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A New Procedure for Finding Critical Slip Surfaces in Slope Stability Analysis

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Introduction

It is now well appreciated that slope stability analysis by limit equilibrium methods is essentially a problem of optimisation wherein the shape and location of the critical slip surface is determined, which corresponds to the minimum factor of safety subject to the conditions that the shape of the critical slip surface is physically reasonable and that the obtained solution satisfies some acceptability criteria. During the past three decades, a great deal of research has been directed towards refinements in the development of the safety functional. Quite a few methods are currently available (Morgenstern and Price, 1965; Spencer, 1973; Fredlund and Krahn, 1977; Sarma, 1979; Chen and Morgenstern, 1983), which are valid for general slip surfaces and satisfy all conditions of equilibrium. Excellent reviews are available on the accuracy of various limit equilibrium methods of analysis (Duncan, 1996).

Refinements in the method of analysis were followed by the use of sophisticated optimization techniques to search for the critical slip surface, e.g., calculus of variation (Ramamurthy et. al, 1977; Castillo and Revilla, 1977; Baker and Garber, 1977), linear programming (Martins, 1982), dynamic programming (Baker, 1980). While dynamic programming technique is very powerful as it yields the absolute minimum disregarding any local minima that may exist, it suffers from a major drawback known as the curse of dimensionality (Rao, 1984). Baker (1980) has also pointed out other drawbacks of dynamic programming technique when applied to slope stability

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problems. Morgenstern (1977) commented on some difficulties in the application of variational calculus to slope stability problems and cautioned against ignoring these difficulties. According to Martins (1982), variational techniques cannot be used for heterogeneous media. Because the stability analysis of slopes involves nonlinearity, the linear programming technique has not been widely adopted by researchers in this area. The penalty function formulation or the sequential unconstrained minimization technique (SUMT) has found a number of applications in the slope stability computations (Basudhar, 1976; Greco, 1988). The most important merit of the penalty function methods is their flexibility; one can easily add or delete constraints, modify the objective function or constraints and interchange the roles of various parts of the problem (Fox, 1971). A critical appraisal of the application of optimization techniques to slope stability problems has been presented by Bhattacharya (1990).

To obtain a physically acceptable solution, it is essential not only to satisfy the equilibrium and boundary conditions and failure criterion along the shear surface but also to satisfy some conditions of acceptability or criteria for admissibility such that the implied state of stress within the soil mass is feasible. The stresses obtained from the solution should not violate the Mohr-Coulomb failure criterion anywhere within the sliding body, no tension should be implied and the directions of forces should all be kinematically admissible (Morgenstern and Price, 1965). It has been pointed out (Sarma, 1979) that it is not always possible to obtain a completely acceptable solution without going through a lot of iterations. Therefore, experience is needed to seek out a seemingly unacceptable solution that can be treated as acceptable. However, such experience cannot be expected of common users and, moreover, an acceptable solution differs from an unacceptable one by an unknown magnitude and is likely to be case specific.

It is thus evident that there still exists a need to develop a generalised procedure for determination of critical slip surface with adequate built-in provision to ensure that the obtained solution satisfies the prescribed conditions of acceptability. Towards this end, in this paper, an effort has been made to develop a procedure for slope stability analysis in which a rigorous method of evaluation of factor of safety satisfying all conditions of statical equilibrium is coupled with a powerful optimization technique to search for the critical slip surface such that the formulation itself is capable of incorporating appropriate acceptability criteria for the obtained solution.

Of all the rigorous methods, which are valid for an arbitrary or general slip surface, the Spencer method (Spencer, 1973) has been rated as the best as it combines a simple method of solution with an acceptable degree of accuracy (Wright, 1969). The Spencer method is used in conjunction with the sequential unconstrained minimization technique to construct a computational

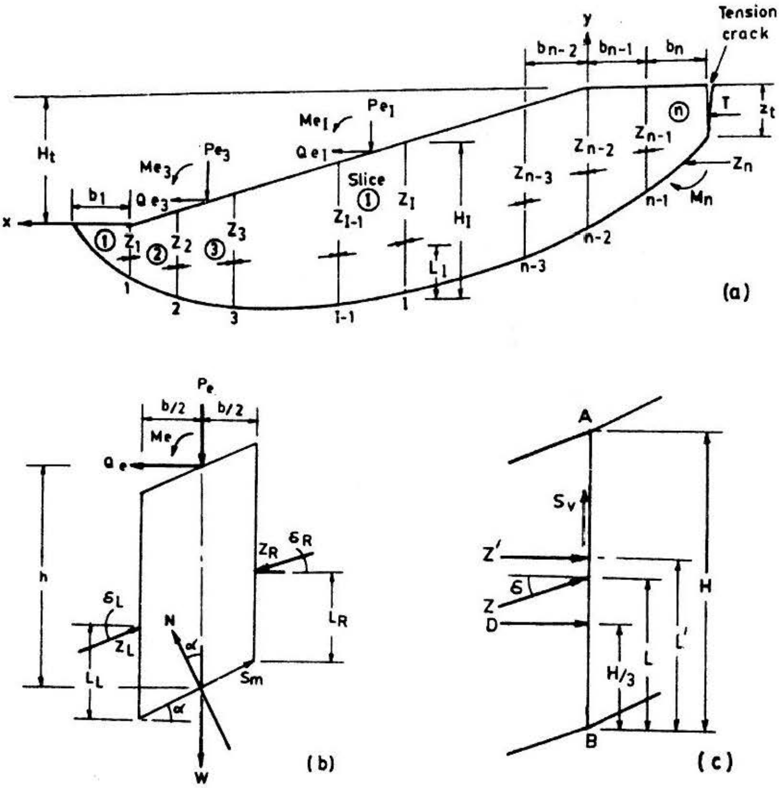

FIGURE 1 : (a) Definitions and Notations; (b) Forces on a Typical Slice; (c) Forces on an Interslice Boundary

procedure by which both the critical slip surface and the minimal factor of safety are determined simultaneously. The mathematical programming formulation of the problem has been taken advantage of in incorporating the acceptability criteria as design constraints. It should be pointed out, however, that the generalized procedure presented here can be coupled with any other rigorous method such as the Morgenstern and Price method (Morgenstern and Price, 1965); the choice of the Spencer method has been made for convenience only. For the sake of simplicity, the formulation of the procedure is explained and elucidated with reference to simple homogeneous slopes. The extension and application of the developed procedure to heterogeneous slopes such as in zoned dams are reported elsewhere (Bhattacharya and Basudhar, 2000).

The Safety Functional for the Spencer Method

In the Spencer method of analysis it is required to solve the following pair of nonlinear equilibrium equations to find the two unknowns F and θ:

Force Equilibrium: $Z_n(F, \theta) = 0$ (1a)

Moment Equilibrium: $M_n(F, \theta) = 0$ (1b)

where, referring to Fig.1, Z_n and M_n are the external balancing force and moment respectively; F is the average factor of safety and θ is a characteristic angle defining the variation of the interslice force inclination, δ, given by :

$$\tan\delta_i = k_i \tan\theta \tag{2}$$

where the suffix i denotes the i^{th} interslice boundary (Fig.1). The coefficient k in the Spencer method is equivalent to the interslice force function $f(x)$ in the Morgenstern and Price method. If n be the number of slices, $(n-1)$ values are chosen or prescribed by the user for k; e.g., if k is taken to be unity throughout, then the interslice forces will be all parallel and their slopes, δ_i will be each equal to θ. Expressions for Z_n and M_n, originally given by Spencer, have been modified by the authors to include external forces and moments (Bhattacharya and Basudhar, 1992).

The method of solution suggested by Spencer (1973) for the pair of nonlinear stability equations stated above is a process of successive approximation in which the values of the external force Z_n and the external moment M_n are gradually reduced to a negligible level. Bhattacharya and Basudhar (1999) have discussed certain limitations of this method and proposed a new powerful and efficient equation solver, which has been coded in a computer program, SOLVE.

Line of Thrust

Figure 1(c) shows, on a typical interslice boundary, the normal component of the effective interslice force, Z' together with the heights L and L' of the points of action of the total and effective inter-slice forces respectively from the slip surface. Accordingly, the lines joining these points at various interslice boundaries are called the line of thrust for total stress and for effective stress respectively. In the Spencer method of analysis, these are obtained as a part of the solution. Expressions for L, L' and Z' are given by Spencer (1973) and by Bhattacharya (1990) for unloaded and loaded slopes respectively.

Acceptable Line of Thrust - Introduction of Tension Crack

In those cases in which the positions of the lines of thrust (obtained as part of the solution) are not satisfactory, Spencer (1973) has recommended the assumption of a water-filled vertical tension crack running parallel to the

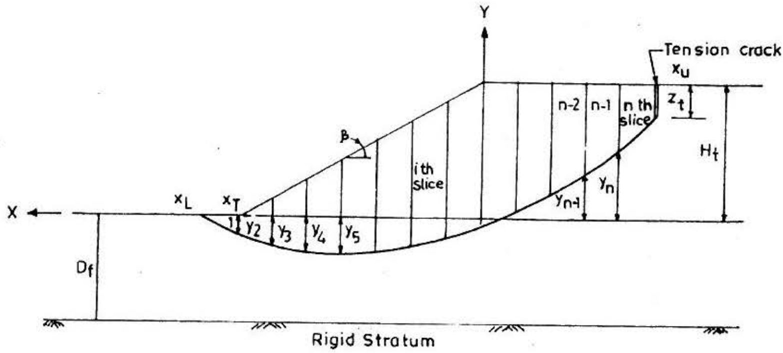

FIGURE 2 : Discretization Model for Homogeneous Slopes

crest of the embankment. The depth of the tension crack, z_t, can be assumed as the depth of zero active earth pressure, z_0, given by:

$$z_0 = \frac{2c'}{\gamma F(1-r_u)}\sqrt{\frac{1+\sin\phi'_m}{1-\sin\phi'_m}} \tag{3}$$

The above expression is, however, applicable only to slopes in homogeneous soils in which the pore water pressure increases with depth in direct proportion to the overburden.

Depth of Tension Crack as Design Variable

Equation (3) requires an iterative procedure to solve for the depth of tension crack z_0. Spencer (1973) suggested that the value of F occurring in this expression can be obtained from a preliminary trial taking k = 1 throughout and with no tension crack. In the search for critical slip surface, the value of F changes from one trial shear surface to another. For slip circle analysis, Spencer (1968) has provided a chart to obtain z_0 for various homogeneous slopes and soil and pore pressure coefficient values. In the search for critical slip surfaces of general shapes, the iteration can be conveniently done by treating z_t as a design variable together with an upper limit for z_t as z_0. This aspect is further discussed in a later section.

Minimization of the Safety Functional

Slice Discretization

The potential sliding mass is divided into n vertical slices of uniform width (Fig.2). Let y_1, y_2, y_i, ..., y_{n+1} be the y co-ordinates of the shear surface at the slice boundaries. The shear surface terminates at the bottom of a vertical tension crack of depth z_t. If x_1, x_2, ..., x_i, x_{n+1} be the corresponding x co-ordinates, then, $y_{n+1} = (H_t - z_t)$; $x_1 = x_L$ and, $x_{n+1} = x_U$. From these, the angle α_i that the base of the i^{th} slice makes with the horizontal can be calculated.

Design Vector

The shape and location of a shear surface is completely defined by y_2, y_3, ..., y_i, ..., y_n, z_t, x_L, and x_U and, for a given soil, the factor of safety can be expressed as a function of the above co-ordinates. The search for the critical surface is to find these co-ordinates, which minimizes the factor of safety. The design vector in this case is, therefore, as follows:

$$\mathbf{D} = \left[y_2, y_3, \ldots, y_i, \ldots y_n, x_L, x_U, z_t\right]^T \text{ and } n_{dv} = n+2 \tag{4}$$

where, n_{dv} is the total number of design variables. Clearly, the number of design variables is directly proportional to the number of slices adopted in the computation. When z_t is not considered as a design variable, $n_{dv} = n + 1$.

Objective Function

Since the objective is to minimize the safety functional, F, it is identified as the objective function and can be expressed in terms of the design vector as:

$$F = f(\mathbf{D}) \tag{5}$$

Design Constraints

In order to ascertain that the shape and location of the slip surface are physically reasonable and kinematically compatible, the following restrictions or constraints need to be imposed on the choice of the design variables. The constraints enumerated below are all inequality constraints.

Boundary Constraints

1. The shear surface must lie within the slope geometry; this will be satisfied if the following restrictions are imposed.

(i) $$g_j(\mathbf{D}) = \frac{y_i}{H_t} \leq 0 \tag{6a}$$

$i = 2, 3, \ldots, m_1 + 1;\quad j = 1, 2, \ldots, m_1.$

where m_1 is the number of interslice boundaries lying to the left of the toe.

(ii) $$g_j(\mathbf{D}) = \frac{y_i\, x_T}{\{(x_T - x_i)H_t\}} - 1 \le 0 \tag{6b}$$

$$(m_1 + 2) \le i \le (m_1 + m_2 + 1)$$

$$\text{and } (m_1 + 1) \le j \le (m_1 + m_2)$$

where, m_2 is the number of interslice boundaries within the inclined portion (including the toe) of the slope surface and x_T is the x-coordinate of the toe.

(iii) $$g_j(\mathbf{D}) = \frac{y_i}{H_t} - 1 \le 0 \tag{6c}$$

$$(m_1 + m_2 + 2) \le i \le n$$

$$\text{and } (m_1 + m_2 + 1) \le j \le (n - 1)$$

2. The shear surface should not penetrate any rigid stratum below. Assuming that the rigid stratum boundary is a horizontal one at a depth D_f, as in Fig.2, the normalized form of the above requirement is given by:

$$g_j(\mathbf{D}) = \frac{|y_i|}{D_f} - 1 \le 0 \tag{7a}$$

When the hard stratum boundary is an irregular one, the above constraint is given by:

$$g_j(\mathbf{D}) = \frac{|y_i|}{|z_i^f|} - 1 \le 0 \tag{7b}$$

where, z_i^f represents the corresponding ordinates of the irregular rigid boundary. If, however, the lowest point y_M of this boundary is found out, then, instead of putting a constraint on all negative y_i, only one constraint would do. This can be expressed as:

$$g_j(\mathbf{D}) = \frac{|y_M|}{D_f} - 1 \le 0 \tag{7c}$$

3. When the depth of tension crack, z_t is a design variable, an upper limit for z_t is set at z_0, the depth of zero active earth pressure (Eqn.3).

$$g_j(\mathbf{D}) = \frac{z_t}{z_0} - 1 \leq 0 \tag{8}$$

Curvature Constraints

4. For the shear surface to be concave upward, the following relationship should be satisfied.

$$g_j(\mathbf{D}) = \frac{-\left(y_{i-1} - 2y_i + y_{i+1}\right)}{H_t} \tag{9}$$

Side Constraints

5. To ensure reasonable values and to avoid unnecessary search, an appropriate lower bound on the design variable F may be imposed as follows:

$$g_j(\mathbf{D}) = -F + F_o \leq 0 \tag{10}$$

where F_o is the specified lower bound on F. Similarly, appropriate upper and lower bounds may be imposed on the design variable θ. A detailed discussion on this has been given by Bhattacharya and Basudhar (1999).

Acceptability Constraints

To obtain a physically acceptable solution, the following constraints need to be imposed :

1. No state of tension should be implied to exist above the slip surface. For this, control is effected on the position of the line of thrust. The line of thrust, which is obtained as a part of the solution should lie within the middle thirds of the heights of the interslice boundaries. However, it has been observed that imposition of such constraints may become too stringent for smooth progress of the minimization scheme. To allow more flexibility, therefore, the line of thrust is restricted to lie within the sliding mass. The following normalized forms of the constraints are considered:

$$g_j(\mathbf{D}) = \frac{-L_i}{H_i} \le 0 \qquad i = 1, 2, \ldots, n-1 \quad (11a)$$

$$g_j(\mathbf{D}) = \frac{L_i}{H_i} - 1 \le 0 \qquad i = 1, 2, \ldots, n-1 \quad (11b)$$

where, referring to Fig.1(c), L_i is the height of the point of application of the interslice forces from the shear surface and H_i is the height of the i^{th} interslice boundary.

However, situations may arise where the above constraints are found hard to satisfy particularly in locations near the crest of a slope. In some cases it also happens that the above constraints are satisfied, yet, the line of thrust for effective stress is not satisfactory. In such cases introduction of tension crack generally results in acceptable line of thrust. In some cases, in addition to tension crack, other assumptions regarding the slopes of the interslice forces are to be tried in order to obtain reasonable lines of thrust (Spencer, 1973).

2. The internal forces obtained from the solution should not violate the Mohr-Coulomb failure criterion anywhere within the sliding body. This can be ensured by checking that the values of the factors of safety along vertical interfaces are not less than the overall factor of safety of the slope. However, it has been demonstrated (Spencer, 1981) that in those cases in which the obtained line of thrust is satisfactory, the solutions generally show good agreement between the average factor of safety against shearing on the slip surface and the factors of safety on the critical shear planes (which may not be vertical). On the other hand, it is not desirable to burden the numerical scheme with too many constraints, unless they are essential, so that the progress of minimization is not unduly affected. Keeping the above observations in view, no such constraint has been imposed; however, the obtained solution is checked for any violation of this requirement.

3. The directions of internal forces obtained as part of the solution should be kinematically admissible. Following the discussion above, in order to avoid using too many constraints, no such constraint has been imposed. However, in all cases the obtained solutions are checked for the admissibility of signs of the forces, consistent with the method of analysis.

Proposed Direct Procedure for Determination of Critical Slip Surface

Principle

The procedure of determination of critical shear surfaces for which formulation has been presented above may be referred to as the Indirect Procedure as it involves, in the process of arriving at the critical slip surface, numerous attempts to solve the stability equations for a large number of trial slip surfaces. Rigorous methods of analysis such as the Spencer method or the Morgenstern and Price method call for an elaborate numerical technique for solving the pair of nonlinear stability equations. It has been reported that apart from being slow, such techniques occasionally meet with convergence difficulties (Soriano, 1976; Bhattacharya and Basudhar, 1997).

It has, therefore, been felt that it would be very useful if the slope stability problem be formulated in such a manner that the critical slip surface is determined directly obviating, thereby, the tedium of solving a couple of nonlinear equations every time a trial slip surface is generated by the auto search technique employed in the minimization scheme. This may be achieved by including both F and θ in the design vector along with the slip surface co-ordinates while putting the force equilibrium and the moment equilibrium requirements as equality constraints. The optimal design vector would now give not only the shape and location of the critical slip surface but also the factor of safety, F, and the interslice force angle, θ, associated with the critical shear surface. However, the objective function, namely, the factor of safety of the slope, remains unchanged. Thus, in this new formulation, the objective function, F, also appears as a design variable. The procedure of determination of critical slip surface based on this new formulation will henceforth be referred to as the direct procedure. The basic problem may be stated as follows:

Find the shear surface as well as the corresponding factor of safety, F, and the interslice force angle, q, such that the factor of safety of the slope is minimized subject to the conditions that:

(i) the force equilibrium condition is satisfied i.e.., $Z_n = 0$

(ii) the moment equilibrium condition is satisfied i.e.., $M_n = 0$

(iii) the shape of the critical slip surface is kinematically admissible and that the obtained solution satisfies the acceptability criteria discussed in the preceding section.

Design Vector and Constraints

From the above, it is clear that for the direct procedure, the design vector defined earlier is extended by the inclusion of two additional design variables, F and q, and is given by :

$$\mathbf{D} = \left[y_2, y_3, \ldots, y_i, \ldots y_n, x_L, x_U, z_t, F, \theta\right]^T \text{ and } n_{dv} = n + 4 \qquad (12)$$

The constraints associated with the direct formulation includes all the inequality constraints associated with the indirect formulation discussed earlier. In addition, two equality constraints are required to be imposed in the form of the two equilibrium requirements (Z_n = 0; M_n =0) which are inherent in the formulation itself. These two conditions can be combined to form a single normalized equality constraint as:

$$l_j(\mathbf{D}) = \frac{\left(Z_n^2 + S_f M_n^2\right)}{\left(\gamma b H_t\right)^2} \qquad (13)$$

where, l_j stands for the j^{th} equality constraint function, H_t is the height of the slope, g is the unit weight of soil, b is the width of each slice and S_f is a scale factor. The scale factor S_f is introduced to make the function well behaved without any eccentricity resulting from the possible large difference in the magnitude in the values of Z_n and M_n. Otherwise the iterative scheme to find the minimum may not converge. In this analysis, the scale factor has been chosen as given below and used earlier by Morgenstern and Price (1967).

$$S_f = \frac{\left[\dfrac{\partial Z_n}{\partial F}\right]^2 + \left[\dfrac{\partial Z_n}{\partial \theta}\right]^2}{\left[\dfrac{\partial M_n}{\partial F}\right]^2 + \left[\dfrac{\partial M_n}{\partial \theta}\right]^2} \qquad (14)$$

Mathematical Programming Formulation and Solution Procedure

The constrained minimization problem stated above can be cast as a mathematical programming problem of the following general form:

Find **D** such that

$$f(\mathbf{D}) \rightarrow \text{Min.} \qquad (15a)$$

$$\text{subject to : } g_j(\mathbf{D}) \leq 0 \qquad j = 1, 2, \ldots, n_i. \tag{15b}$$

$$l_j(\mathbf{D}) \leq 0 \qquad j = 1, 2, \ldots, n_e. \tag{15c}$$

where, n_i and n_e are the total number of inequality and equality constraints respectively. $\mathbf{D}$, $f(\mathbf{D})$, $g(\mathbf{D})$ and $l(\mathbf{D})$ represent the design vector, the objective function, the inequality and the equality constraint functions respectively.

Since it is difficult to obtain an initial feasible decision vector, a method, which accepts infeasible initial design vector is advantageous. The extended penalty function method enunciated by Kavlie (1971) is adopted here because of the fact that this method readily accepts infeasible decision points, but the optimal solution lies in the feasible region. In this method, the modified objective function is formed as:

$$\psi(\mathbf{D}, r_k) = f(\mathbf{D}) - r_k \sum_{j=1}^{p} G_j\left[g_j(\mathbf{D})\right] \tag{16}$$

where function G is chosen as follows:

$$G\left\lfloor g_j(\mathbf{D}) \right\rfloor = \begin{cases} \dfrac{1}{g_j(\mathbf{D})} & \text{for } g_j(\mathbf{D}) \leq \varepsilon \\[2ex] \dfrac{\left[2\varepsilon - g_j(\mathbf{D})\right]}{\varepsilon^2} & \text{for } g_j(\mathbf{D}) > \varepsilon \end{cases} \tag{17}$$

where, the tolerance, ε, is given by:

$$\varepsilon = -\frac{r_k}{\delta_t} \tag{18}$$

δ_t is a parameter defining the transition between the two types of penalty terms and p is the total number of constraints. General guidelines for appropriate choice of the parameters δ_t and, ε are available in the literature (Kavlie and Moe, 1971; Cassis and Schmit 1976). r is a positive constant called penalty parameter and r_k is its value corresponding to the k^{th} cycle of minimization. Using a reduction factor c (usual value is 0.10), the penalty parameter r_k is made successively smaller in order to obtain the constrained minimum value of the objective function $f(\mathbf{D})$. Thus,

$$r_{k+1} = c\, r_k \tag{19}$$

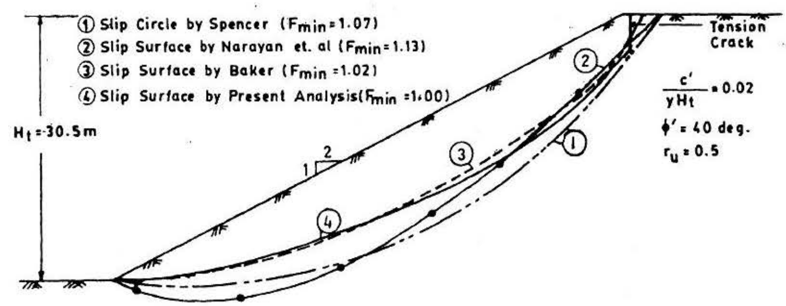

FIGURE 3 : Critical Slip Surfaces obtained from Various Techiques for the Example Problem

The composite function $\psi(\mathbf{D})$ so generated, is then minimized by using Powell's method of conjugate directions for multidimensional search and quadratic interpolation technique for unidimensional search (Fox, 1971).

The proposed Direct Procedure formulated above has been coded in a Fortran program SUMSTAB and all computations reported herein have been carried out using the same.

Illustrative Example

Figure 3 presents a section of a homogeneous slope. The soil properties viz., the effective angle of shearing resistance ϕ', the stability ratio ($c'/\gamma H_t$) and the pore pressure ratio, r_u are as indicated in the figure. The actual value of γ and c' are taken as 20.0 kN/ m^3 and 12.20 kPa respectively. The same slope was earlier analysed by the following investigators, using different minimization techniques as given below:

1. Spencer (1967) using slip circle analysis and grid search technique.

2. Narayan et al. (1976) using general slip surface and variational technique.

3. Baker (1980) using general slip surface and dynamic programming technique.

The purpose of selecting this example problem is to present a comparative study of the developed Direct Procedure with the other existing techniques such as based on grid search, variational calculus and dynamic programming technique etc and thereby validate the program SUMSTAB with respect to the previously reported results.

Present Solution

It is well known that any nonlinear programming problem is starting point dependent and engineering judgement is required to choose a good starting point. In the present analysis, therefore, three widely different starting points or initial design vectors (which correspond to three different shear surfaces) have been considered. A total of 13 slices have been used in the computation. Such a selection of the number of slices is based on observations in earlier studies (Bhowmik and Basudhar, 1989) that in noncircular analysis of homogeneous slopes, when the number of slices exceeds 12, there is no significant variation in the results. To start with, it has been assumed that there is no tension crack and that the interslice forces are all parallel i.e., $k = 1$ throughout. As expected, three different initial design vectors have resulted in three different local minima i.e., three different F_{min} values corresponding to three different final shear surfaces. Out of these, the shear surface, which is associated with the least of the F_{min} values ($F_{min} = 1.03$) has been selected as the critical shear surface; however, the acceptability of this surface requires to be checked.

On examining the detailed results, it has been observed that although the line of thrust for total stress remains entirely within the middle-third, the solution is unacceptable as the effective interslice force (Z') at the last interslice boundary is negative, which implies development of tension. As a remedial measure, presence of a water-filled tension crack running parallel to the crest of the slope has been assumed. In the re-analysis, the depth of tension crack has been treated as a design variable with an initial depth (z_t) of 3.0 m based on Spencer's chart (Spencer, 1968). The revised solution is found to be acceptable. The values of the minimum factor of safety and the corresponding interslice force angle have been obtained as : $F_{min} = 1.004$ and $\theta = 0.4164$ rad. The final depth of the tension crack associated with the acceptable solution has been obtained as $z_t = 4.893$ m. The critical shear surface thus obtained is presented in Fig.3 and the associated line of thrust and other internal variables are presented in Table 1. It is seen from the Table that the line of thrust for total stress as well as that for effective stress (indicated by the ratios L/H and L'/H respectively) lie within the middle-third of the heights of the interslice boundaries. Furthermore, values of the factors of safety along vertical interfaces, F_v, are seen to be greater than the value of F_{min}, as it should be. The Table also shows that the effective normal and shear stresses at slice bases and, also, the normalized horizontal component of the effective interslice force (indicated by σ', τ and $Z'/\gamma b H_t$ respectively) are all positive as required for an acceptable solution.

It may be mentioned that a total of seventeen design variables (corresponding to thirteen slices) have been considered in the present analysis, including the depth of tension crack. The design vector and constraints at the

Table 1 : Calculated Responses Associated with the Solution to the Example Problem

Slice No.	σ' kPa	τ kPa	L/H	L'/H	$\frac{Z'}{\gamma} b H_t$	F_v
1	21.89	30.17				
			0.57	0.61	0.05	3.43
2	44.68	49.19				
			0.42	0.45	0.09	2.96
3	55.08	58.00				
			0.39	0.41	0.13	2.71
4	59.50	62.54				
			0.37	0.39	0.16	2.56
5	65.00	64.47				
			0.36	0.37	0.18	2.46
6	63.00	64.45				
			0.35	0.36	0.18	2.40
7	60.00	62.99				
			0.35	0.36	0.17	2.38
8	58.50	60.29				
			0.34	0.35	0.15	2.39
9	56.15	55.66				
			0.34	0.35	0.13	2.45
10	44.19	48.21				
			0.34	0.35	0.09	2.61
11	41.08	45.91				
			0.35	0.36	0.06	2.83
12	29.71	35.31				
			0.37	0.48	0.03	3.45
13	14.10	21.89				

beginning and at the end of minimization are given in Table 2. The Table indicates that even though the initial design vector is infeasible with regard to the equality constraint, the proposed scheme, utilizing the Extended Interior Penalty Function method, brings out a feasible optimal solution quite efficiently.

Comparison with Earlier Solutions

Table 3 presents the values of the minimum factors of safety F_{min} along with the interslice force inclinations θ, as reported by various investigators using different techniques. It is seen that the present analysis has yielded the least value of F_{min} and that obtained by Baker (1980) using dynamic programming is closest to this value. The F_{min} values obtained in the other two solutions (Spencer, 1967; Narayan et al., 1976) are a little higher. For the sake of comparison, the critical slip surfaces for all these solutions are also presented in Fig.3 alongside the acceptable critical slip surface obtained in the present solution.

Table 2 : Design Vector and Constraints for the Example Problem

STARTING POINT

δ_t	ε_o	f	ψ	F	θ	Z_n	M_n
0.001	−0.1	1.1	1.3014	1.1	0.40	-0.2534×10^3	-0.1425×10^4

Design Variables : 17 Variables for 13 Slices

−0.51	0.00	0.515	1.50	2.50	4.25	6.40	9.00
11.95	15.00	18.85	22.75	61.20	−5.00	1.10	0.40
27.50							

Constraints (inequality) :

−0.9941	−0.7775	−0.6685	−0.5857	−0.5242	−0.4698	−0.4163	−0.3618
−0.2960	−0.2342	−0.1637	−0.0953	−1.0200	−0.0050	−0.4700	−0.0150
−0.7500	−0.4000	−0.4500	−0.3500	−0.0999	−0.8000	−0.0500	−0.8500
−0.4505	−0.3854	−0.3532	−0.3424	−0.3325	−0.3316	−0.3333	−0.3408
−0.3630	−0.4186	−0.7233	−18.82	−0.5496	−0.6146	−0.6468	−0.6576
−0.6675	−0.6684	−0.6667	−0.6592	−0.6370	−0.5814	−0.2767	17.82
−0.1001	−0.3990	−0.0625					

Constraint (equality) : 0.800E−02

Table 2 : Continued

OPTIMAL POINT

δ_t	ε_o	f	ψ	F	θ	Z_n	M_n
0.001	-1.0×10^{-8}	1.0041	1.0061	1.0041	0.4164	0.3719×10^{-4}	0.3048×10^{-2}

Design Variables : 17 Variables for 13 Slices

−0.2338	0.2190	1.0750	2.2580	3.7229	5.4468	7.4141	9.6194
12.1122	15.0213	17.9337	21. 400	62.1160	−0.0448	1.0041	0.4164
25.6071							

Constraints (inequality) :

−0.7357	−0.6213	−0.5381	−0.4811	−0.4286	−0.3796	−0.3329	−0.2866
−0.2389	−0.1883	−0.1322	−0.0672	−0.6866	−0.4032	−0.3270	−0.2819
−0.2590	−0.2433	−0.2381	−0.2874	−0.4164	−0.0033	−0.6939	−0.4607
−0.5682	−0.4226	−0.3864	−0.3695	−0.3592	−0.3520	−0.3467	−0.3428
−0.3404	−0.3395	−0.3451	−0.3729	−0.4318	−0.5774	−0.6136	−0.6305
−0.6408	−0.6480	−0.6533	−0.6572	−0.6596	−0.6605	−0.6549	−0.6271
−0.0141	−0.4164	−0.4611					

Constraint (equality) : 0.7723E−14
No. of r-minimization required = 7

Note : Out of a total of 17 design variables, the first 12 denote y-coordinates, the next two x-coordinates of the two ends of a shear surface, the next two denote F and θ while the last variable denote $(H_t - z_t)$. Out of a total of 51 inequality constraints, the first 12 are boundary constraints, next 12 are curvature constraints, next 24 are constraints on the line of thrust, the next 3 are side constraints.

Table 3 : Comparison of Various Solutions to the Example Problem

S.No.	Methodology used		Minimum Factor of safety F_{min}	Interslice force angle in degree θ	Reference
	Shape of slip surface	Minimization technique			
1.	Circular	Grid search	1.07	22.50	Spencer (1967)
2.	General	Variational technique			
		(i) Direct	1.13	23.41	Narayan et al. (1976)
		(ii) Indirect	1.12	23.72	
3.	General	Dynamic programming	1.02	22.50	Baker (1980)
4.	General	SUMT			
		(i) Acceptable	1.00	23.85	Present solution
		(ii) Unacceptable	1.03	24.66	Present solution

From Fig.3, it is observed that the shapes and locations of the critical slip surfaces obtained from various techniques are markedly different. Among these, the critical surface obtained from the present analysis and that from Baker's analysis based on dynamic programming are close to each other except at the upper end. This corresponds well with the closeness of their F_{min} values. However, for a meaningful comparison, it is required to check whether the earlier solutions satisfy the prescribed acceptability criteria. Now, so far as an acceptable line of thrust is concerned, except in the case of solution using variational technique (Narayan et al., 1976), only the line of thrust for total stress were reported and shown to be satisfactory. This, however, does not necessarily imply that the line of thrust for effective stresses would also be satisfactory. To investigate this aspect further, the critical slip circle reported by Spencer (1967) has been reanalyzed using the program SOLVE. The results of the reanalysis indicate perfect agreement with the factor of safety value of 1.07 reported by Spencer (1967). Moreover, the line of thrust for total stress is seen to lie within the middle third of the heights of the interslice boundaries, as reported; yet, the line of thrust for effective stress and the resultant effective interslice forces are far from satisfactory (details of these results are not presented here for the sake of brevity). Had this acceptability been checked in the reported solutions (Spencer, 1967; Baker, 1980), perhaps assumption of tension crack would have been necessary, as has been done in the present solution. It may be recalled that in the initial part of the present investigation also, a critical surface has been obtained, without considering tension crack, for which the obtained line of thrust for total stress is found to be satisfactory; yet, the corresponding line of thrust for effective stress is not acceptable. In the solution using variational technique (Narayan et al., 1976), unlike the present solution, an acceptable solution was obtained without introducing any tension

Table 4 : Progress of Minimization in the Example Problem

No. of r-minimization	Value of r	Objective function f	Composite function ψ	Z_n kN/m	M_n kN-m/m
0 (Starting Point)	1×10^{-4}	1.1000	1.3014	−0.2534 E +03	−0.1425 E +04
1	1×10^{-4}	1.0007	1.0236	0.1204 E +04	0.4662 E +02
2	1×10^{-5}	0.9999	1.0056	0.1270 E +03	0.1312 E +03
3	1×10^{-6}	1.0011	1.0051	0.9338 E +02	0.2525 E +03
4	1×10^{-7}	1.0002	1.0047	0.4021 E +02	0.1864 E +03
5	1×10^{-8}	1.0033	1.0069	0.1137 E +02	0.2605 E +02
6	1×10^{-9}	1.0042	1.0063	0.9508 E +00	0.9785 E +00
7	1×10^{-10}	1.0041	1.0061	0.1432 E +00	0.3439 E +00
8 (Optimal Point)	1×10^{-11}	1.0041	1.0061	0.3719 E −04	0.3048 E −02

crack. This may be attributed to the difference in the definition of the interslice force functions in the two analyses. It may be noted that in the former case the interslice force function was based on effective stresses whereas in the present analysis this function (k-distribution) is based on total stresses.

As regards the other acceptability criteria that the values of factors of safety along vertical interfaces (F_v) cannot be less than that for the critical slip surface (F_{min}) while the present solution and the solution by variational technique satisfy this criteria, this criterion has been apparently ignored in the other two solutions.

Progress of Minimization

Table 4 presents the variation of the objective function f and the composite function ψ with the penalty parameter r. It is seen that unlike interior penalty function method and like exterior penalty function method the decrease in the function values with the progress of minimization is not monotonic. On examination of the detailed results (not presented here) it has been observed that in the first few cycles of r-minimization, the function values (f and ψ) decrease continually till they reach some minimum values (usually controlled by the lower bound for F used in the analysis) and the scheme shows a tendency to keep all the inequality constraints satisfied. However, at these stages the equality constraint remains violated i.e., its value does not decrease significantly, as is evident from the values of Z_n and M_n

presented in the Table. Afterwards, the minimization procedure attempts to effectively satisfy the equality constraint by re-adjusting the design variables F and θ. This eventually results in subsequent increase in the value of F till convergence is achieved. It may be mentioned that similar trends have been observed in the case of a wide range of problems studied by the authors, in which the direct procedure has been used.

Conclusions

From the studies contained in this paper the following conclusions are drawn:

1. The proposed direct procedure for determination of critical slip surface promises to be a powerful tool for the stability analysis of homogeneous slopes. However, as is usual with any nonlinear programming approach, the developed numerical scheme is starting point dependent. As such, to locate the global minimum, it may be necessary to try with a few different starting points; however, the effort can be reduced if the initial design vector is based on engineering judgement.

2. The extended interior penalty function method adopted in the proposed direct procedure appears to be a powerful algorithm in handling infeasible design points as well as equality constraints for this class of problems. Acceptance of infeasible starting points is an advantage with the proposed procedure to solve a wide variety of problems.

3. The results obtained by using the proposed procedure compares well with those obtained by using other techniques e.g., calculus of variation and dynamic programming technique. However, the proposed technique is more flexible in incorporating the acceptability criteria in the analysis.

4. In most of the analysis using Spencer's method, introduction of tension crack is necessary to get an acceptable critical shear surface. The method developed has the provision for including the depth of the tension crack in the design vector, which increases its potential for obtaining acceptable solutions.

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Notation

b = width of slice

c' = effective cohesion

D = force due to pore water pressure on an interslice boundary

$\mathbf{D}$ = design or decision vector

δ_t = transition parameter in the extended penalty function formulation

F	=	factor of safety associated with a given or trial shear surface
F_{min}	=	minimum factor of safety corresponding to the critical shear surface
F_v	=	factor of safety against shear along a vertical interface
$f(\mathbf{D})$	=	objective function
$g_j(\mathbf{D})$	=	j^{th} inequality constraint function
H	=	height of an interslice boundary
H_t	=	height of embankment
h	=	mean height of a slice
k	=	co-efficient of slope of interslice forces
L	=	height of the point of action of an interslice force for total stress above the base of the interslice boundary on which it acts
L'	=	height of the point of action of the interslice force for effective stress above the base of the interslice boundary on which it acts
$l_j(\mathbf{D})$	=	j^{th} equality constraint
M_e	=	externally applied moment
M_n	=	external moment required to be applied to the n^{th} slice to stabilize the embankment
n	=	number of slices
n_{dv}	=	number of design variables
n_e	=	number of equality constraints
n_i	=	number of inequality constraints
N	=	normal force at the base of a slice
P_e	=	external vertical force on a slice
Q_e	=	external horizontal force on a slice
r_k	=	penalty parameter in the k^{th} cycle
r_u	=	pore pressure ratio
S_m	=	mobilized shear force along base of a slice
S_v	=	mobilized shear force along an interslice boundary

T = force due to water pressure in tension crack

W = weight of a slice

x_L = x co-ordinate of the lower intersection point of the shear surface with the ground surface

x_U = x co-ordinate of the upper intersection point of the shear surface with the ground surface

y_i = y co-ordinate of the intersection of the i^{th} interslice boundary with the shear surface

Z = resultant interslice force on any interslice boundary

Z' = force due to effective stress normal to a interslice boundary

z_t = depth of tension crack

z_0 = depth of zero active effective stress

Z_n = external force required to be applied to the n^{th} slice to stabilize the embankment

α = inclination of slice base with the horizontal

γ = bulk density of soil

γ_w = density of water

θ = characteristic angle defining the slope of resultant interslice forces

δ = slope of resultant interslice forces

δ_t = transition parameter in the extended penalty function method

ε = tolerance in the extended penalty function method

σ = total stress normal to base of a slice

σ' = effective stress normal to base of slice

τ = shear stress along the base of a slice

ϕ' = effective angle of shearing resistance

ϕ'_m = mobilised angle of shearing resistance

ψ = penalty function or composite function