Indian Geotechnical Journal, 31 (3), 2001

Prediction of Stress-Strain Behaviour of Soil using Hypoelasticity Constitutive Model

Krishnamoorthy* and N.B.S. Rao†

In recent years great interest has been developed in modelling the behaviour of soils and hence a wide range of models are available. Some of the models are so simple that essential soil behaviour like nonlinearity prior to yielding and dilatancy are not considered whereas some of the models are too complex to use for practical problems. Therefore nowadays research is directed in such a way that the resulting model can represent the behaviour of soil realistically without involving much mathematical complexity. Hypoelasticity is one such approach in modelling the behaviour of soils.

Hypoelasticity describes the behaviour of materials in which stress and strain are related by coefficients, which in their simplest form are functions of stress, strain or both. The behaviour is infinitesimally reversible. More advanced formulations in this class introduce density as a parameter in the behavioural equation and postulate the existence of the critical state at which the material flows under a constant stress.

Using the theory of hypoelasticity, Yin et al. (1989) have developed a constitutive model for soil on the basis of incremental theory and generalised Hook's law. The stress-strain relation is formed in incremental and three-dimensional form. The model considers the important soil properties like nonlinearity, dilatancy and coupled behaviour. It requires six parameters, which can be easily determined from isotropic consolidation and conventional drained or undrained triaxial compression tests (CTC).

---

* Professor, Department of Civil Engineering, Manipal Institute of Technology, Manipal – 576 119, India.

† Professor, Department of Applied Mechanics, Karnataka Regional Engineering College, Surathkal – 574 157, India.

However it is found from the available literature on the model that the applicability of the model was verified only for the results obtained from conventional triaxial compression test and not verified for other stress paths, the model was formed only to obtain the behaviour of soil under drained condition of loading and the applicability of the model for overconsolidated soils was not verified. Also during verification it is found that the model does not predict satisfactorily the behaviour of soil along the paths other than the path followed by conventional triaxial test. Hence the model is modified so as to make it applicable for all the stress paths. A procedure of determining the model parameters both for normally and overconsolidated soils is developed. The model is also modified so as to make it applicable for undrained condition of loading. A brief description of the modified model and the method of evaluating the model parameters are presented in this paper. Further, the results of stress strain behaviour of soil samples with different stress history and tested under different stress paths are also presented. The capability of the model to predict the stress strain behaviour of soil samples having different stress history and stress path has been demonstrated by comparing the above results with available results in literature.

Description of The Model

The model consists of three stress dependent modulus functions. They are:

1. Bulk modulus K
2. Shear modulus G
3. The coupling modulus J that relates effectve mean stress p' and shear strain ε_s as well as shear stress q versus volumetric strain ε_v.

The change in volumetric strain $d\varepsilon_v$ and shear strain $d\varepsilon_s$ corresponding to the change in effective mean stress dp' as well as shear stress dq as proposed by Yin et al. (1989) are, expressed by the relationships.

$$d\varepsilon_v = dp'/K + dq/J \tag{1}$$

$$d\varepsilon_s = dp'/J + dq/3G \tag{2}$$

In the formulation of the model it is assumed that dp', $d\varepsilon_s$ coupling and dq, $d\varepsilon_v$ coupling are controlled by the same modulus. Equations 1 and 2 can be generalised in a tensor form as:

$$d\varepsilon_{ij} = \left(\frac{1}{9\kappa} - \frac{1}{3qJ}\sigma'_{mm} - \frac{1}{6G}\right)\delta_{ij}\delta_{kl} + \frac{1}{2qj}\sigma'_{kl}\delta_{ij} + \frac{1}{2qJ}\sigma'_{ij}\delta_{kl} + \frac{1}{4G}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)d\sigma'_{kl}$$

or in matrix form

$$\begin{vmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\varepsilon_{12} \\ d\varepsilon_{23} \\ d\varepsilon_{31} \end{vmatrix} = \begin{vmatrix} a_1+2b_1 & a_2+b_1+b_2 & a_2+b_1+b_3 & c_1 & c_2 & c_3 \\ a_2+b_1+b_2 & a_1+2b_2 & a_2+b_2+b_3 & c_1 & c_2 & c_3 \\ a_3+b_1+b_3 & a_2+b_2+b_3 & a_1+2b_3 & c_1 & c_2 & c_3 \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} & \frac{1}{2G} & 0 & 0 \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} & 0 & \frac{1}{2G} & 0 \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} & 0 & 0 & \frac{1}{2G} \end{vmatrix} \begin{vmatrix} d\sigma'_{11} \\ d\sigma'_{22} \\ d\sigma'_{33} \\ d\sigma'_{12} \\ d\sigma'_{23} \\ d\sigma'_{31} \end{vmatrix}$$

.... (3)

where

$$a1 = 1/9K + 1/3G$$

$$a2 = 1/9K - 1/6G$$

$$b1 = \left(2\sigma'_1 - \sigma'_2 - \sigma'_3\right)/\left(6qJ\right)$$

$$b2 = \left(2\sigma'_2 - \sigma'_1 - \sigma'_3\right)/\left(6qJ\right)$$

$$b3 = \left(2\sigma'_3 - \sigma'_1 - \sigma'_2\right)/\left(6qJ\right)$$

$$c_1 = d\sigma'_{12}/\left(qJ\right)$$

$$c_2 = d\sigma'_{23}/\left(qJ\right)$$

$$c_3 = d\sigma'_{31}/\left(qJ\right)$$

p' and q = effective mean stress and shear stress respectively,

σ'_{11}, σ'_{22}, σ'_{33} = normal stresses, and

σ'_{12}, σ'_{23}, σ'_{31} = shear stresses.

The bulk modulus K can be determined from isotropic consolidation test. The coupling modulus J and shear modulus G can be determined from conventional undrained triaxial compression test.

Bulk Modulus K

The bulk modulus K gives the relationship between changes in volumetric strain corresponding to the change in effective mean stress. Figure 1 shows the typical relationship between effective mean stress p' and volumetric strain ε_v. The slope of ε_v versus log $\log p'$ is λ/Vi and is given by the equation

$$\lambda/Vi = (\varepsilon_{vo} - \varepsilon_v)/\log(1.0) - \log(p') \tag{4}$$

Differentiating Eqn.4 one gets

$$K = p'/(\lambda/Vi)$$

Determination of Model Parameter K

Isotropic consolidation test provides data that relates effective mean stress p' and volumetric strain ε_v. This relationship consists of two straight lines as shown in Fig.2. The slope of the line λ/Vi is considered for loading paths whereas the slope of the line κ/Vi is considered for unloading and recompression paths. The value of p' corresponding to the intersection of these two lines gives the values of preconsolidation pressure p'_{cons}. The bulk modulus K for any value of p' can then be determined from equations

$K = p'/(\lambda/Vi)$ for loading paths (first time loading)

$K = p'/(\kappa/Vi)$ for unloading and recompression paths

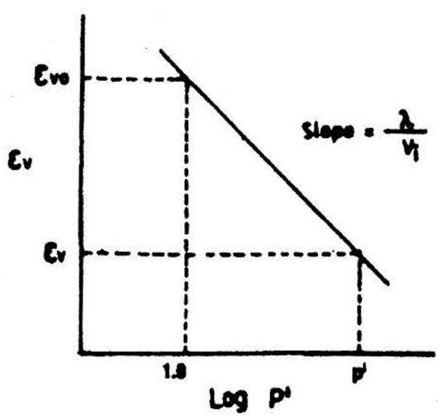

FIGURE 1 : Effective Mean Stress vs. Volumetric Strain Relationship

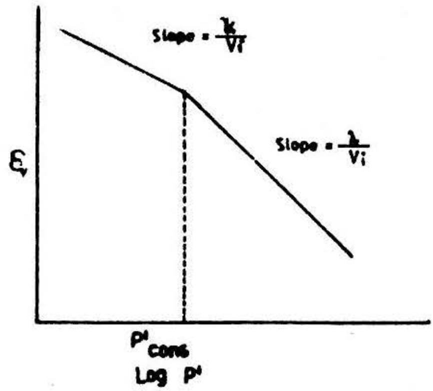

FIGURE 2 : Effective Mean Stress vs. Volumetric Strain Relationship

Coupling Modulus J

Coupling modulus J relates p' and ε_s as well as q and ε_v behaviour. This can be determined from conventional undrained triaxial compression test (CTC). Figure 3 shows the typical relationship between q/p'_{cons} versus p'/p'_{cons} for the data obtained from conventional undrained triaxial test. This relationship is modelled in the form

$$q/p'_{cons} = A\left(1 - p'/p'_{cons}\right)^n \tag{5}$$

Differentiating Eqn.5 with respect to p' and substituting $d\varepsilon_v = 0$ for undrained triaxial test we get

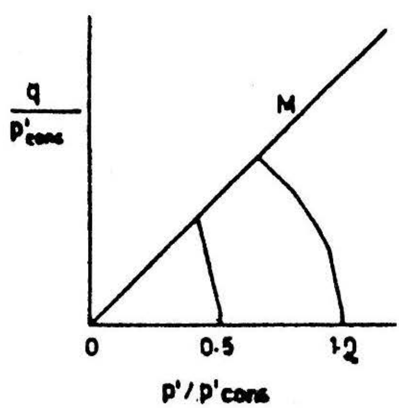

FIGURE 3 : q/p'_{cons} **vs.** p'/p'_{cons} **Relationship**

$$J = KnA^{1/n}\left(q/p'_{cons}\right)^{(n-1/n)} \tag{6}$$

Determination of Model Parameter J

The results of an undrained triaxial compression test conducted along path A are used to obtain the relationship between q/p'_{cons} and $(1-p'/p'_{cons})$. This relationship plotted on log – log plot is a straight line as shown in Fig.4. The slope of this line gives the value of n. Value of q/p'_{cons} corresponding to $(1-p'/p'_{cons})$ equal to 1.0 gives the value of A. Value of J can then be obtained from Eqn.6.

Shear Modulus G

Figure 5 shows a typical relationship between ε_s versus q/p' which can be obtained from the data of drained or undrained triaxial compression test conducted along path A. This relationship is approximated by a hyperbolic equation

$$\varepsilon_s/(q/p') = E + F\varepsilon_s \tag{7}$$

Using the relationship $d\varepsilon_s = dp'/J + dq/3G$ from Eqn.2 and $dp'/dq = -K/J$ from Eqn.1 ($d\varepsilon_v = 0$ for undrained test)

$$G = DJ^2/\left(J^2 + 3DK\right) \tag{8}$$

where $D = G - (Gdp')/(Jd\varepsilon_s)$

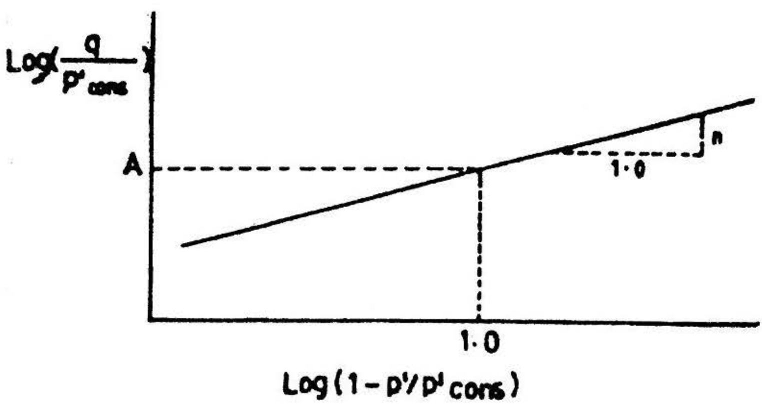

FIGURE 4 : q/p'_{cons} **vs.** $(1-p'/p'_{cons})$ **Relationship**

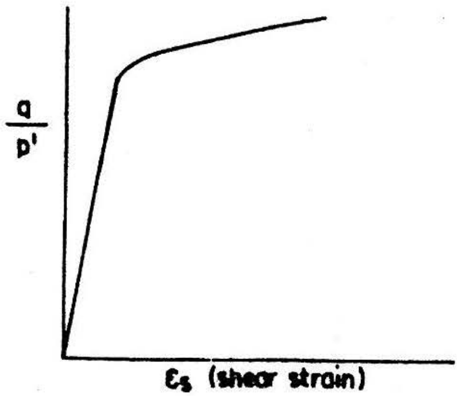

FIGURE 5 : ε_s vs. q'/p' Relationship

From Eqn.7

$$\varepsilon_s/(q/p') = E + F\varepsilon_s$$

Differentiating with respect to ε_s and simplifying,

$$D = p'(1 - Fq/p')^2/3E \tag{9}$$

Determination of Shear Modulus G

Consolidated undrained triaxial compression test along path A provides the relationship between ε_s and $\varepsilon_s/(q/p')$. This relationship is a straight line as shown in Fig.6. The slope of the line gives the value of F. The intersection

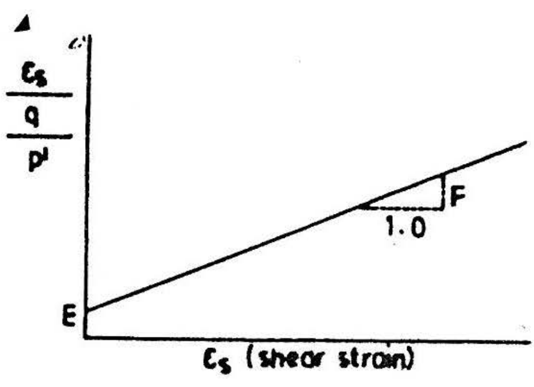

FIGURE 6 : ε_s vs. $\varepsilon_s/(q'/p')$ Relationship

of this line with $\varepsilon_s/(q/p')$ axis gives the value of E. The parameter D can then be obtained from Eqn.9. The shear modulus G can then be determined from Eqn.8.

Thus all the parameters can be determined from simple tests on soil samples and there is no need for any specialised testing procedure.

Experimental Verification of the Model

Stress controlled drained and undrained triaxial compression tests on isotropically consolidated soil samples along various stress paths are conducted in the laboratory to verify the applicability of the model. The stress strain behaviour of soil samples for each stress path is predicted using the model explained above and compared with the observed behaviour. The applicability of the model is also verified using the results published by Rao (1982) for anisotropically consolidated and lightly overconsolidated soil samples. The results published by Kim et al. (1994) for lightly over-consolidated soil sample tested under drained and undrained conditions of loading are also used to further verify the applicability of the model.

Applicability of the Model for Normally Consolidated Soils

Stress controlled drained and undrained triaxial compression tests along various stress paths are conducted on soil samples prepared in the laboratory. Locally available soil after passing through 4.75 mm sieve is used for testing. The physical properties of the soil are : Specific gravity = 2.55, Liquid limit = 40.85%, Plastic limit = 25.32%, Uniformity Coefficient = 3.60 and Coefficient of Curvature = 1.024. As per the I.S. classification the soil can be classified as Sandy clay.

All the soil samples used for testing are saturated by applying a back-pressure. The soil samples are consolidated to a cell pressure of 0.15 MPa. Isotropic consolidation test, drained and undrained triaxial compression tests along the paths A, B and C (These paths are shown in respective figures) on these soil samples are conducted. The parameter λ/Vi is determined from isotropic consolidation test and the other required model parameters are determined from the undrained triaxial compression tests conducted along path A. The model parameters obtained are A = 1.8, n = 0.25, E = 0.002, F = 0.66 and λ/Vi = 0.022. These parameters are used to predict the behaviour of soil samples tested along paths A, B and C.

Prediction of Stress-Strain Behaviour for Drained Tests

Figure 7 show the results predicted from the model and those obtained from tests for drained tests conducted along paths A, B and C for

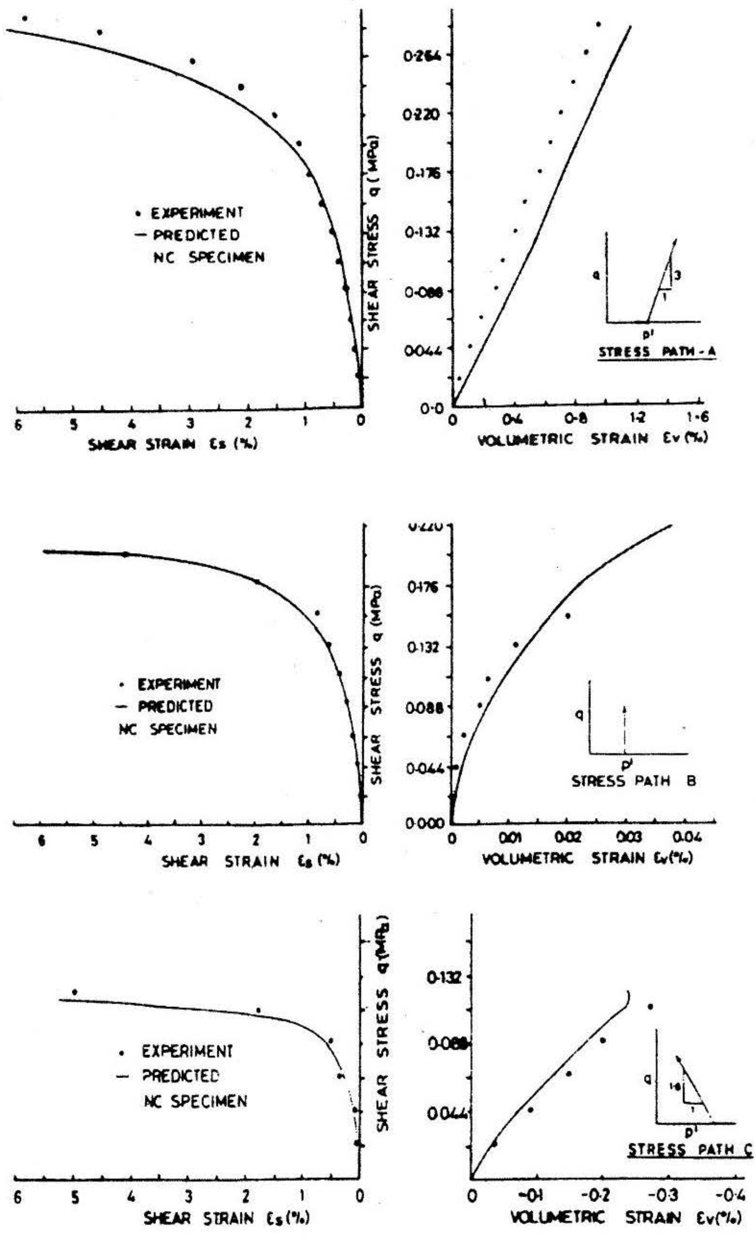

FIGURE 7 : Observed and Predicted Stress-Strain Behaviour for Drained Test Conducted along Paths A, B and C

isotropically normally consolidated soils. It can be seen from these figures that the results predicted by the model and those obtained from experiment agree well. The shear strains predicted by the model and those obtained from experiment agree very well for all the stress paths considered. However along path A the volumetric strain predicted by the model is slightly higher than that obtained from experiment. Along path B, the volumetric strain measured from experiment and predicted from model are very small. The difference between the predicted and measured volumetric strain as seen from the figure is negligibly small. Along path C the volumetric strain predicted by the model and that obtained from experiment are negative and match very well.

Prediction of Stress-Strain Behaviour for Undrained test

Figure 8 show the shear stress versus shear strain and shear stress versus pore pressure obtained from experiment for undrained test. The values predicted from the model are also shown in the same figures. It can be seen from the figures that the results predicted from the model and those obtained from experiment agree well. Along path A, since the pore pressure is positive, the effective stresses are smaller than total stresses. Hence the soil sample reaches the critical state earlier than that tested under drained condition. Along path B, since the pore pressure developed is very small the total and effective stresses are almost the same. Hence the shear strain predicted is almost same as that of drained test. Along path C, since the pore pressure predicted is negative, the effective stresses are higher than total stresses. Hence the soil sample fails at higher values of shear stress than in drained condition. Thus all the soil samples tested along paths A, B and C under undrained condition of loading reach the critical state line at the same value of shear stress. This agrees with the statement of Atkinson and Bransby (1978) that the value of shear stress at which the soil reaches the critical state line is same for all stress paths when the soil sample is tested under undrained condition of loading.

Verification of the Model using Published Data

The applicability of the model is also verified for the data presented by Rao (1982) for normally and anisotropically consolidated soil samples for a stress ratio (q_o/p'_o) of 0.85. The model parameters A, n, E and F are determined from stress-strain relationship presented by Rao (1982) for drained test along path A. The calculated model parameters are A = 3.5, n = 0.16, E = 0.0015, F = 0.72, λ = 0.0016 and κ = 0.0003. λ is considered for loading path (path A) where as κ is considered for unloading path (path B and path C). The initial specific volume V_i is 1.8. These parameters are used to predict the stress strain relationship for the soil samples tested under drained condition of loading along paths A, B and C.

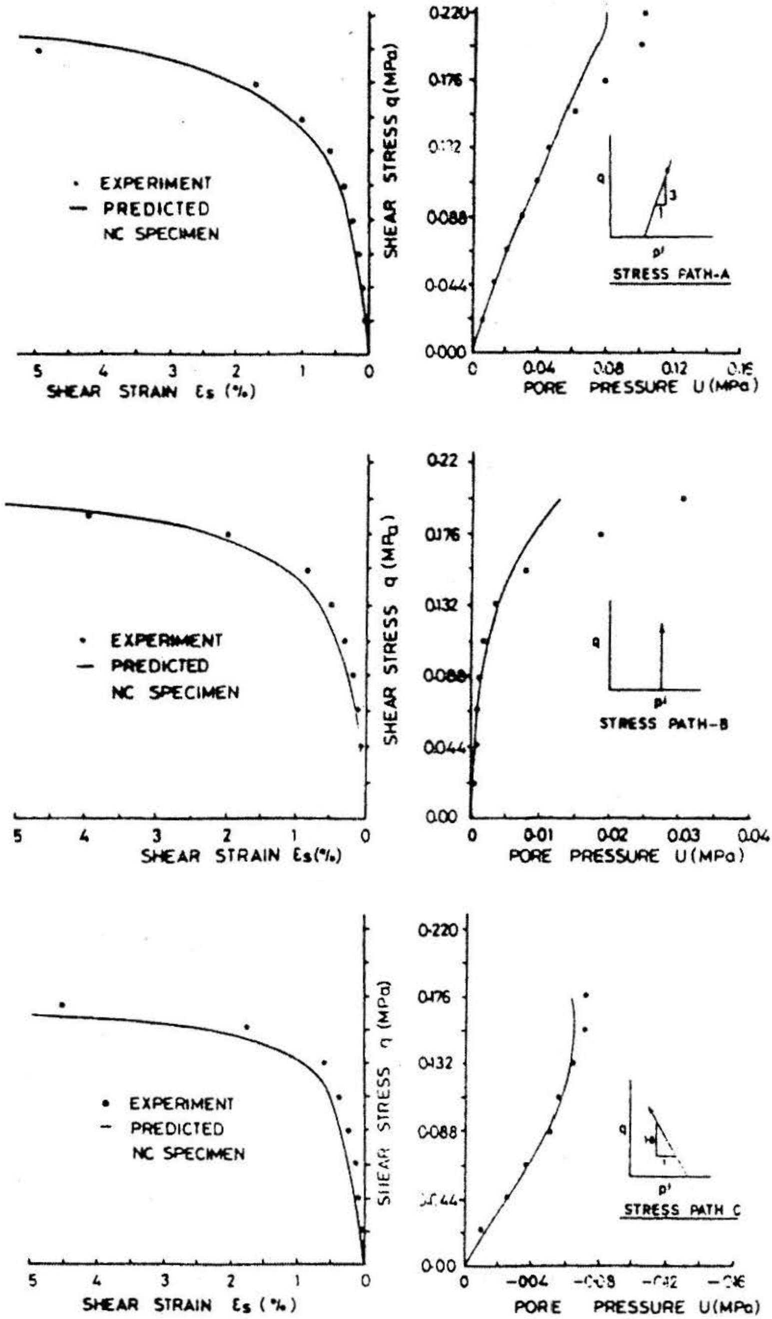

FIGURE 8 : Observed and Predicted Stress-Strain Behaviour for Undrained Test Conducted along Paths A, B and C

Figure 9 show the stress ratio (q/p') versus shear strain and stress ratio versus volumetric strain predicted by the model and that obtained (Rao, 1982) from experiment for path A, path B and path C respectively. It can be seen from these figures that the results predicted by the model and those obtained from the experiment agree well except for the relationship between stress ratio and volumetric strain along path B. The volumetric strain predicted by the model along path B is lower than that obtained from the experiment from the beginning.

Verification of the Model for Overconsolidated Soils

The applicability of the model for overconsolidated soils is also studied. The experimental data required for the verification of the model along various stress paths is taken from the results presented by Rao (1982) for the soil samples with OCR = 1.6 and tested along paths A, B and C. As reported by Rao (1982) these soil samples were tested at a mean stress of 0.25 MPa and shear stress of 0.2125 MPa (Anisotropic consolidation with $q_o/p'_o = 0.85$). The parameters A, n, E and F are determined using the data of drained test conducted along path A. These are $\kappa = 0.0003$. $V_i = 1.80$, $A = -3.0$, $n = 0.20$, $E = 0.00015$ and $F = 0.70$. $\lambda = 0.0016\kappa$ is considered for both recompression (path A) and unloading path (paths B and C). These parameters are used to predict the behaviour of soil samples tested under drained condition of loading along paths A, B and C.

Figure 10 shows the relationships between stress ratio and shear strain and stress ratio versus volumetric strain obtained from experiment as well as predicted by the model. It can be seen from these figures that the shear strain obtained from the experiment and that predicted from the model agree well. Along path A, the volumetric strain obtained from experiment and predicted by the model agrees well upto a stress ratio of 1.1. Beyond this stress ratio, the volumetric strain obtained from experiment decreases upto a stress ratio of 1.35 and then increases to a large value. However for the model, the volumetric strain increases to a large value above a stress ratio of 1.1. Along path B, the volumetric strain predicted from the model is lower than that obtained from experiment beyond a stress ratio of 1.35. Along path C, the volumetric strain predicted by the model agrees satisfactorily upto a stress ratio of 1.1. Beyond this stress ratio, the volumetric strain predicted from the model are lower than that obtained from experiment.

Study of Applicability of the Model for the Data Presented by Kim et al. (1994)

The applicability of the model for isotropically overconsolidated soils is also verified using the results published by Kim et al. (1994) for the soil samples with OCR = 2.15 tested under drained and undrained condition of

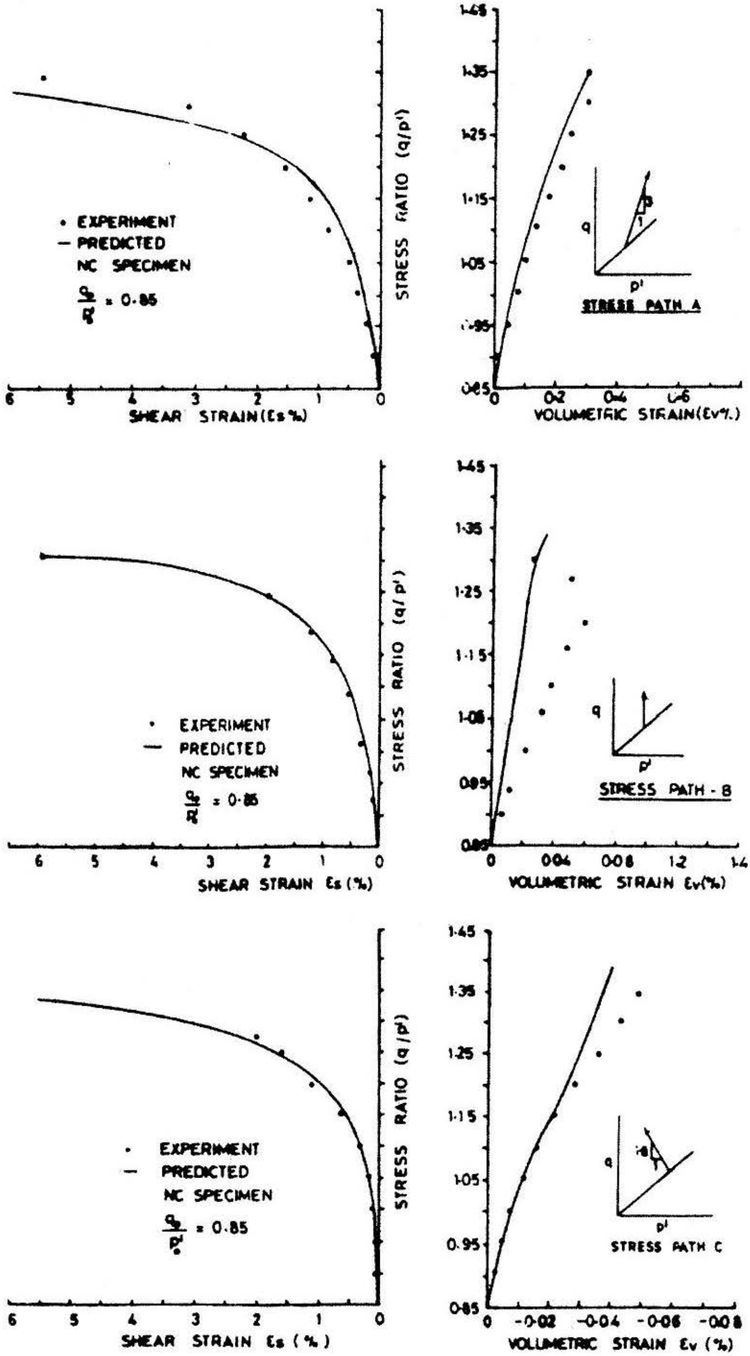

FIGURE 9 : Observed and Predicted Stress-Strain Behaviour for Anisotropically Normally Consolidated Soil Samples

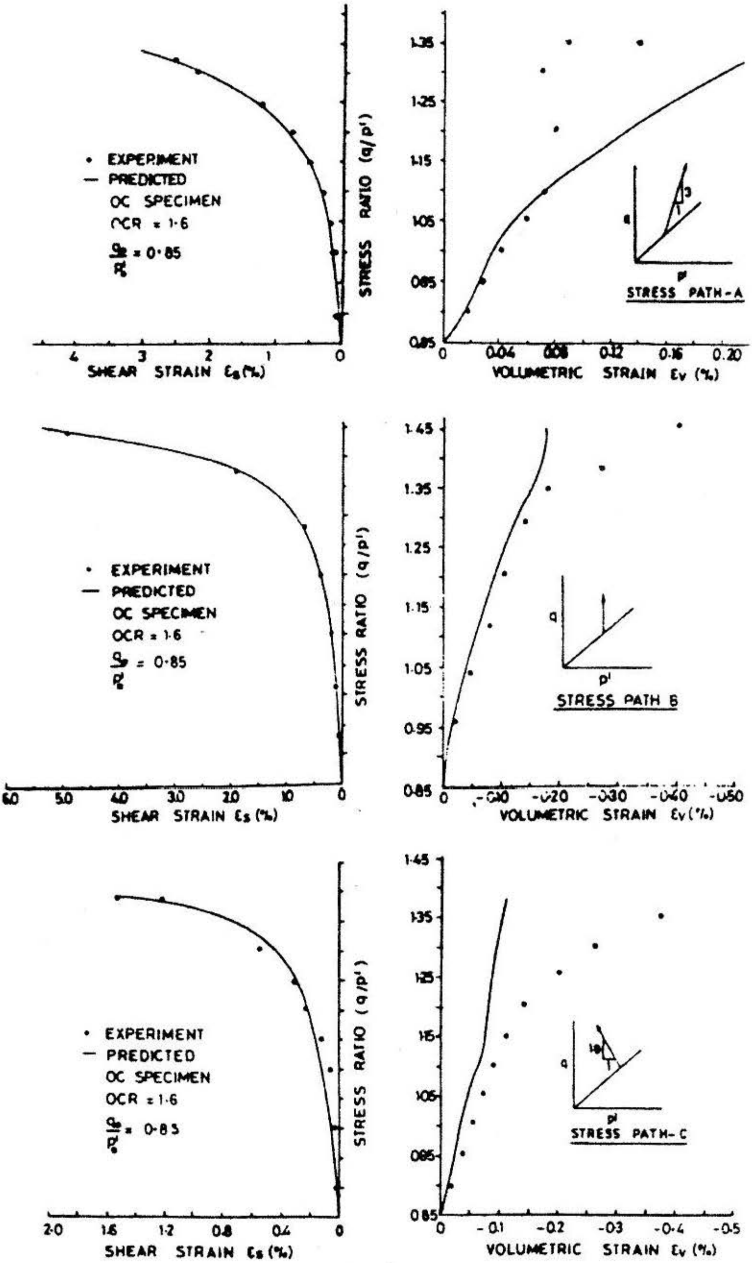

FIGURE 10 : Observed and Predicted Stress-Strain Behaviour for Anisotropically Overconsolidated Soil Samples

loading along path A. The required model parameters for drained tests are determined using the results obtained from drained tests. The parameters obtained are A = 10.0, n = 0.58, E = 0.008 and F = 1.03. For undrained tests the parameters are determined using the results of undrained tests. The parameters obtained are A = 45.0, n = 2.0, E = 0.005 and F = 0.90. The value of κ is 0.146.

Figure 11(a) shows the relationship between shear stress and shear strain and Fig.11(b) shows shear stress versus volumetric strain relationship predicted by the present model for drained test. The experimental results reported by

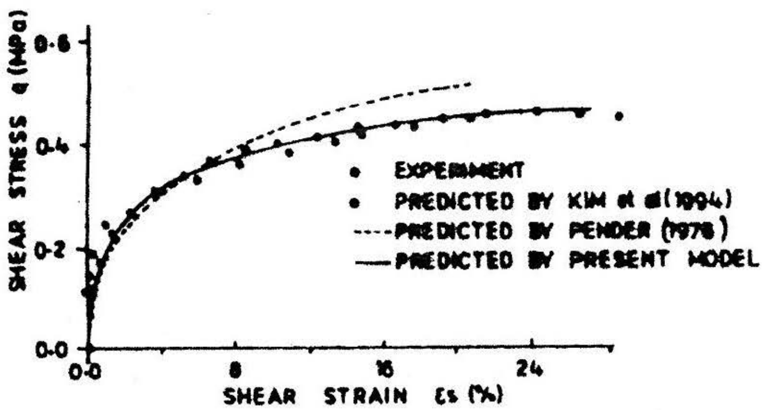

FIGURE 11(a) : Observed and Predicted Shear Stress vs. Shear Strain Behaviour for Isotropically Overconsolidated Soil Sample

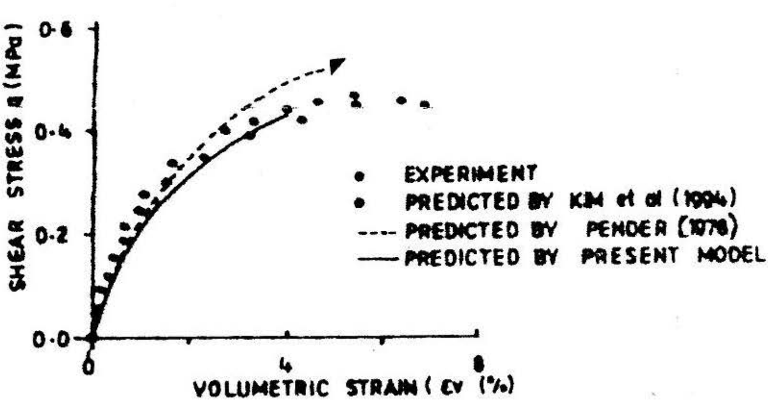

FIGURE 11(b) : Observed and Predicted Shear Stress vs. Volumetric Strain Behaviour for Isotropically Overconsolidated Soil Sample

Kim et al. (1994) as well as predicted by the models proposed by Kim et al. (1994) and Pender (1978) are also shown in the same figure. It can be seen from these figures that the results predicted by the present model agree very well with the results obtained from experiment.

The relationship between shear stress and effective mean stress predicted by the model for undrained test is shown in Fig.12(a). Figure 12(b) shows the ratio of shear stress and preconsolidation pressure versus shear strain predicted by the model. The relationship obtained from the experiment (Kim et al., 1994) is also shown in the same figure. It can be seen from these figures that the results predicted by the present model agree very well with

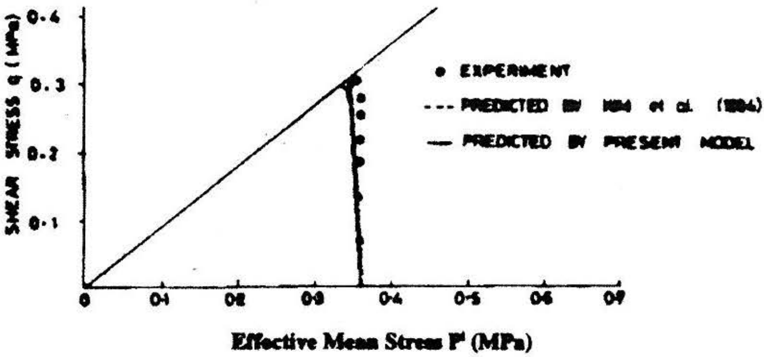

FIGURE 12(a) : Observed and Predicted Effective Stress Paths for Isotropically Overconsolidated Soil Sample

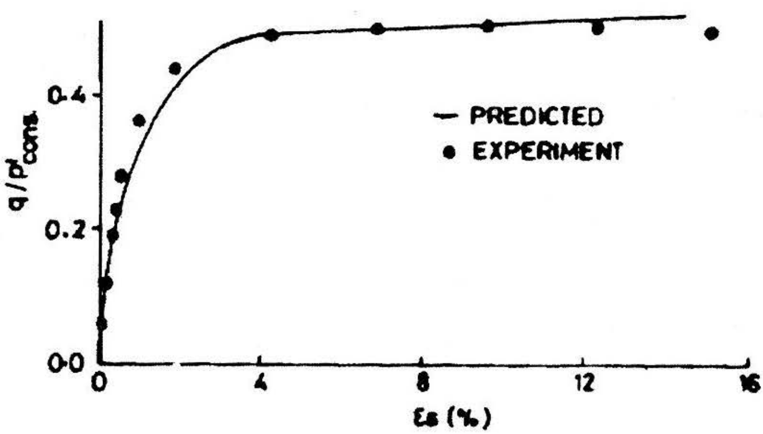

FIGURE 12(b) : Observed and Predicted (Undrained) Shear Stress vs. Shear Strain Behaviour for Isotropically Overconsolidated Soil Sample

the results obtained from experiment. The effective stress predicted by Kim et al. (1994) slightly deviates from the experimental data as the stress path reaches a critical state line. However the effective stresses predicted by the present model agree well from the beginning upto critical state line.

Summary and Conclusions

The model proposed by Yin et al. (1989) is modified so as to make it more general and versatile. The parameters required for the model are determined for normally consolidated soil samples prepared in the laboratory from isotropic consolidation and conventional undrained triaxial compression test. The applicability of the model along various stress paths is studied for the soil samples with different stress histories tested under different conditions of loading. Based on the above study, the following conclusions are drawn.

1. The proposed model is simple; the model parameters can be easily determined from isotropic consolidation and triaxial compression tests, which are simple tests.

2. The model can be used to predict the stress-strain behaviour of soil tested under drained condition of loading.

3. The model can also be used to predict the pore pressure and stress-strain behaviour of soil under undrained condition of loading.

4. The behaviour of soil samples with different stress histories, can also be predicted by the proposed model.

References

ATKINSON, J.H. and BRANSBY, P.L. (1978) : *The Mechanics of Soils - An Introduction to Critical State Soil Mechanics*, McGraw-Hill Book Company, U.K.

KIM, S.R., SEAH, T.H. and BALASUBRAMANIUM, A.S. (1994) : "Formulation of Stress-Strain Behaviour Inside the State Boundary Surface", *Proc. 13th Int. Conf. Soil Mech. Found. Engineering*, Vol. , pp.51-56.

RAO, N.B.S. (1982) : "Studies on Empirical Modelling of Soil Behaviour", *Ph.D. Thesis*, IIT Kanpur, India.

YIN, J.H., GRAHAM, J., SAADAT, F. and AZIZI, F. (1989) : "Constitutive Modelling of Soil Behaviour Using Three Modulus Hypoelasticity", *Proc. 12th Int. Conf. Soil Mech. Found. Engineering*, Vol.1, p. 143-147.