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In-Situ Hydraulic Conductivity of Highly Permeable Soils using Slug Tests

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Introduction

In-situ tests, such as pumping tests, packer (or pressure) tests, and slug tests are commonly performed to determine the hydraulic conductivity (K) of soils for a variety of purposes ranging from water resource investigations to remedial investigations. Among these tests, the slug tests are preferred because they are easy to perform, economical, provide acceptable accuracy, and can be repeated over a short period of time.

Slug tests are generally performed in small-diameter wells using a cylindrically-shaped solid slug with flat ends. The typical test procedure involves either introducing or withdrawing a slug to displace water in the well, and then measuring the displacement of the well water versus time until initial equilibrium conditions are reached. The data are then analyzed to calculate the hydraulic conductivity using either the Bouwer and Rice (1976), Cooper et al. (1967) or Hvorslev (1951) data reduction procedures. The data reduction procedures used normally involve matching a type curve or straight line to the data as part of calculating hydraulic conductivity (Freeze and Cherry, 1979). A schematic diagram of water displacement versus time for a typical test condition is shown in Fig. 1.

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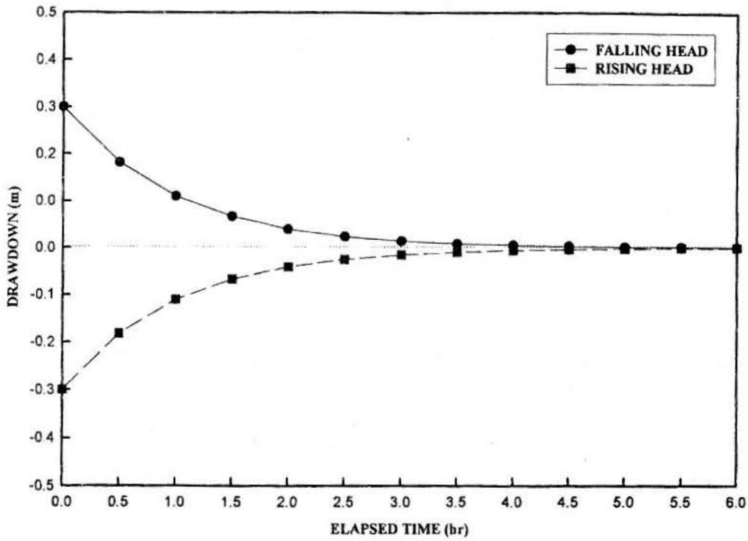


FIGURE 1 : Typical Slug Test Results in Low to Medium Permeable Soils

Although slug testing has been successfully used to provide reasonable estimates of hydraulic conductivity of low to medium permeable soils, difficulties arise when these tests are performed in highly permeable soils. Specific difficulties include: (1) the oscillation of water levels with time, and (2) the stabilization of the water level in a short time, usually within a few seconds. A schematic diagram of these short duration, oscillatory water level conditions is depicted in Fig. 2. An analysis of the data in Fig. 2 using the conventional data reduction methods of matching a curve or straight line, may not be possible or may provide erroneous values of hydraulic conductivity.

In order to accurately determine the hydraulic conductivity of highly-permeable soils using slug tests, it is necessary to understand the factors which contribute to the oscillating water level response which occurs in these soils during slug testing. It is also important to investigate if the oscillating water level response is due to water turbulence which came about from either the insertion or extraction of the slug and/or if the slug shape and size can be modified so that it will cause less water disturbance and as a result, the oscillating water level response can be eliminated. If the oscillating water level response cannot be eliminated, the existing common data reduction methods such as the Bouwer and Rice (1976), Cooper et al. (1967) or Hvorslev (1951) methods cannot be used. Therefore, the use of more rigorous analysis techniques such as the Van der Kamp (1976), Kipp (1985) or Springer and Gelhar (1991) methods would be warranted.

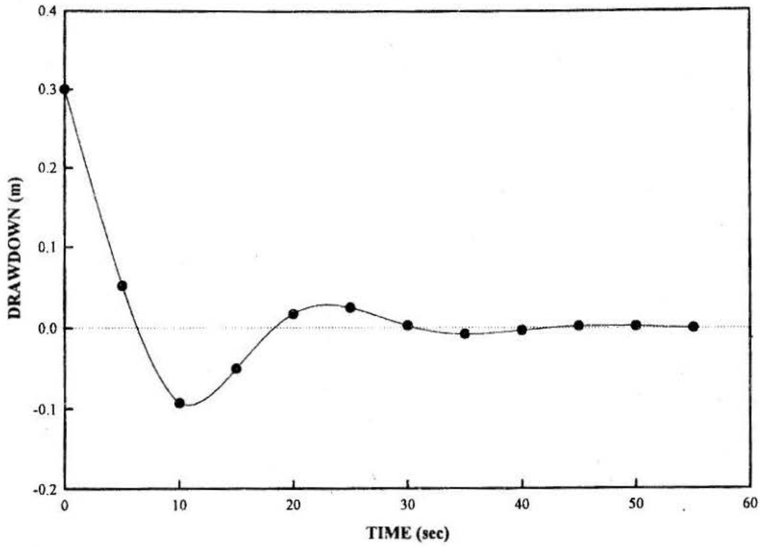


FIGURE 2 : Typical Slug Test Results in Highly Permeable Soils

This paper presents the results of a field investigation performed to assess the effects of slug shape and to determine the accurate slug test data analysis technique to use for the calculation of the hydraulic conductivity of a highly permeable soil formation. The field testing was performed with differently shaped slugs to determine if it is possible to develop a hydro-dynamically-efficient slug which will cause minimum disturbance during both insertion and extraction from a well and, as a result, will eliminate the oscillating water level response in the well. The most common analysis techniques, which address the oscillatory water level data from slug tests were reviewed and assessed for their applicability to calculate the hydraulic conductivity of highly permeable soils.

Field Investigation Methodology

The purpose of the field investigation was to determine whether a hydro-dynamically-efficient slug that would eliminate the oscillating water level response could be developed. This determination required designing and fabricating slugs of different shapes and proportions. These slugs were then used to perform tests in a highly-permeable soil formation at a site located in southern Illinois. A previous pumping test, which was performed at the site, provided reliable information on the hydraulic conductivity of the soil formation. The slug tests conducted for this study were used to determine the influence of the slug geometry and slug shape on the measured water level

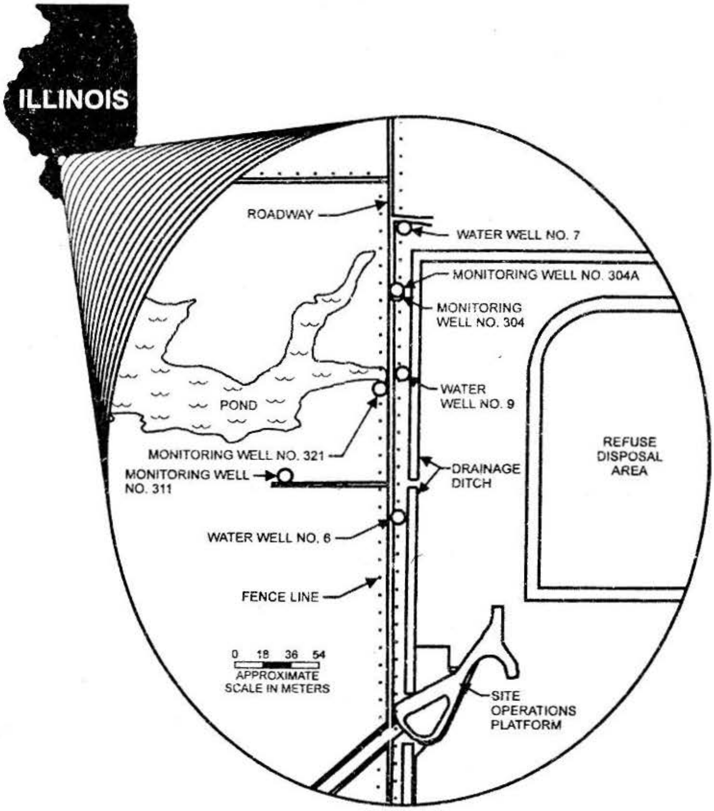


FIGURE 3 : Site Location Map

response in the wells. The slug test data were also needed to develop and validate an analysis technique that could account for the oscillating water level response.

Site Description

The site selected for this field investigation is located in southern Illinois (Fig. 3). This site was selected for the following reasons:

1. The subsurface stratigraphy consists of a thick layer of a highly permeable sand and gravel formation.

2. Several monitoring and pumping wells were previously installed in this formation.

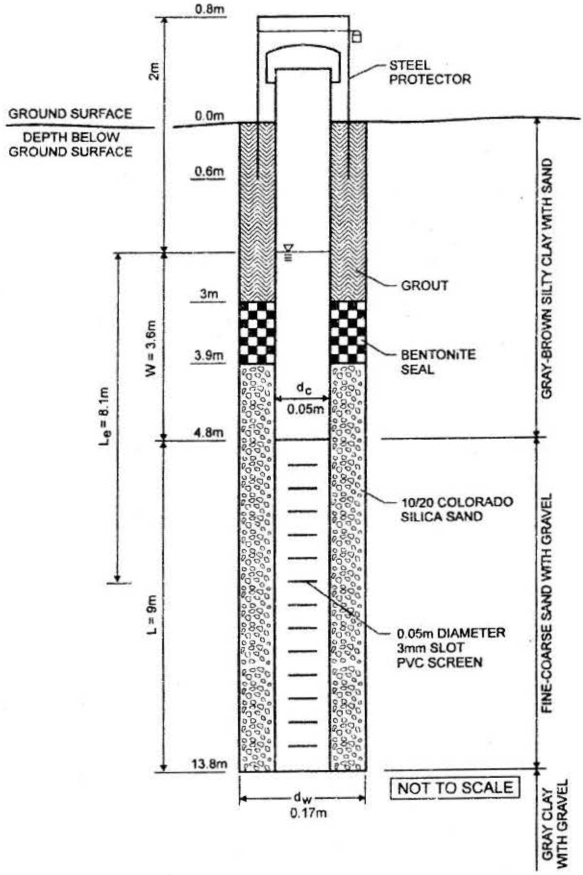


FIGURE 4 : Well W-321 As-Built

3. Pumping tests had already been conducted at the site so the hydraulic conductivity of the soil formation was known and could be used for comparison with the hydraulic conductivity value determined from the slug tests.

When the pumping tests were performed, Well W-9 was used as a pumping well, while Well W-321 was used as observation well (see Fig. 3). For this investigation, Well W-321 was used for all of the slug tests. This well was selected because the K value, which was calculated based on the observed water level data in Well W-321 during pumping test, provided an accurate value of the hydraulic conductivity of the soil formation. The as-built drawing of Well W-321 is shown in Fig. 4. The geology at this well location is also shown in this figure.

Before the pumping test was performed, an extensive subsurface investigation was conducted to determine the site subsurface conditions. At this site, clayey soils and loess were encountered from the ground surface to depths ranging from 4.5 to 5.1 m. Below these clayey soils and loess lies a sand and gravel formation, locally known as the Pearl Sand aquifer, which ranges from 6.0 to 9.3 m in thickness. Clayey soils were encountered below the sand and gravel formation. Based on the water level measurements in wells, which are located at the site, the groundwater in the sand and gravel formation exists under confined conditions.

Slug Testing Equipment and Procedure

A total of 28 slug tests were performed in Well W-321 using a total of seven differently shaped slugs. The slug tests were performed under both falling and rising head conditions, which resulted in four sets of data for each slug. The first set of data was the falling head data, and it was obtained when the slug was inserted into the well. The second set of data was the rising head data and it was obtained when the slug was extracted from the well. The third and fourth sets of data were obtained by repeating this same procedure of inserting and extracting the slug. These data helped assess the repeatability of the testing procedure. The following equipment was used for the slug testing:

1. Data Logger (Hermit 2000, Model SE2000, In-Situ Inc.)
2. Water Level Indicator (Model 51453, Slope Indicator Co.)
3. Pressure Transducer (Model PTX161, In-Situ Inc.)
4. Toshiba Laptop 486 Computer
5. Slugs with ropes

A sketch of the different slugs used for this investigation is shown in Fig. 5. The standard shaped slug, which is commonly used in slug testing, consisted of a 0.9 m long cylinder with a flat surface at both ends. The shapes of the other slugs were developed based on empirical hydrodynamic considerations. A summary of volume measurements for these slugs is provided in Table 1.

The actual slug test procedure consisted of the following steps:

1. Measuring the distance from the top of the well casing to the water surface using a water level indicator.

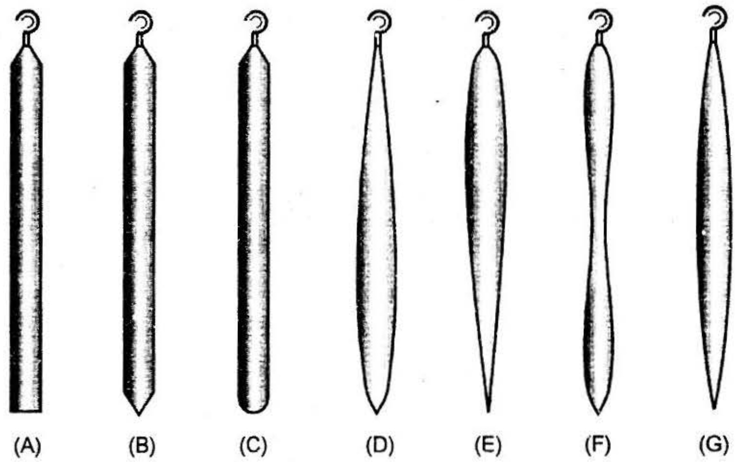


FIGURE 5 : Different Shaped Slugs Used in the Study: (a) Standard (Flat), (b) Cone, (c) Spherical, (d) Tear-drop, (e) Reverse Tear-drop, (f) Wasp-waist and (g) Javelin

2. Inserting the pressure transducer into the well to about 3 m below the water surface to avoid damage due to contact with slug, yet close enough to the water surface to provide accurate water level measurements.

3. Measuring the length of the slug and the length of the rope which holds the slug so that the slug will be completely submerged but will be safely above the transducer when inserted into the well.

4. Inserting the slug while simultaneously starting the data logger in order to collect the water level data.

Table 1
Characteristics of Different Slugs

Slug Shape	Standard	Spherical	Cone	Wasp-waist	Javelin	Tear-drop
Volume (cm^3)	805	795	782	612	601	606
Initial Water Level Change in Well W-321 (m)	0.391	0.386	0.380	0.297	0.292	0.294

5. Extracting the slug from the well while simultaneously recording the water level data using the data logger after the water level reaches equilibrium.

6. Repeating the above test procedure to confirm the results initially obtained for each slug shape.

Results and Analysis

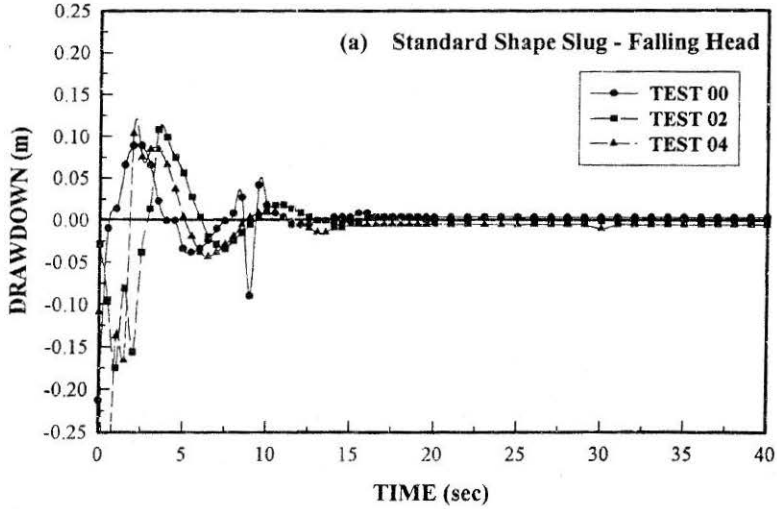
Slug Test Results

The field test data consisted of water level measurements which were recorded at different time intervals. The falling head and rising head data, which were obtained using the standard slug, are shown in Figs. 6a and 6b, respectively. The repeatability of the testing procedure is evident from the matching data obtained from repeated tests. These results show that the water levels in the well reached equilibrium in less than 15 seconds for all tests. Fig. 6 indicates that, unlike conventional slug test data for low- to medium-permeability soils, the water level response for a highly permeable soils will oscillate with time. Although water level fluctuations are a common occurrence in all slug tests, these effects are less significant for low to moderate permeability soils because the test interval is longer and the fluctuation affects only a small portion of the early data. In the low to medium permeability soils, the oscillating data points can then be discarded because there is still usually enough data remaining to define either a curve or a straight line. However, this is not the case for highly permeable soils.

The slug test results for the more streamlined shaped slugs under falling head test conditions are shown in Fig. 7. The test results for the same slugs under rising head conditions are given elsewhere (Reddy et al., 1995). These test results are similar to the results which were obtained for the standard shaped slug in that: (1) the water levels stabilized in less than 15 seconds; and, (2) the water levels oscillated with time. It should also be noted that the amount of either drawdown or rise at any given time for different slugs was not the same. This difference was observed because the volume of each slug was different and the amount of initial drawdown or rise is a direct result of the volume of water which is displaced (see Table 1).

Based on the field testing of the seven different slugs and the visual inspection of the water level versus time data, the following conclusions on the effect of slug shape on the water level recovery can be made:

1. The oscillatory water level response in a small-diameter well cannot be eliminated by varying the shape of the slug.



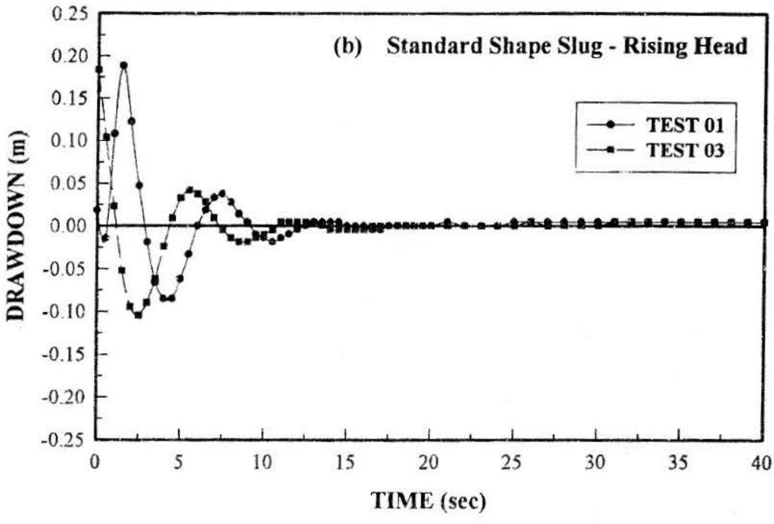


FIGURE 6 : Slug Test Data with Standard Shape Slug

2. Since the shape of the slug does not eliminate the oscillatory water level response in the small diameter wells, the standard slug can be used for slug testing in small-diameter wells which are installed in highly permeable formations.

3. The oscillatory response which is observed during the water level recovery may be attributed to the following two reasons: (a) the moment

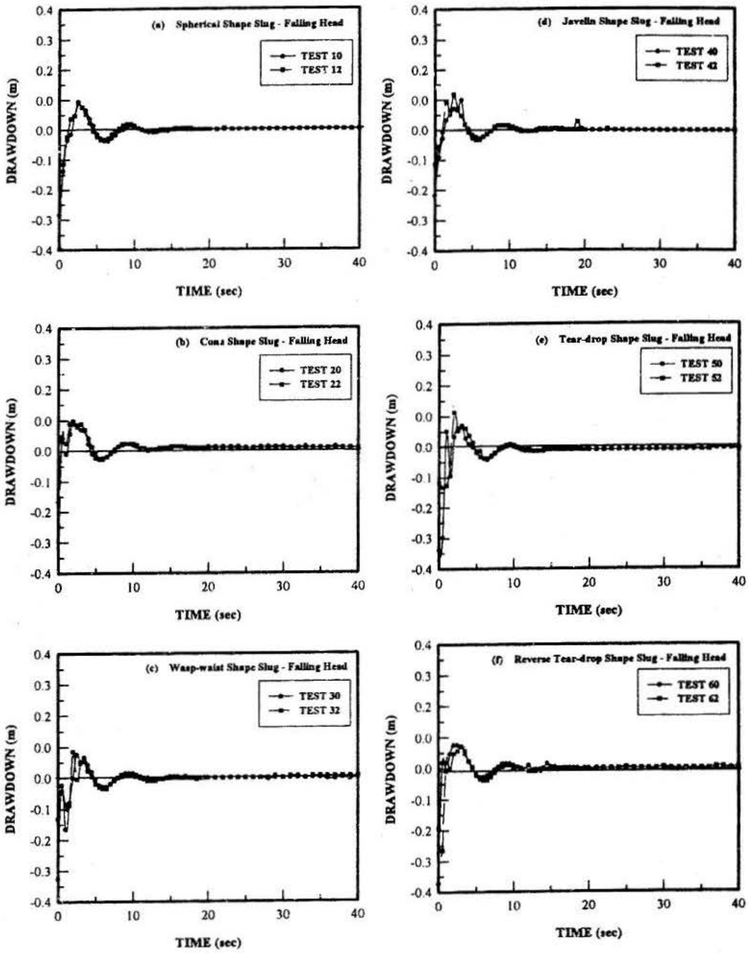


FIGURE 7 : Slug Test Data with Different Shaped Slugs Using the Falling Head Method

of inertia, i.e., when a mass of water is suddenly introduced or extracted from a small diameter well, the water column still has a velocity when it reaches the original water level. As such, the water column will continue to travel beyond the original water level due to its moment of inertia, and (b) casing pipe friction which contributes to the settling down of the water column.

4. The data analysis technique which incorporates the effects of both inertia and friction, and describes the oscillatory water level response, should be used to calculate the hydraulic conductivity (K) of the soil.

Data Analysis Technique

Since the water levels oscillate and stabilize in a very short time, the common data analysis techniques such as the Cooper et al. (1967) method cannot be used to accurately calculate the hydraulic conductivity. Several new rigorous methods are reported in the literature to analyze the oscillating water level data (Van der Kamp, 1976; Kipp, 1985; Springer, 1991). Van der Kamp (1976) developed an approximate method which takes into account the inertia of the water in the well column and describes the oscillatory water level response measured during slug testing. This method supplements the Cooper et al. (1967) method which is applicable only for non-oscillatory water responses during slug testing. The Van der Kamp (1976) method assumes a decaying sinusoid to describe the oscillatory water level response. Because of this assumption,

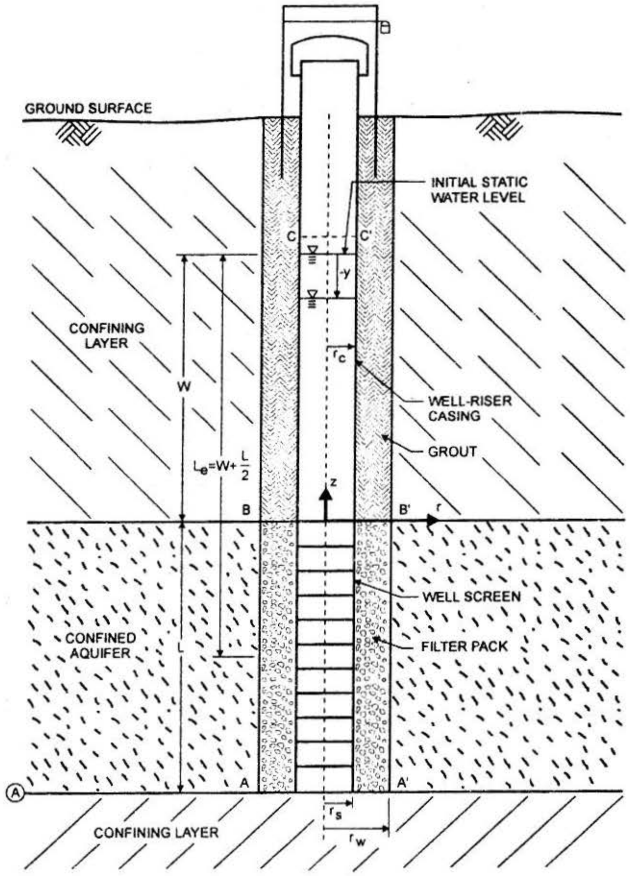


FIGURE 8 : Well Geometry Used in Analysis (modified after Springer, 1991)

this method was found to be inaccurate when the oscillatory response damps out in a short time. In addition, this method does not account for the effect of friction in the well-riser casing which may have significant influence in very small diameter wells or when oscillatory water response damps out very slowly.

Kipp (1985) presented a unified analysis technique, which can be used for the analysis of both the non-oscillatory and oscillatory water level responses. This method takes into account the inertial effects of the water column in the well. When the inertial effects are dominant, oscillatory water level response develops. The application of this method is complicated by the requirement of using appropriate type curves, which are not readily available. In addition, this method does not account for the frictional effects caused by the well-riser casing.

Springer and Gelhar (1991) presented a simplified technique based on the Kipp (1985) method, but incorporated the friction effects of the well-riser casing in order to analyze the oscillatory water level response measured during slug testing. This technique is used to analyze the slug test results obtained from this study.

The well geometry used for the Springer and Gelhar (1991) analysis technique is shown in Fig. 8. This technique assumes that: (1) the aquifer has infinite areal extent, (2) the aquifer is homogeneous, isotropic and has uniform thickness, (3) the potentiometric surface is initially horizontal, (4) the volume of water is injected into or extracted from the well instantaneously, (5) flow to the well is horizontal, and (6) the diameter of well is very small so that storage in the well can be neglected. Based on these assumptions, the oscillatory water drawdown or rise (y_0) in response to the initial water drawdown or rise (y_0) after an elapsed time (t) is mathematically expressed as:

$$\frac{y}{y_0} = \exp\left(-f\bar{t}\right)\left[\cos\left(\beta\bar{t}\right) + \frac{f}{\beta}\sin\left(\beta\bar{t}\right)\right] \quad (1)$$

where

$$\bar{t} = t\sqrt{\frac{g}{L_e}} \quad (2)$$

$$\beta = \sqrt{1 - f^2} \quad (3)$$

$$f = \frac{1}{2}\sqrt{\frac{g}{L_e}}\left[\frac{r_c^2 Y}{4KL} + \frac{2L_e \cos\left(\pi/4\right)\sqrt{\omega\nu}}{g r_c}\right] \quad (4)$$

A detailed explanation of the different terms and derivation of the above equation is provided in Appendix.

Using the new solution defined by Eqn.(1), difficulty arises in constructing type curves in terms of nondimensional drawdown (y/y_0) on log scale and nondimensional time (t') on arithematic scale because of some negative y/y_0 values. In order to circumvent this difficulty, Springer and Gelhar (1991) used squared nondimensional drawdown values $(y/y_0)^2$ for constructing the type curves. An alternative approach for construction of the type curves was followed in this study which involves using the absolute nondimensional drawdown values, which makes it computationally easier. The new type curves using the absolute nondimensional drawdown versus nondimensional time are shown in Fig. 9 for values of damping coefficient f = 0. 1, 0. 2, 0.3, 0.5, 0.6, 0.7, 0.8 and 0. 9. As it is easier to match the peak amplitudes as compared to the entire time-drawdown, lines defining the peaks of the amplitudes are drawn. These sloping lines can then be matched

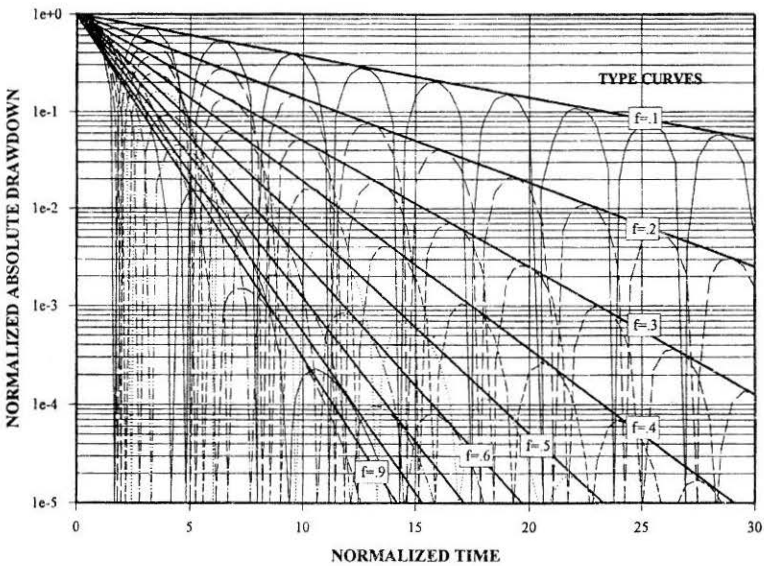


FIGURE 9 : Typical Curves Based on Springer and Gelhar (1991) Analysis Technique

with the time versus drawdown or time versus rise data from actual slug tests to determine the appropriate damping coefficient (f) for the test conditions.

Hydraulic Conductivity Values

The slug test data is analyzed using the new type curves shown in Fig. 9. The procedure used to calculate the hydraulic conductivity of the aquifer using these type curves is as follows:

1. Calculate the initial well drawdown or rise (y_0) based on the volume of the slug used.

2. Divide the measured drawdown or rise (y) by y_0 to calculate either the absolute normalized drawdown or the absolute normalized rise.

3. Convert time into normalized time using Eqn.(2).

4. Plot the normalized time on the x-axis to arithmetic scale and normalized absolute drawdown or rise on the y-axis to logarithmic scale. The plot should be in the same scale as the type curves shown in Fig. 9.

5. Draw a best-fit straight line connecting the peaks of the normalized absolute drawdown or rise.

6. Match this plot with the type curves. To do this, match the lines defining the peaks on the data plot and the type curve. Then, determine the value of f.

7. Calculate the hydraulic conductivity (K) value of the aquifer using the following equation which was obtained by rearranging Eqn.(4):

$$K = \frac{r_c^2 Y}{4L}\left[\frac{2f}{\sqrt{g/L_e}} - \frac{2L_e\left(\pi/2\right)\sqrt{\omega\nu}}{g r_c}\right] \tag{5}$$

where

$$\omega = \sqrt{g/L_e}\cdot\sqrt{1-f^2} \tag{6}$$

and

$$Y = 2\ln\left(\frac{R_e}{r_w}\right) \tag{7}$$

To determine the effective radius (R_e), the following equation derived by Bouwer and Rice (1976) is used:

$$\ln\left(\frac{R_e}{r_w}\right) = \left[\frac{1.1}{\ln\left(\frac{L}{r_w}\right)} + \frac{C}{\frac{L}{r_w}}\right]^{-1} \tag{8}$$

The values of C are given by Bouwer and Rice (1976). Although the Bouwer and Rice (1976) method is applicable for unconfined conditions, Bouwer (1989) reported that the method is also applicable for confined conditions because water displacement in the well during slug testing is dissipated in the vicinity of the well around the screen.

A typical data plot which was generated using this procedure for the standard shape slug is shown in Fig. 10. Similar plots for different shaped

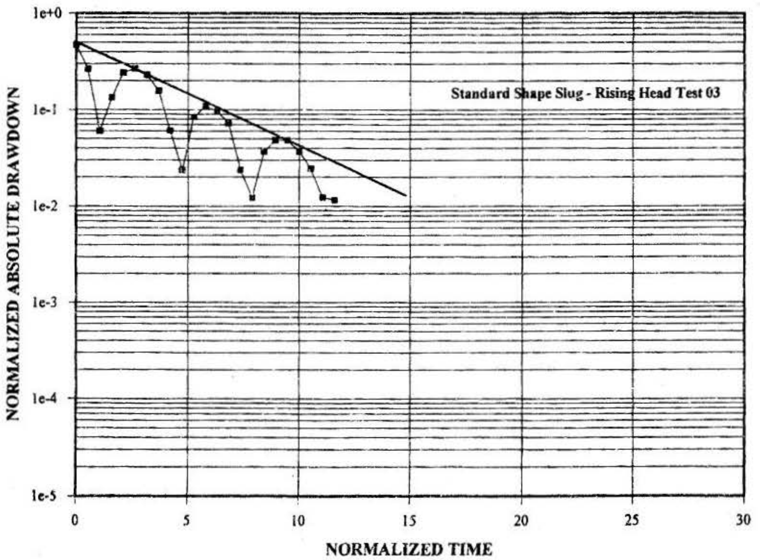


FIGURE 10 : Analysis of Slug Test Data with Standard Shaped Slug by the Rising Head Method

slugs are provided by Reddy et al. (1995). For the value of L/r_w = 107, the value of C calculated from Bouwer and Rice (1976) was equal to 5.5. Substituting this value in Eqn. (8), R, was calculated to be 2.8 m. The values of Y were then calculated using Eqn. (7). All of this data were then substituted into Eqn. (5) to calculate hydraulic conductivity (K) values. The same procedure was followed to calculate the K values from the tests, which used different shaped slugs under both falling head and rising head conditions. A summary of K values which were calculated, is provided in Table 2. As shown in Table 2, the calculated K values for the different shaped slugs were approximately the same. The mean value of K calculated is equal to 3.67×10^{-2} cm/s.

Comparison of Slug Test and Pump Test Results

Pumping tests were performed using Well W-321 as an observation well and Well W-9, which is located 18 m away from Well W-321, as the pumping well. Based on the analysis of this data, the hydraulic conductivity of the soii formation was calculated. The hydraulic conductivity determined at Well W-1321 was 8.4×10^{-2} cm/s. The values of K which were determined from the slug tests ranged from 2.73×10^{-2} cm/s to 5.64×10^{-2} cm/s with an average value

Table 2
Summary of Calculated Hydraulic Conductivity Values

Slug Shape	Test Condition	No. of Tests	Average K (cm/s)
Cone	Falling Head	2	3.32×10^{-2}
	Rising Head	2	4.25×10^{-2}
Spherical	Falling Head	2	3.42×10^{-2}
	Rising Head	2	3.68×10^{-2}
Wasp-waist	Falling Head	2	3.61×10^{-2}
	Rising Head	2	3.89×10^{-2}
Javelin	Falling Head	2	3.26×10^{-2}
	Rising Head	2	3.59×10^{-2}
Tear-drop	Falling Head	2	4.82×10^{-2}
	Rising Head	2	4.00×10^{-2}
Reverse Tear-drop	Falling Head	2	3.07×10^{-2}
	Rising Head	2	4.06×10^{-2}

that is within one half an order of magnitude of the pump test value. This result shows that the analysis technique provides fairly representative values of K for high permeability soils.

Conclusions

Based on this study, the following conclusions can be drawn:

1. The oscillating water level response is a common phenomena that occurs when slug tests are performed in small diameter wells which are installed in highly permeable soils.

2. The shape of the slug does not significantly affect the oscillating water level data for highly permeable formations.

3. The oscillating water response is attributed to the moment inertia of the water column in the well and the friction effect from the well-riser casing.

4. A rigorous analysis technique which accounts for the above mentioned inertia and friction effects should be used to calculate the hydraulic conductivity of the aquifer (K) using slug test data in highly-permeable soils. The K values calculated using the Springer and Gelhar (1991) analysis technique agree reasonably well with the hydraulic conductivity value which was obtained from the pumping tests.

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Appendix
Derivation of Springer and Gelhar (1991) Analytical Solution

To facilitate moment balance determination, the well is divided into two zones: the first zone, which is known as the well screen, is located between planes AA′ and BB′; and the second zone, which is known as the well-riser casing, is located between planes BB′ and CC′ as shown in Fig. 8. For the purpose of this analysis, the plane CC′ is placed above the highest water level attained in the well riser casing, and the radius of the riser pipe is assumed equal to that of the well screen ($r_c = r_s$). The moment change at the junction between the well screen and riser casing is assumed to be insignificant.

For uniform flow through the well screen over the entire aquifer thickness (L), Kipp (1985) showed, based on the mass balance analysis, that the water velocity distribution varies linearly from zero at the screen bottom (AA′) to a value of V_B at the screen top (BB′). Based on this mass balance, the water velocity V(z) at any location (z) along the well screen, with respect to the assumed r-z coordinate system (see Fig. 8), can be expressed as follows:

$$V(z) = \frac{V_B}{L}(L + z) \tag{9}$$

It should be noted here that all force and velocity vectors are positive upward. The momentum balance for the zone between AA′ and BB′ has been expressed by Kipp (1985) as follows:

$$\frac{d}{dt}\int_{-L}^{0} \pi r_s^2 \rho V(z) dz = \left(-\rho V_B^2 - P_B - \rho g L + P_A\right)\pi r_s^2 \tag{10}$$

where ρ = density of water,

g = acceleration due to gravity,

P_A = pressures at the screen bottom

P_B = pressures at the screen top, and

r_s = well screen radius.

Combining Eqn. (9) and Eqn. (10) and then integrating, the momentum balance for the well screen is obtained as follows:

$$\frac{d}{dt}\left(\frac{V_B L}{2}\right) = -V_B^2 - \frac{P_B}{\rho} - gL + \frac{P_A}{\rho} \tag{11}$$

The above Eqn. (11) is equivalent to Eqn. (4g) in Kipp (1985).

The momentum balance for the zone between BB′ and CC′ is expressed as follows (Springer, 1991):

$$\frac{d}{dt}\left[\int_0^{W-y} \rho V(z) \pi r_c^2 \, dz\right] = \begin{Bmatrix} P_B - P_C + \rho V_B^2 \\ -\rho V_C^2 - \rho g(W-y) \end{Bmatrix} \pi r_c^2 - \tau_f 2\pi r_c (W-y) \tag{12}$$

where $V(z)$ = water velocity at any location (z) in the well-riser casing,

P_C and V_C = pressure and velocity above the highest water level in the well-riser, respectively,

r_c = radius of well-riser casing,

W = static water column height above the top of aquifer,

τ_f = friction force per unit wall area of the well-riser casing, and

y = water displacement, either drawdown or rise. For the purpose of analysis, y is defined negative downward (drawdown) and positive upward (rise).

The average water velocity in the well-riser casing is assumed constant. Based on the continuity condition, the water velocity at any location in the well-riser, V(z). However, since there is no momentum flux across CC′, V_C must be equal to zero. Also, the pressure at CC′ is atmospheric, which implies that $P_C = 0$. Substituting these parameters in Eqn. (12) and integrating, the following equation, which is equivalent to Eqn. (4k) in Kipp (1985), is obtained:

$$\frac{d}{dt}\left[V_B\left(W-y\right)\right] = \frac{P_B}{\rho} + V_B^2 - g\left(W-y\right) - \tau_f \frac{2\left(W-y\right)}{\rho r_c} \tag{13}$$

In order to quantify friction in the well-riser casing, Springer (1991) adopted the frictional relationship for oscillatory laminar flow developed by Letelier and Leutheusser (1976), which is mathematically given as:

$$\tau_f = \rho \cos\left(\frac{\pi}{4}\right)\sqrt{\omega \nu}\, V_B \tag{14}$$

where w = frequency of the oscillations, and

x = viscosity of water.

In Eqn. (12), the length over which friction acts is taken equal to (W – y). In order to account for friction in the well screen also, the length over which friction acts is approximated by $L_f = W - y + 0.5 L$. Thus, Eqn. (13) is rewritten as:

$$\frac{d}{dt}\left[V_B\left(W-y\right)\right] = \frac{P_B}{\rho} + V_B^2 - g\left(W-y\right) - 2\left[\frac{\cos\left(\frac{\pi}{4}\right)\sqrt{\omega \nu}}{r_c}\right] L_f\, V_B \tag{15}$$

The combined momentum balance equation for the well screen and riser casing system is obtained by adding Eqn. (11) and Eqn. (15) as given below:

$$\frac{d}{dt}\left[V_B\left(W-y+\frac{L}{2}\right)\right] = g\left(y-W-L+\frac{P_A}{\rho g}\right) - 2\left[\frac{\cos\left(\frac{\pi}{4}\right)\sqrt{\omega \nu}}{r_c}\right] L_f\, V_B \tag{16}$$

Let the original, non-stressed head in the aquifer $h_0 = W$, and the pressure head at the top of well screen $h_s = \left(P_A/\rho g\right) - L$. Substituting these in the second term of above equation:

$$\frac{d}{dt}\left[V_B\left(W-y+\frac{L}{2}\right)\right] = gy+g\left(h_s-h_0\right)-2\left[\frac{\cos\left(\frac{\pi}{4}\right)\sqrt{\omega\nu}}{r_c}\right]L_f\,V_B \qquad (17)$$

Substituting $V_B = -dy/dt$

$$-\frac{d^2y}{dt}\left(W-y+\frac{L}{2}\right) = gy+g\left(h_s-h_0\right)-2\left[\frac{\cos\left(\frac{\pi}{4}\right)\sqrt{\omega\nu}}{r_c}\right]L_f\,V_B \qquad (18)$$

It should be noted that the Kipp (1985) formulation includes the effects of compressible storage in the aquifer, which is why the storage coefficient (S) appears in equations. The formulation presented herein assumes quasi-steady state flow conditions in the aquifer and therefore neglects aquifer storage effects. The storage effects in small-diameter wells with very small water displacements are insignificant, thus, the formulation leads to a simple and reasonably accurate solution. By using Thiem's equation (Freeze and Cherry, 1979):

$$Q = -\frac{2\pi KL\left(h_s-h_0\right)}{\ln\left(\frac{R_e}{r_w}\right)} \qquad (19)$$

Further simplification of this equation leads to:

$$h_s-h_0 = -\left(\frac{Q}{4\pi KL}\right)Y = \frac{r_c^2\,Y}{4KL}\frac{dy}{dt} \qquad (20)$$

where $Y = 2\ln\left(R_e/r_w\right)$, is known as the geometry factor. Substituting Eqn. (20) in Eqn. (18) and rearranging terms:

$$\frac{d^2y}{dt^2}\left(W-y+\frac{L}{2}\right)+2\left[\frac{\cos\left(\frac{\pi}{4}\right)\sqrt{\omega\nu}}{r_c}\right]L_f\frac{dy}{dt}+\frac{r_c^2\,Yg}{4KL}\frac{dy}{dt}+gy = 0 \qquad (21)$$

Let the effective water column length (L) is equal to $W+0.5L$. Since water level displacements (y) are small, $W-y+0.5L = L_e$ and $L_f = L_e$. Thus, Eqn. (21) reduces to:

$$\frac{L_e}{g}\frac{d^2y}{dt^2}+\left[\frac{r_c^2\,Y}{4KL}+\frac{2L_e\cos\left(\frac{\pi}{4}\right)\sqrt{\omega\nu}}{g\,r_c}\right]\frac{dy}{dt}+y = 0 \tag{22}$$

Define time in nondimensional form:

$$T_0 = \sqrt{\frac{L_e}{g}} \tag{23}$$

and

$$\bar{t} = \frac{t}{T_0} = t\sqrt{\frac{g}{L_e}} \tag{24}$$

In terms of nondimensional time, Eqn. (22) is given by:

$$\frac{d^2y}{d\bar{t}^2}+2f\frac{dy}{dt}+y = 0 \tag{25}$$

where

$$f = \frac{1}{2}\sqrt{\frac{g}{L_e}}\left[\frac{r_c^2\,Y}{4KL}+\frac{2L_e\cos\left(\frac{\pi}{4}\right)\sqrt{\omega\nu}}{g\,r_c}\right] \tag{26}$$

Equation (25) is similar to the classic free-force vibration problem in which f represents the damping coefficient. Three possible solutions are possible for Eqn. (17): (1) $f > 1$ overdamped condition, (2) $f = 1$ critical damping condition, or (3) $f < 1$ underdamped condition. The decreasing oscillatory conditions are representative of the underdamped condition. The solution for the underdamped case ($f < 1$) with initial conditions of:

$$\frac{dy}{dt}(\bar{t}=0) = 0, \quad y(\bar{t}=0) = y_0 \tag{27}$$

is given by:

$$\frac{y}{y_0} = \exp(-f\bar{t})\left[\cos(\beta\bar{t}) + \frac{f}{\beta}\sin(\beta\bar{t})\right] \tag{28}$$

where

$$\beta = \sqrt{1-f^2} \tag{29}$$