{r}_o \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{Bmatrix}
$$

$$+\sum_{i=1}^{l}\begin{bmatrix} c_{xi} & 0 & 0 & 0 & \left(c_{xi}z_i+c_{xi}\right) & -c_{xi}y_i \\ 0 & c_{yi} & 0 & \left(-c_{yzi}-c_{yi}z_i\right) & 0 & c_{yi}x_i \\ 0 & 0 & c_{zi} & c_{zi}y_i & -c_{zi}x_i & 0 \\ 0 & \begin{pmatrix} h_{xyi} \\ +c_{yi}z_i \end{pmatrix} & c_{zi}y_i & (F) & -x_iy_ic_{zi} & \begin{pmatrix} x_ih_{xyi} \\ -x_iz_ic_{yi} \end{pmatrix} \\ \begin{pmatrix} h_{xyi} \\ +y_ic_{xi} \end{pmatrix} & 0 & -x_ic_{zi} & -x_iy_ic_{zi} & (G) & \begin{pmatrix} -y_ih_{yxi} \\ -y_iz_ic_{xi} \end{pmatrix} \\ -y_ic_{xi} & x_ic_{yi} & 0 & \begin{pmatrix} x_ic_{yxi} \\ +x_iz_ic_{xi} \end{pmatrix} & \begin{pmatrix} y_ic_{xyi} \\ +y_iz_ic_{xi} \end{pmatrix} & (H) \end{bmatrix}\begin{Bmatrix} \dot{p}_o \\ \dot{q}_o \\ \dot{r}_o \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix}$$

$$+\sum_{i=1}^{l}\begin{bmatrix} k_{xi} & 0 & 0 & 0 & \left(k_{xi}z_i+k_{xi}\right) & -k_{xi}y_i \\ 0 & k_{yi} & 0 & \left(-k_{yzi}-k_{yi}z_i\right) & 0 & k_{yi}x_i \\ 0 & 0 & k_{zi} & k_{zi}y_i & -k_{zi}x_i & 0 \\ 0 & \begin{pmatrix} s_{xyi} \\ +k_{yi}z_i \end{pmatrix} & k_{zi}y_i & (I) & -x_iy_ik_{zi} & \begin{pmatrix} x_is_{xyi} \\ -x_iz_ik_{yi} \end{pmatrix} \\ \begin{pmatrix} s_{xyi} \\ +y_ik_{xi} \end{pmatrix} & 0 & -x_ik_{zi} & -x_iy_ik_{zi} & (J) & \begin{pmatrix} -y_is_{yxi} \\ -y_iz_ik_{xi} \end{pmatrix} \\ -y_ik_{xi} & x_ik_{yi} & 0 & \begin{pmatrix} x_ik_{yxi} \\ +x_iz_ik_{xi} \end{pmatrix} & \begin{pmatrix} y_ik_{xyi} \\ +y_iz_ik_{xi} \end{pmatrix} & (K) \end{bmatrix}\begin{Bmatrix} p_o \\ q_o \\ r_o \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}$$

$$=\begin{Bmatrix} \bar{F}_x \\ \bar{F}_y \\ \bar{F}_z \\ \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{Bmatrix} \tag{6}$$

where

$$(F) = \left[h_{xi}+z_i^2c_{yi}+y_i^2c_{zi}-z_ic_{yxi}-z_ih_{yxi}\right]$$

$$(G) = \left[h_{yi}+z_i^2c_{xi}+x_i^2c_{xi}+z_ic_{xyi}+z_ih_{yxi}\right]$$

$$(H) = \left[h_{zi}+x_i^2c_{xi}+y_i^2c_{xi}\right]$$

$$(I) = \left[s_{xi}+z_i^2k_{yi}+y_i^2k_{zi}-z_ik_{yxi}-z_is_{yxi}\right]$$

$$(J) = \left[s_{yi} + z_i^2 k_{xi} + x_i^2 k_{xi} + z_i k_{xyi} + z_i s_{yxi}\right]$$

$$(K) = \left[s_{zi} + x_i^2 k_{xi} + y_i^2 k_{xi}\right]$$

c_{ki}, $k_{si} \ldots h_{xi} \ldots$, $s_{yi} \ldots$ etc. are damping and stiffness parameters of the individual pile along and about the directions specified by suffixes (Novak, 1974, 1978; Saul, 1968) and are given in Appendix 2 and 1 is the number of piles.

Equations (5) and (6) need to be solved to get the response for foundations directly supported on soil and for pile foundations respectively. It may be difficult to obtain analytical solutions of these coupled equations as forces and moments are usually not known in the form of continuous functions, but their values are given at discrete intervals of time and each of them could be having different period/frequency.

Solutions using Fourier Analysis

Fourier Series

Any periodic load F(t) with a period tð can be expressed as

$$F(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

where

$$a_o = \frac{2}{\tau}\int_0^{r} F(t)\,dt \qquad (7)$$

$$a_n = \frac{2}{\tau}\int_0^{r} F(t)\cos n\omega t \cdot dt$$

$$b_n = \frac{2}{\tau}\int_0^{r} F(t)\sin n\omega t \cdot dt$$

$$\omega = \frac{2\pi}{\tau}$$

Integration in Eqn.7 can be carried out exactly if $F(t)$ is known in the form of an integrable function, otherwise numerical techniques are used for evaluating Fourier coefficients a_o, a_n, b_n.

The number of terms considered in the above summation depends upon the degree of accuracy desired in approximation of the actual function by the Fourier series.

Fourier Analysis of a Set of Coupled, Linear Differential Equations

The governing equations of motion of a multi degree of freedom (MDF) system such as Eqns.5 and 6 can be written in matrix form as

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\} = \{F(t)\} \tag{8}$$

where M, C, K are mass, damping and stiffness matrices respectively of the system of size *nxn* (*n* is the number of degrees of freedom). {X} response vector $\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ of the system for force vector $\begin{Bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{Bmatrix}$.

Since $F_i(t)$, $i = 1$ to n are periodic in nature, they can be expressed in Fourier series form as

$$F_i(t) = \frac{a_{io}}{2}+\sum_{k=1}^{\infty}\left(a_{ik}\cos k\omega_i t+b_{ik}\sin k\omega_i t\right) \quad (i = 1 \text{ to } n) \tag{9}$$

where $\omega_i = \dfrac{2\pi}{\tau_i}$ [where τ_i is the period of $F_i(t)$].

The solutions of Eqn.8 are sought, using the principle of superposition, in the form

$$X = \sum_{i=1}^{n} X_i \tag{10}$$

where $X_i \left(= \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \right)$ is obtained by putting the value of $F_i(t)$ from Eqn.9

into Eqn.8 and taking other values of $F(t)$, i.e. $F_1(t)$, $\cdots$, $F_{i-1}(t)$,

$F_{i+1}(t)$, $\cdots$, $F_n(t)$, to be zero. Since Eqn.8 are linear differential equations, by superposition principle (Ayers, Jr., 1981) complete steady state solution will be the sum of all X_i as given by Eqn.10.

Since forcing functions are in the form of Fourier series, the responses X_i for $F_i(t)$ are also sought to be obtained in the form of Fourier series as

$$\begin{aligned}
x_{1i} &= \frac{A^i_{1o}}{2} + \sum_{k=1}^{\infty}\left(A^i_{1k}\cos k\omega_i t + B^i_{1k}\sin k\omega_i t\right) \\
&\vdots \\
x_{ji} &= \frac{A^i_{jo}}{2} + \sum_{k=1}^{\infty}\left(A^i_{jk}\cos k\omega_i t + B^i_{jk}\sin k\omega_i t\right) \qquad (11) \\
&\vdots \\
x_{ni} &= \frac{A^i_{no}}{2} + \sum_{k=1}^{\infty}\left(A^i_{nk}\cos k\omega_i t + B^i_{nk}\sin k\omega_i t\right)
\end{aligned}$$

Now for each i (from 1 to n), the values of x_{ji} (j = 1 to n) from Eqn.11 and of $F_i(t)$ from Eqn.9 [and $F_1(t)$ = 0 for $1 \neq i$] are substituted in Eqn.8. Equation 8 are then integrated from 0 to τ_i to obtain A^i_{1o}, $\cdots$, A^i_{no}. To make use of orthogonality of harmonic functions (sine and cosine terms), Eqn.8 are multiplied by $\cos m\omega_1 t$ and then integrted from 0 to τ_i. Similarly they are multiplied by $\sin m\omega_1 t$ and integrted from 0 to τ_i. We obtain linear algebraic equations which can be solved by Gauss-Elimination Method for getting the Fourier coefficients of the response.

By varying m from 1 to ∞ and i from 1 to n, one can obtain all the Fourier coefficients of the responses x_1, x_2, $\cdots$, x_n. Thus complete steady state solution of Eqns. (8) can now be written as

$$x_j(t) = \sum_{i=1}^{n}\left[\frac{A^i_{jo}}{2} + \sum_{k=1}^{\infty}\left(A^i_{jk}\cos k\omega_i t + B^i_{jk}\sin k\omega_i t\right)\right] \qquad (12)$$

$$(j = 1 \text{ to } n)$$

The coefficients in the above equations can be obtained by solving algebraic equations of the form

$$[K]\begin{Bmatrix} A_{1o}^i \\ A_{2o}^i \\ \vdots \\ A_{no}^i \end{Bmatrix} = \begin{Bmatrix} a_{1o}\delta_{1i} \\ a_{2o}\delta_{2i} \\ \vdots \\ a_{no}\delta_{ni} \end{Bmatrix}$$

and

$$\begin{bmatrix} \left(-k^2\omega_i^2 M + K\right) & \left(k\omega_i C\right) \\ \left(-k\omega_i C\right) & \left(-k^2\omega_1^2 M + K\right) \end{bmatrix} \begin{Bmatrix} A_{1k}^i \\ A_{2k}^i \\ \vdots \\ A_{nk}^i \\ B_{1k}^i \\ B_{2k}^i \\ \vdots \\ B_{nk}^i \end{Bmatrix} = \begin{Bmatrix} a_{1k}\delta_{1i} \\ a_{2k}\delta_{2i} \\ \vdots \\ a_{nk}\delta_{ni} \\ b_{1k}\delta_{1i} \\ b_{2k}\delta_{2i} \\ \vdots \\ b_{nk}\delta_{ni} \end{Bmatrix} \tag{13}$$

$$i = 1 \text{ to } n, \quad k = 1 \text{ to } \infty$$

δ_{ij} is the Kronecker delta, defined as

$$\begin{aligned} \delta_{ij} &= 0 \qquad \text{for } i \neq j \\ &= 1 \qquad \text{for } i = j \end{aligned}$$

If any of the six forcing functions in Eqn.8 has components of different frequencies, then solution should be obtained for each frequency, taking the components of other frequencies to be zero. The final solution will be the sum of all such independently obtained solutions.

Response Analysis of Machine-Foundation-Soil System and of Machine-Pile-Foundation-Soil System

Equations of motion, Eqns.5 and 6, can be written in the form of Eqn.8. For a machine assembly the forces and moments $\bar{F}_x$, $\bar{F}_y$, $\bar{F}_z$, $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ are given at discrete time intervals over a period of time. These are converted into Fourier series by evaluating the coefficients by numerical integration. Eqn.8 are then solved for these forcing functions for six degrees of freedom system ($n = 6$) using the procedure of last section.

Now the solution p_o, q_o, r_o, θ_x, θ_y, θ_z can be obtained from Eqn.12 by evaluating their Fourier coefficients from Eqn.13. Response at any time at the combined C.G. can be obtained by substituting that value of time in Eqn.12. Displacements and rotations at any other point of the machine assembly can then be obtained using Eqn.1.

Examples and Results

While the Fourier series method developed above is applicable for any value of n in Eqn.8, for the sake of relevance and simplicity, solutions of above equations are obtained from n = 6, i.e., for six degrees of freedom system, which is usually the case for machine foundations. Also, the method developed in the last section can be applied to any multi degree of freedom (MDF) problem, if steady state solution is required for periodic loading of any nature. A computer programme has been developed based on the above analysis, which can be used not only for the machine foundation analysis, but also for the solution of equations of motion of any MDF system.

To demonstrate the viability and efficiency of the method, the results are obtained for the following problems and are compared with the existing ones.

Solution of a MDF System's Equations

For demonstrating the above method, following 3-degrees of freedom system is analysed (Kumar, 1994) and the responses are plotted along with time in Fig.3.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & -2 \\ -1 & 3 & 5 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 10 & 0 & 10 \\ -100 & 50 & -25 \\ 75 & 50 & 20 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}$$
$$+100 \begin{bmatrix} 50 & 10 & 0 \\ 30 & 10 & 30 \\ 0 & 5 & 20 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 10 + 20\sin 10t + 30\cos 50t \\ 75 \\ 100\cos 40t \end{Bmatrix} \tag{14}$$

The above equations can not easily be solved by Mode-superposition method or by any of the approximate methods (Hurty, 1964; Warburton, 1977) as the coefficient matrices are not symmetric and the forcing functions consist of components of different frequencies. If the response is required at very large t, then number of calculations becomes very large, if one uses any numerical method. On the other hand, by Fourier series method the solution is easily obtained.

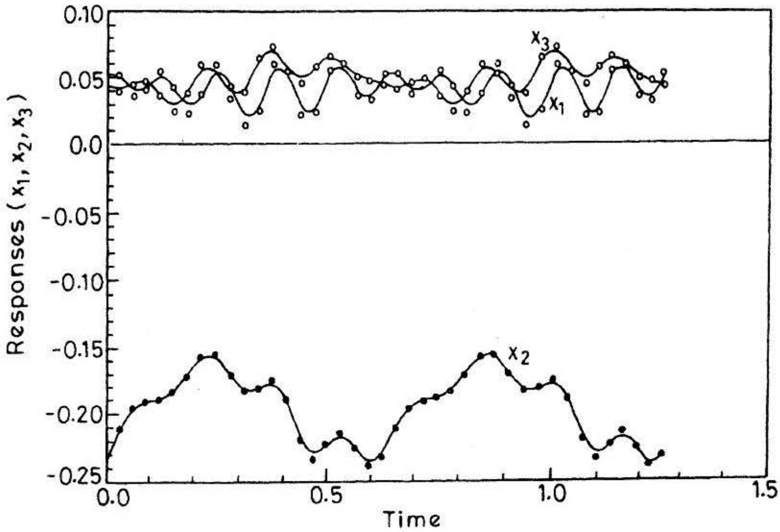

FIGURE 3 : Responses of Three Degrees of Freedom System

The method has also been tried out successfully in several other problems of general nature to demonstrate its versatility (Kumar, 1994).

The application of above method to a few practical problems is illustrated below.

Analysis of Machine-Foundation-Soil System

As a sample problem, computations have been carried out for the eight cylinder diesel engine power generating set manufactured by M/s Kirloskar Oil Engines Ltd., Pune, India. For the proper design of the foundation for above engine, it is required to find out displacements and rotations at the various points of the machine-foundation assembly due to periodically varying dynamic loads and moments coming from the engine. The machine-foundation assembly is shown in Fig.1. Some relevant details of the engine, generator set assembly, dynamic forces and moments coming from the engine are given in Appendix 3. Foundation block of the shape of a rectangular parallelopiped is considered, having dimensions 2 m and 5.985 m in horizontal direction. Analysis is carried out for 2 values of thickness, 1.5 m and 0.3 m. The unit weight and Poisson's ratio of the soil are 1700 kg/m^3 and 0.3 respectively. Values of shear modulus, G, of the soil are considered from 0.1×10^7 kg/m^2 (9.81×10^6 N/m^2) to 1×10^7kg/m^2 (9.81×10^7 N/m^2). Based on above data, the progrmme first computes the elements of matrices of Eqn.5 and then use Fourier analysis to solve Eqn.5.

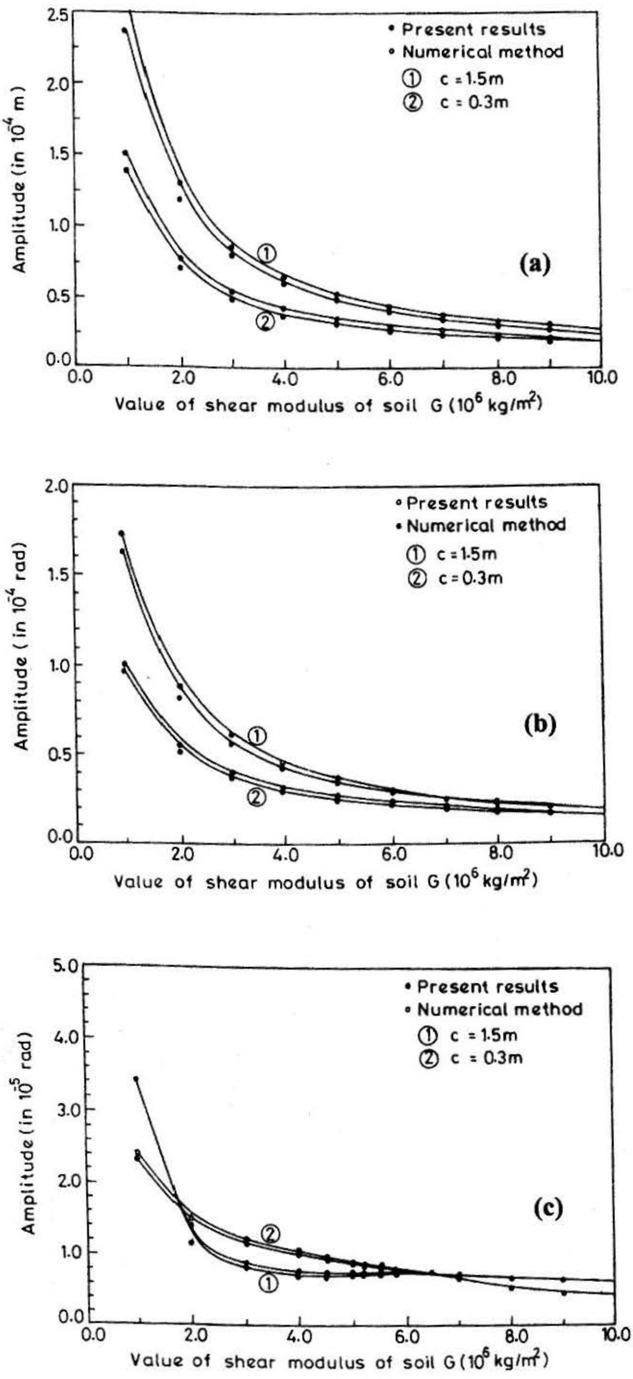

FIGURE 4 : Variation of Steady State Amplitudes of (a) P_o, (b) θ_y and (c) θ_z with G (10^6 kg/m^2) (Block Foundation)

The forces and moments are of general nature and hence the programme carries out the analysis using Fourier series and steady state response at combined C.G. is plotted with time to obtain the steady state amplitude for a given *G*. Varying the value of *G*, steady state amplitude is plotted against *G* along with the response obtained using numerical techniques (Venugopal Rao, 1994) (numerical integration of equations of motion using Runge-Kutta method), in Figs.4a, 4b and 4c.

Analysis of Machine-Pile-Foundation-Soil System

For the above diesel engine installation, pile foundation needs to be provided in case of weak soils. Four circular reinforced cement concrete (R.C.C.) piles each of the radius 0.2 m and length 6m are used for illustration, which are located symmetrically with respect to combined C.G. in top view at a distance of 0.65 m in x-direction and 2.6425 m in y-direction as shown in Fig.5. Young's modulus of elasticity and Poisson's ratio of pile material are taken as 0.14×10^{10} kg/m^2 (1.37×10^{10} N/m^2) and 0.25 respectively. The values of G are considered from 0.05×10^7 kg/m^2 (0.49×10^7 N/m^2) to 2×10^7 kg/m^2 (1.96×10^8 N/m^2).

Steady state amplitudes are plotted with *G* and are compared with numerically obtained results in Figs.6a, 6b and 6c.

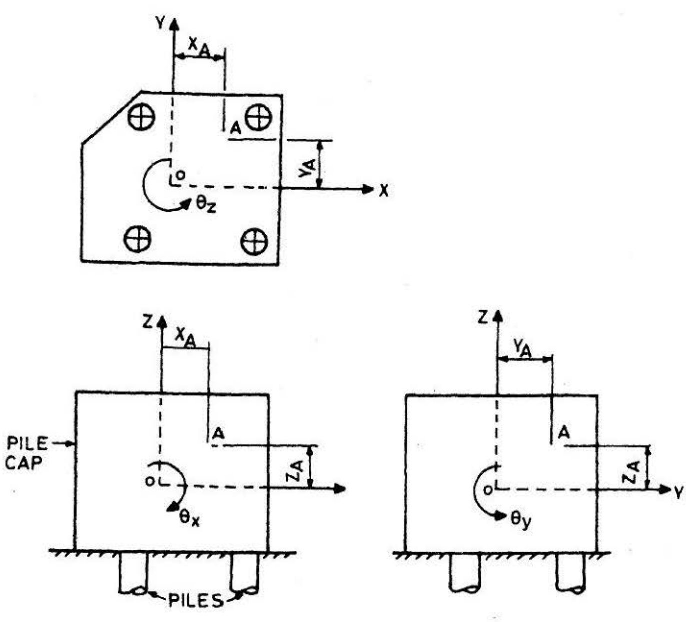

FIGURE 5 : Pile-Foundation Block Supporting the Machine

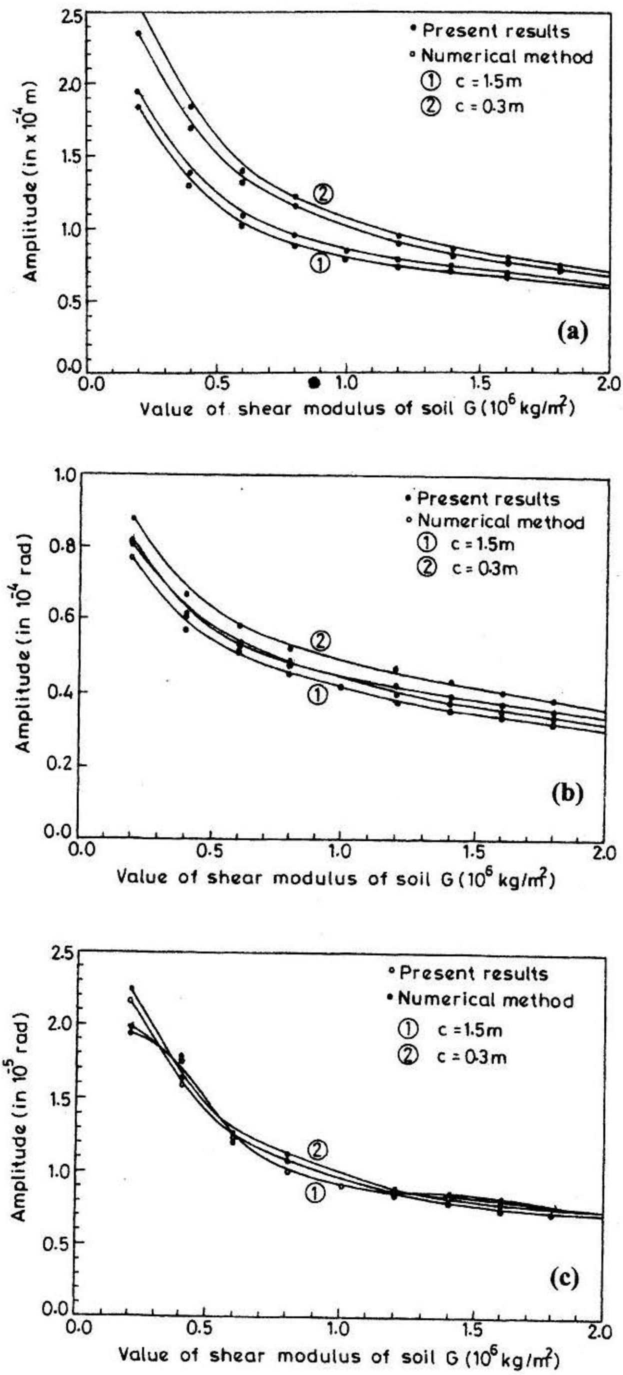

FIGURE 6 : Variation of Steady State Amplitudes of (a) P_o, (b) θ_y and (c) θ_z with G (10^6 kg/m^2) (Pile Foundation)

Conclusions

Results obtained by present method compare favourably with those obtained by numerical integration of equations of motion (Venugopal Rao, 1994) (the maximum difference between them being less than 10%). This slight difference can be attributed to the fact that the values of forces and moments have been given at discrete intervals. Thus at any intermediate time, the values used by two methods may differ slightly. The choice of step size in numerical integration and the number of terms considered in the summation of Fourier series may be the other reasons for the difference in results.

As compared to the existing methods of approximate solutions (Craig, 1981) and numerical integration (James, 1989) for practical dynamic problems, the present Fourier series method offers a simple yet efficient way to get the dynamic response of the system. Convergence is much faster than Runge-Kutta method for the same order of accuracy. Even the problems involving asymmetric matrices, or considerable eccentricity can be easily handled by this method. However, the present method cannot be used for non-periodic loads. Barring the above exceptions, for all cases of machine foundations, the given method is most efficient in terms of its simplicity and versatility of application to most of the practical problems.

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Appendix 1

Expressions for Stiffness and Damping Parameters for Block Foundations

The expressions are given in Table A1.1. The notations used in table are:

b	=	width of foundation (along axis of rotation for the case of rocking).
a	=	length of foundation (in the plane of rotation for rocking)
G	=	Shear modulus of soil
	=	$E/2(1+\nu)$
E	=	Young's Modulus of elasticity of soil
n	=	Poisson's ratio of soil.
m	=	combined mass of the machine and foundation block
r	=	unit weight of the soil
r_o	=	radius of equivalent rigid circular foundation block
I_x, I_y, I_z	=	mass moments of inertia about the respective axes of rotation, passing through the combined center of gravity of the machine-foundation assembly.
c	=	damping coefficients
c_c	=	critical damping
	=	$2(km)^{1/2}$ for translational motion
	=	$2(kI)^{1/2}$ for rotational motion
k	=	spring constant

TABLE A1.1 : Constants of Equivalent Analogue Systems in all Six Modes of Vibration (Kameswara Rao, 1977)

Motion	Spring Constant k	Mass Ratio B	Damping Ratio $\rho = c/c_c$
Horizontal along x-axis	$k_x = \dfrac{32(1-\nu)Gr_o}{7-8\nu}$	$B_x = \dfrac{(7-8\nu)mg}{32(1-\nu)\rho r_o^3}$	$\rho_x = \dfrac{0.288}{B_x^{1/2}}$
Horizontal along y-axis	$k_y = \dfrac{32(1-\nu)Gr_o}{7-8\nu}$	$B_y = \dfrac{(7-8\nu)mg}{32(1-\nu)\rho r_o^3}$	$\rho_y = \dfrac{0.288}{B_x^{1/2}}$
Vertical along z-axis	$k_z = \dfrac{4Gr_o}{1-\nu}$	$B_x = \dfrac{(1-\nu)mg}{4\rho r_o^3}$	$\rho_z = \dfrac{0.425}{B_z^{1/2}}$
Rocking about x-axis	$s_x = \dfrac{8Gr_o^3}{3(1-\nu)}$	$B_x = \dfrac{3(1-\nu)I_x g}{8\rho r_o^5}$	$\rho_x = \dfrac{0.15}{(1+B_x)B_x^{1/2}}$
Rocking about x-axis	$s_y = \dfrac{8Gr_o^3}{3(1-\nu)}$	$B_y = \dfrac{3(1-\nu)I_y g}{8\rho r_o^5}$	$\rho_y = \dfrac{0.15}{(1+B_y)B_y^{1/2}}$
Torsion about z-axis	$s_z = \dfrac{16}{3}Gr_o^3$	$B_z = \dfrac{I_z g}{\rho r_o^5}$	$\rho_z = \dfrac{0.50}{1+2B_z}$

Appendix 2

Expressions for Stiffness and Damping Parameters of Pile Foundation

The expressions are given in Table A2.1. The notations used in the table are:

E_p = Young's modulus of elasticity of pile material.

G_p = Shear modulus of pile material.

ν_p = Poisson's ratio of pile material.

ρ_p = unit weight of pile material.

l = length of pile.

r_o = radius of pile for circular piles (effective radius for piles of other shapes)

G_s = Shear modulus of supporting soil medium.

ν_s = Poisson's ratio of soil medium.

ρ_s = unit weight of soil medium.

V_p = shear wave velocity in pile = $\sqrt{\dfrac{G_p g}{\rho_p}}$

V_c = longitudinal wave velocity in pile = $\sqrt{\dfrac{E_p g}{\rho_p}}$

V_s = shear wave velocity in soil medium = $\sqrt{\dfrac{G_s g}{\rho_p}}$

A_p = area of cross-section of pile

I_{xx}, I_{yy}, I_{zz} = area moment of inertia of pile cross section about x, y, z axes respectively passing through the C.G. of the pile cross section.

$= \dfrac{\pi r_o^4}{4}$ (for I_{xx} and I_{yy}) and $\dfrac{\pi r_o^4}{2}$ (for I_{zz})

The values of interaction functions $f_{7.1}$, $f_{18.2}$ etc. in the table depend on the slenderness ratio of pile and on the ratio of shear wave velocity in soil medium to longitudinal wave velocity in the pile. These can be found from the available charts (Novak, 1974).

TABLE A2.1 : Dynamic Stiffness and Damping of Piles (Novak, 1977, 1978)

Parameter	Pile Head Fixed and Pile Tip Hinged	Pile Head Fixed and Pile Tip Fixed	Case of Static Load
c_x	$\frac{E_p I_{yy}}{r_o^2 V_s} f_{11,2}$	$\frac{E_p I_{yy}}{r_o^2 V_s} f_{6,2}$	$\frac{E_p I_{yy}}{r_o^2 V_s} \bar{f}_{11,2}$
c_y	$\frac{E_p I_{xx}}{r_o^2 V_s} f_{11,2}$	$\frac{E_p I_{xx}}{r_o^2 V_s} f_{6,2}$	$\frac{E_p I_{xx}}{r_o^2 V_s} \bar{f}_{11,2}$
c_z	$\frac{E_p A_p}{V_s} f_{18,2}$	$\frac{E_p A_p}{V_s} f_{18,2}$	$\frac{E_p A_p}{V_s} f_{18,2}$
c_{xy}	$\frac{E_p I_{yy}}{r_o V_s} f_{9,2}$	$\frac{E_p I_{yy}}{r_o V_s} f_{4,2}$	$\frac{E_p I_{yy}}{r_o V_s} \bar{f}_{9,2}$
c_{yx}	$\frac{-E_p I_{xx}}{r_o V_s} f_{9,2}$	$\frac{-E_p I_{xx}}{r_o V_s} f_{4,2}$	$\frac{-E_p I_{xx}}{r_o V_s} \bar{f}_{9,2}$
k_x	$\frac{E_p I_{yy}}{r_o^3} f_{11,1}$	$\frac{E_p I_{yy}}{r_o^3} f_{6,1}$	$\frac{E_p I_{yy}}{r_o^3} \bar{f}_{11,1}$
k_y	$\frac{E_p I_{xx}}{r_o^3} f_{11,1}$	$\frac{E_p I_{xx}}{r_o^3} f_{6,1}$	$\frac{E_p I_{xx}}{r_o^3} \bar{f}_{11,1}$
k_z	$\frac{E_p A_p}{r_o} f_{18,1}$	$\frac{E_p A_p}{r_o} f_{18,1}$	$\frac{E_p A_p}{r_o} f_{18,1}$
k_{xy}	$\frac{E_p I_{yy}}{r_o^2} f_{9,1}$	$\frac{E_p I_{yy}}{r_o^2} f_{4,1}$	$\frac{E_p I_{yy}}{r_o^2} \bar{f}_{9,1}$
k_{yx}	$\frac{-E_p I_{xx}}{r_o^2} f_{9,1}$	$\frac{-E_p I_{xx}}{r_o^2} f_{4,1}$	$\frac{-E_p I_{xx}}{r_o^2} \bar{f}_{9,1}$

TABLE A2.1 : Contd.

Parameter	Pile Head Fixed and Pile Tip Hinged	Pile Head Fixed and Pile Tip Fixed	Case of Static Load
h_x	$\frac{E_p I_{xx}}{V_s} f_{7,2}$	$\frac{E_p I_{xx}}{V_s} f_{2,2}$	$\frac{E_p I_{xx}}{V_s} \bar{f}_{7,2}$
h_y	$\frac{E_p I_{yy}}{V_s} f_{7,2}$	$\frac{E_p I_{yy}}{V_s} f_{2,2}$	$\frac{E_p I_{yy}}{V_s} \bar{f}_{7,2}$
h_z	$\frac{G_p I_{zz}}{V_s} f_{T,2}$	$\frac{G_p I_{zz}}{V_s} f_{T',2}$	$\frac{G_p I_{zz}}{V_s} f_{T,2}$
h_{xy}	$\frac{-E_p I_{xx}}{r_o V_s} f_{9,2}$	$\frac{-E_p I_{xx}}{r_o V_s} f_{4,2}$	$\frac{-E_p I_{xx}}{r_o V_s} \bar{f}_{9,2}$
h_{yx}	$\frac{E_p I_{yy}}{r_o V_s} f_{9,2}$	$\frac{E_p I_{yy}}{r_o V_s} f_{4,2}$	$\frac{E_p I_{yy}}{r_o V_s} \bar{f}_{9,2}$
s_x	$\frac{E_p I_{xx}}{r_o} f_{7,1}$	$\frac{E_p I_{xx}}{r_o} f_{2,1}$	$\frac{E_p I_{xx}}{r_o} \bar{f}_{7,1}$
s_y	$\frac{E_p I_{yy}}{r_o} f_{7,1}$	$\frac{E_p I_{yy}}{r_o} f_{2,1}$	$\frac{E_p I_{yy}}{r_o} \bar{f}_{7,1}$
s_z	$\frac{G_p I_{zz}}{r_o} f_{T,2}$	$\frac{G_p I_{zz}}{r_o} f_{T',2}$	$\frac{G_p I_{zz}}{r_o} f_{T,2}$
s_{xy}	$\frac{-E_p I_{xx}}{r_o^2} f_{9,1}$	$\frac{-E_p I_{xx}}{r_o^2} f_{4,1}$	$\frac{-E_p I_{xx}}{r_o^2} \bar{f}_{9,1}$
s_{yx}	$\frac{E_p I_{yy}}{r_o^2} f_{9,1}$	$\frac{E_p I_{yy}}{r_o^2} f_{4,1}$	$\frac{E_p I_{yy}}{r_o^2} \bar{f}_{9,1}$

Appendix 3

Details of Machine Assembly and Load data.

Machine Assembly

The engine is eight cylinder W8V-TC with rpm speed of 1000 and brake horse power of 372. Weights and coordinates of the various units of the assembly with reference to coordinate axes passing through the C.G. of the diesel engine are given in the following table.

TABLE 3 : Specifications of the Engine Assembly and Foundation

Unit	Weight (kg)	Co-ordinates with respect to the C.G. of the engine (in meters)		
		x	y	z
Engine	36 03	0.000	0.000	0.000
Alternator	2445	0.000	2.545	−0.300
Base Plate	1500	0.000	0.125	−0.721
Foundation Block:				
i) depth = 1.5 m	43092	0.000	0.425	−1.846
ii) depth = 0.3 m	8618.4	0.000	0.425	−1.246

The loads and moments at the C.G. of the engine are given in Tables A3.2 and A3.3 respectively. These consists of force along x-axis, F_{ex} with a period of 90 degrees (4000 cycles per minute), moment about y-axis, M_{ey} with a period of 90 degrees, and moment about z-axis, M_{ez} with a period of 720 degrees (500 cycles per minute). The values of these forces and moments are provided by the manufacturer at the intervals of 8.33×10^{-4} seconds (equivalent to 5 degrees of rotation of the crank angle). Knowing these values and the coordinates of the C.G. of the engine with reference to x, y, z axes passing through the combined C.G. of the machine and foundation, the forcing functions at 0, the combined C.G., $\bar{F}_x$, $\bar{F}_y$, $\bar{F}_z$, $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ can be easily computed which are used in the analysis.

Table A3.2 : The Variation of F_{ex} and M_{ey} with the Crank Angle

Crank Angle (Degree)	F_{ex} (in kg)	M_{ey} (in kgm)
0	−375	−28.3
5	−685	−81.2
10	−1031	−141.8
15	−1398	−207.2
20	−1658	−253.0
25	−1745	−267.9
30	−1663	−253.1
35	−1483	−221.9
40	−1216	−177.9
45	−928	−132.8
50	−648	−89.8
55	−429	−56.7
60	−242	−27.1
65	−73	2.8
70	41	26.6
75	101	42.8
80	49	39.4
85	−123	13.2
90	−375	−28.3

TABLE A3.3 : The Variation of M_{ez} with the Crank Angle

Crank Angle (Degree)	M_{ez} (kgm)	Crank Angle (Degree)	M_{ez} (kgm)	Crank Angle (Degree)	M_{ez} (kgm)
0	−51.5	120	684.7	240	−430.6
5	201.4	125	673.6	245	−337.4
10	480.7	130	662.6	250	−267.7
15	777.7	135	674.5	255	−213,7
20	1006.3	140	689.9	260	−195.8
25	1122.3	145	691.9	265	−215.4
30	1122.7	150	701.9	270	−258.8
35	1052.6	155	726.6	275	−317.5
40	924.8	160	730.6	280	−384.6
45	790.8	165	697.2	285	−455.1
50	660.1	170	583.2	290	−489.2
55	558.4	175	379.4	295	−465.4
60	478.1	180	118.7	300	−388.7
65	414.9	185	−182.1	305	−288.6
70	372.5	190	−501.4	310	−155.0
75	341.6	195	−854.0	315	1.5
80	337.4	200	−1117.3	320	155.1
85	361.0	205	−1252.4	325	269.6
90	403.9	210	−1252.6	330	373.6
95	460.5	215	−1165.3	335	478.9
100	527.6	220	−1006.9	340	542.2
105	602.1	225	−836.1	345	551.5
110	659.8	230	−667.6	350	448.4
115	684.7	235	−537.3	355	222.5

Crank Angle (Degree)	M_{ez} (kgm)	Crank Angle (Degree)	M_{ez} (kgm)	Crank Angle (Degree)	M_{ez} (kgm)
360	−83.8	485	−1207.8	610	282.6
365	−448.0	490	−1100.0	615	250.4
370	−852.0	495	−1008.9	620	213.7
375	−1281.1	500	−923.2	625	171.0
380	−1603.5	505	−846.4	630	123.4
385	−1750.8	510	−789.3	635	70.9
390	−1721.3	515	−753.0	640	13.3
395	−1586.8	520	−715.6	645	−48.3
400	−1362.8	525	−660.5	650	−107.9
405	−1125.2	530	−656.3	655	−163.1
410	−893.4	535	−423.8	660	−210.0
415	−713.0	540	−254.0	665	−245.7
420	−565.5	545	−64.5	670	−283.0
425	−441.3	550	138.1	675	−335.9
430	−357.5	555	350.6	680	−388.4
435	−304.9	560	520.2	685	−424.1
440	−319.5	565	623.9	690	−461.1
445	−405.4	570	653.9	695	−505.3
450	−539.3	575	631.1	700	−527.3
455	−707.1	580	568.9	705	−514.8
460	−898.9	585	501.7	710	−430.5
465	−1105.5	590	434.3	715	−266.9
470	−1257.0	595	382.8	720	−51.5
475	−1313.2	600	343.1		
480	−1283.2	605	311.0		