Indian Geotechnical Journal, 26 (4), 1996

Technical Note

A Note on the Back Analysis of Slope Failure

G. Bhattacharya* and P.K. Basudhar†

Introduction

In geotechnical engineering, back analysis is frequently used for the estimation of soil and rock parameters. As the analysis is generally carried out in an ad-hoc manner, controversial issues are generated (Leroueil and Tavenas, 1981). Optimisation techniques are now frequently used to evolve a systematic and rational approach for estimation of such parameters (Gioda and Sakurai, 1987). Mathematical programming formulation of the analysis has certain advantages. For instance, depending on the crude estimation of the concerned parameters, prior information regarding the bound within which these are most likely to lie, may be imposed easily as side constraints.

The geotechnical parameters are the shear strength parameters (c', ϕ'), the pore water pressure coefficient (r_u) and the soil unit weight (γ) for all the concerned strata. In such an analysis the numbers of unknown parameters that are to be estimated, are generally more than the number of equations that can be derived from static equilibrium; so the problem of parameter estimation is under-determined and a unique solution can not be obtained. This fact has been recognised by Sauer and Fredlund (1988); they have stated that it is not possible to back analyse a landslide and obtain a unique combination of effective cohesion and effective angle of shearing resistance. The efforts in this direction are generally limited to the back calculation of only two geotechnical parameters namely, the shear strength parameters c' and ϕ'. Nguyen (1984a, b) has applied nonlinear programming approaches namely, the simplex reflection technique and the secant method to the back calculation of slope failure. He has reported that such techniques, also do not provide unique solutions. In spite of this realisation, there has been some

---

* Assistant Professor, Civil Engineering Department, Bengal Engineering College, Howrah – 711103, West Bengal, India.

† Professor, Civil Engineering Department, Indian Institute of Technology, Kanpur – 208016, U.P., India.

attempts toward finding a unique solution. Saito (1980) has recommended a numerical-graphical procedure for determination of c′ and ϕ'. But his procedure is based on the Ordinary method of slices applicable to circular failure surfaces only.

Yamagami and Ueta (1987) have presented a procedure based on nonlinear programming technique for the back analysis of strength parameters. They formulated the problem as a constrained optimisation problem with mixed equality and inequality constraints. The objective function has been chosen as the sum of the squares of the difference between the corresponding ordinates of the known failure surface and the chosen shear surface. In this procedure, starting from an arbitrary set of values for the design parameters, the minimisation of the objective function is carried out and the corresponding critical shear surface is determined. But, unless the chosen parameters are true, the critical shear surface so determined will not match with the actual failure surface. In other words, only those values of parameters which results in a critical shear surface coinciding with the failure surface are the true or actual values. The whole idea is that in order to ensure a correct solution the back analysed parameters should not result in a factor of safety equal to unity at the incipient failure condition but also yield a critical slip surface which perfectly matches with the actual failure surface. This is an improvement over the early practice of arbitrarily varying these parameters in order to match the given value of factor of safety of unity, completely disregarding the fact that several combinations of the parameters are possible, all of which yield the same value of factor of safety (the value being equal to unity); but each combination will result in a different critical slip surface, some of which may be widely different from the actual failure surface.

To pick the right combination, the early practice was to apply engineering judgement. In spite of the relevance of prudent engineering judgement in such an analysis, these alone can not be the panacea to obtain the best possible solution for which it is essential to match the obtained critical slip surface with the actual failure surface. Dhawan (1986) has also stressed on this point. Based on several studies using the program SUMSTAB (Satyam Babu, 1986; Basudhar et al., 1988), he has observed that the critical slip surface of a slope is most sensitive to c′ and relatively less sensitive to ϕ' and r_u. He, therefore, concludes that the conventional procedure of back analysis by arbitrarily varying c′ and ϕ' to match F equal to unity without matching the surface, is likely to give erroneous results. He recommended that as ϕ' and r_u can be estimated with reasonable accuracy from laboratory and in situ tests respectively, it is better to vary c′ and match the predicted critical slip surfaces with the observed failure surface for good prediction of c′. However, manual matching of the predicted critical slip surface observed in the field is very tedious and monotonous and, as such, prone to errors.

Even though the Yamagami and Ueta's method (1987) is in the right direction, there is a need to iterate for several critical shear surfaces during the optimisation process. This is avoidable as the shear surface is known and hence there is no need to iterate the matching it with the actual failure surface. The problem of indeterminacy has not been addressed in this study.

Thus, there is a need for an effective and efficient procedure for back analysis addressing all these aspects. In this paper, an attempt has been made to develop a back analysis algorithm based on the Janbu's generalised procedure of slices coupled with the sequential unconstrained minimisation technique wherein no iteration is needed to match the predicted and the observed slip surface. Furthermore, as the problem is in general under-determined, the importance of imposing some side constraints defining the interval within which the parameters are most likely to lie are studied.

Principal of the Proposed Method of Back Analysis

Statement of the Inverse Problem

Find, for a given slope with a given slip surface and the associated factor of safety, the value of the concerned geotechnical parameters which yield the same value of the factor of safety subject to some physical and behavioural constraints.

Problem Formulation

The geotechnical parameters, such as c', ϕ' and r_u to be estimated from the back analysis must satisfy the following relationship :

$$F = F_o$$

where F stands for the factor of safety expression used in the analysis and F_o is the factor of safety of the given slip surface. For a slope at the incipient failure condition, F_o is equal to unity.

The back analysis problem then can be formulated as one of nonlinear constrained optimisation as follows :

$$\text{Find } D = \left[c',\ \phi',\ r_u\right]^T \tag{1}$$

such that

$$f(D) = (F - F_o)^2 \rightarrow \text{Min.} \tag{2}$$

subject to the constraints :

$$c'_{min} \leq c' \leq c'_{max} \tag{3}$$

$$\phi'_{min} \leq \phi' \leq \phi'_{max} \tag{4}$$

$$\left(r_u\right)_{min} \leq r_u \leq \left(r_u\right)_{max} \tag{5}$$

The inequalities given by the Eqns. 3, 4 and 5 are more explicitly expressed as follows :

$$c' - c'_{max} \leq 0 \tag{6a}$$

$$c'_{min} - c' \leq 0 \tag{6b}$$

$$\phi' - \phi'_{max} \leq 0 \tag{7a}$$

$$\phi'_{min} - \phi' \leq 0 \tag{7b}$$

$$\left(r_u\right) - \left(r_u\right)_{max} \leq 0 \tag{8a}$$

$$\left(r_u\right)_{min} - \left(r_u\right) \leq 0 \tag{8b}$$

The Eqns. 6, 7 and 8 can be written in a general form as follows :

$$g_j(D) \leq 0; \qquad j = 1, 2 \ldots\ldots, n \tag{9}$$

The subscript min. and max. refer to the anticipated minimum and maximum values of the unknown parameters respectively. These may be obtained from shear test results, visual examination and experience. As the problem is under-determined, the imposition of the side constraints are necessary; these are effective in guarding against situations wherein the design variables may become highly unreasonable during iteration apart from saving a lot of unnecessary computations by restricting the search zone.

In this formulation, as there is no need to iterate for finding the critical slip surface, the number of design variables are generally reduced and thus a lot of computation is saved.

Method of Solution

The optimisation problem as stated in the above section can then be cast in a general form as follows :

Find D_m such that,

$$F(D_m) \text{ is minimum} \tag{10}$$

$$\textit{subject to } g_j(D_m) \le 0 \tag{11}$$

A composite function $\psi(D, r_k)$ is constructed as follows :

$$\psi(D, r_k) = F(D) - r_k \sum \frac{1}{g_j(D)} \tag{12}$$

where r_k is called the penalty parameter. Interior penalty function method is used to carry out the sequential unconstrained minimisation of the composite function for decreasing value of the penalty parameter using the Powell's method and the quadratic interpolation technique for multidimensional and one dimensional unconstrained searches respectively. In each successive stage r_k is generally reduced to one tenth of its value in the previous step, till a convergent solution is obtained. These techniques are available in standard text book on optimisation (Fox, 1971) and, therefore, not discussed here.

Even for a homogenous slope in a general back analysis, there are more than two design parameters. More the number of design parameters, more will be the number of possible combinations which will yield the same factor of safety for the given slip surface. As such, the studies conducted in this paper have been initially restricted to the two parameter back analysis. Subsequently, the accuracy of the method in back calculating more than two parameters has been investigated.

Results and Discussion

Illustrative Example

An example problem of homogenous slope has been chosen to illustrate the proposed method (Fig. 1). The slip surface (not necessarily the critical slip surface) and the corresponding factor of safety value are known (from a forward or conventional stability analysis carried out earlier using the given values for the geotechnical parameters). With respect to the known slip surface, the slope is stable i.e., the safety factor is greater than unity.

Since the procedure is exactly the same for both failed and stable slope, it does not make any difference whether the slope in the example problem is a failed slope or not. To validate the proposed numerical method, the factor of safety of the given slope corresponding to the given slip surface

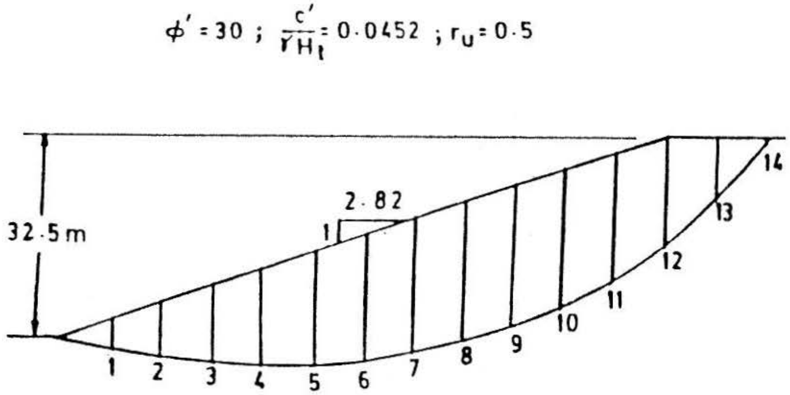

FIGURE 1 : **Slope Section of the Example Problem (After Spencer, 1973)**

and geotechnical parameters has been found out and then the proposed method of back analysis has been applied to check whether it is possible to get back the same values of parameters for that slip surface.

The slope section as shown in Fig. 1 is the one that was taken up by Spencer (1973) to demonstrate his method of stability computations. The soil and pore pressure properties are also shown in the figure. Assuming $\gamma = 20$ kN/m^3 for the given height of the slope (H_t = 32.5 m) the effective cohesion (c') corresponding to the stability number equal to 0.0452 is calculated as 29.38 kPa.

Assuming parallel interslice forces the non-circular shear surface shown in the figure has been analysed by Spencer (1973). The reported value of the factor of safety of the slope is 1.42.

To facilitate the convergence during optimisation, the design variable has been normalised as follows :

Variable	c' (kPa)	ϕ' (deg.)	r_u
Normalized form	$\frac{c'}{\lambda \gamma H_t}$	$\frac{\pi \phi'}{180}$	r_u

λ is a parameter so chosen that the value of the normalised effective cohesion lies between 0.1 and 1.0. In the present study it has been taken to be 0.1.

The side constraints that have been imposed are as follows unless otherwise specified. Units of the effective cohesion is kPa.

$$0.0 \leq c' \leq 100.0 \tag{13a}$$

$$\phi' \geq 0.0 \tag{13b}$$

$$0.0 \leq r_u \leq 0.55 \tag{13c}$$

The line of thrust has been chosen in accordance with Janbu's (1973) suggestions; the same is presented in Table 1. The estimated value of the factor of safety for the given slope using 14 slices and Janbu's GPS (1973) is 1.4668.

For having some physical significance, the lower bound values for the constraints are justified as the design parameters can not be less than zero. The upper bound values for the same may be roughly estimated from very simple laboratory or field tests. No upper limit has been imposed on ϕ', because ϕ' generally does not play any significant role on the position of the critical slip surface. However, it can be easily incorporated in the analysis if it is so desired, without affecting the basic flow of the developed technique and ease of obtaining the solution.

Two-Parameter Back Analysis

Case I : When c' and ϕ' are the unknowns

To start with, a two-parameter back analysis has been attempted. The two parameters considered are c' and ϕ'; it is assumed that r_u and γ are known for this analysis in addition to the shape and location of the shear surface and its factor of safety. As it has already been pointed out that the location of the critical slip surface is dominated by the value of the c', studies have been conducted to find the effect of the uncertainty interval of the effective cohesion on the results.

The results are presented in Table 2. it is observed that unique solution cannot be obtained and the closeness of a solution with the correct solution depends on the narrowness of the uncertainty interval of the design

TABLE 1 Line of Thrust Assumed in the Analysis

Interslice Boundary Number	1	2	3	4 to 12	13
Line of Thrust Position, h_t/z	0.34	0.35	0.36	0.38	0.43

h_t = distance between the shear surface and the line of thrust along an interslice boundary

z = height of an interslice boundary

TABLE 2 Results of Two Parameter (c', ϕ') Back Analysis of the Slope Section

Parameter/ Function	Initial Value	Back Analysed Values				Actual Value
		Set 1 $0 < c' < 150$	Set 2 $0 < c' < 100$	Set 3 $0 < c' < 75$	Set 4 $0 < c' < 60$	
c' (kPa)	10.0	42.65 (45.2)	37.64 (28.1)	29.64 (0.88)	29.38 (0.0)	29.38
ϕ' (degrees)	15.0	25.75 (14.2)	27.35 (8.8)	29.92 (0.3)	30.00 (0.0)	30.00
F	0.7321	1.46698	1.46698	1.46696	1.4668	1.4668
Ψ	0.5398	0.0	0.0	0.0	0.0	

F = Factor of Safety

Ψ = Composite Function (Equation 12)

parameters. The narrower the interval, the more is the likelihood of converging to the actual solution. As shown in the Table, studies have been carried out varying the value of c'_{max}, the upper limit on c'. It is seen that by gradually narrowing down the interval for c' from an initial c'_{max} of 150 kPa, a solution very close to the actual solution has been obtained when c'_{max} is equal to 75 kPa; but a solution identical to the input parameters has been obtained when c'_{max} is equal to 60 kPa. Considering that 60 kPa is more than two times the actual value of 29.38 kPa, it is expected that such a crude estimate may not be difficult to make from the laboratory tests conducted on soil samples collected from the shear zone. As already stated, no such upper limit on ϕ' has been imposed as it has been observed that the cohesion value predominantly controls the predicted response.

The values within the parentheses show the percentage difference of the predicted values of the design parameters from the actual values. It is seen that the solution corresponding to the constraint set no. 4 is identical to the true solution. This highlights the importance of imposing such constraints based on a priori estimation of error bound, however approximate it may be, to achieve meaningful and correct solution.

Case II : When c' and r_u are unknowns

As the friction angle of soils can be determined with better accuracy in the laboratory from samples collected from the field, it can be treated to

TABLE 3 Results of Two Parameter (c', r_u) Back Analysis of the Slope Section

Parameter/ Function	Initial Value		Back Analysed Values		Actual Value
	Set 1	Set 2	Set 1	Set 2	
c' (kPa)	10.00	20.00	18.4750 (37.1)	26.7800 (8.8)	29.38
r_u	0.35	0.35	0.4385 (12.3)	0.4851 (2.98)	0.5
F	1.6054	1.7010	1.4669	1.4669	1.4668
Ψ	0.0192	0.0548	0.0	0.0	—

N.B. For Set 1 : $0 \leq c' \leq 75$; $0.3 \leq r_u \leq 0.55$

Set 2 : $20 \leq c' \leq 75$; $0.3 \leq r_u \leq 0.55$

be known; then the back analysis can be carried out with r_u as unknown along with c'. It can be seen from Table 3 that the dependence of the solution on the bounds imposed on the variables is more pronounced in this case than the previous one.

The studies conducted show that for an accurate analysis the search domain has to be narrowed down further by using good engineering guess and judgement about the existing soil and pore pressure conditions. From the experience gained in the previous analysis involving c' and ϕ', the upper limit for c' has been kept at 75.0 as before. As far as the pore pressure coefficient r_u is concerned, the lower and upper limits have been placed as 0.3 and 0.55 respectively for both the experiments. The lower limit on c' was kept 0.0 in the first set. Observing that this has resulted in a very small value of c', it has been raised to 20.0 in the second set. Though not accurate, a reasonably close result has been obtained in the second case as presented in Table 3, the maximum error in c' and r_u being equal to 8.8% and 2.98% respectively instead of the corresponding values for the set 1.

Three-Parameter Back Analysis (c', f', r_u) :

From the results presented in Table 4 it can be seen that the observations made earlier also hold in the case of the 3-parameter back analysis to find c', ϕ' and r_u. The variables have to be given narrow ranges of variation to obtain an accurate solution. The results show that a reasonable solution has been obtained for set 2 with lower limit of 0.3 on r_u in addition

TABLE 4 Results of Three Parameter (c', ϕ', r_u) Back Analysis

Parameter/ Function	Initial Value		Back Analysed Values		Actual Value
	Set 1	Set 2	Set 1	Set 2	
c' (kPa)	10.00	10.00	28.425 (3.3)	29.20 (0.8)	29.38
ϕ'	15.00	15.00	21.80 (27.3)	26.45 (11.6)	30.00
r_u	0.30	0.35	0.295 (41.0)	0.4269 (14.6)	0.50
F	0.9804	0.9178	1.4669	1.4669	1.4668
Ψ	0.2367	0.3015	0.0	0.0	—

N.B. For Set 1 : $0 \leq c' \leq 75$; $\phi' \geq 0$; $0 \leq r_u \leq 0.55$
Set 2 : $0 \leq c' \leq 75$; $\phi' \geq 0$; $0.3 \leq r_u \leq 0.55$

to the upper limits on c' and r_u. A lower limit of 0.0 has been used for c'. The maximum error is in the prediction of r_u and is equal to 14.62%. A better control on c' may result in achieving a better solution.

Conclusions

From the limited studies presented in this paper the following conclusions are drawn :

1. Back analysis of failed slopes in order to estimate the values of the geotechnical design parameters involves more unknowns than the number of equations that can be derived from the consideration of static equilibrium. As such, realistic solutions cannot be obtained without imposing side constraints.

2. In the proposed scheme of back analysis there is no need to search for the critical slip surface and match it with the given slip surface to obtain the design parameters. The adopted method of slope stability computation is more general and more rigorous than the ones that have been used so far.

3. The solution is sensitive to the prescribed domain of the parameters, c' and r_u.

4. The studies conducted to back predict more than two geotechnical parameters for homogenous slopes indicate that a solution with errors within tolerable limits can be achieved with proper imposition of side constraints.

5. If meaningful upper and lower bounds on the geotechnical parameters are found out from the field and laboratory investigations of the slide zones and are used in the back analysis, realistic values of design parameters can be obtained.

References

BASUDHAR, P.K., YUDHBIR and BABU, N.S. (1988) : "SUMSTAB – A Computer Software for Generalised Stability Analysis of Zoned Dams", *Numerical Methods in Geomechanics (Innsbruck)*, Swoboda (Ed.), Balkema, Rotterdam.

BHATTACHARYA, G. (1990) : "Sequential Unconstrained Minimisation Technique in Slope Stability Analysis", *Ph.D. Thesis*, I.I.T. Kanpur.

DHAWAN, R.K. (1986) : "Parametric Studies in Slope Stability using SUMSTAB Package", *M.Tech. Thesis*, I.I.T. Kanpur.

FOX, R.L. (1971) : *Optimisation Methods for engineering Design*, Addison-Wesley, Reading, Mass.

GIODA, G. and SAKURAI, S. (1987) : "Back Analysis Procedure for the Interpretation of field Measurements in Geomechanics", *Int. J. for Num. and Analyt. Methods in Geomech.*, 11, 555-583.

JANBU, N. (1957) : "Earth Pressure and Bearing Capacity Calculations by Generalised Procedure of Slice", *Proc. of the 4th Int. Conf. on Soil Mechanics and Foundation Engineering*, London.

JANBU, N. (1973) : *Slope Stability Computation, Embankment Dam Engineering, Casagrande*, Volume Edited by R.C. Hirschfeld and S.C. Poulos, John Wiley & Sons, New York, 47-86.

LEROUEIL, S. and TAVENAS, F. (1981) : "Pitfalls of Back Analysis", *Proc. 10th Int. Conf. on Soil Mechanics and Foundation Engineering*, Stockholm, 1, 185-190.

NGUYEN, V.U. (1984a) : "A Technique for the Back Analysis of slope Failures", *Proc. of the Fourth Australia-New Zealand Conf. on Geomechanics*, Perth, 617-622.

NGUYEN, V.U. (1984b) : "Back Calculation of Slope Failures by the Secant Method", *Geotechnique*, 34, No.3, 423-427.

SAITO, M. (1980) : "Reverse Calculation Method to Obtain c and ϕ on a Slip Surface", *Proc. Int. Symp. on Landslides*, Delhi, 2, 281-284.

SATYAM BABU, N. (1986) : "Optimisation Technique in Stability Analysis of Zoned Dams", *M.Tech. Thesis*, I.I.T. Kanpur.

SAUER, E.K. and FREDLUND, D.G. (1988) : Effective Stress Limit Equilibrium Back-Analysis of Failed Slopes : Guidelines, Landslides – Glissements de terrain, 1, Christopher Bonnard ed., *Proc. of the 5th Int. Symp. on Landslides*, Balkema, Rotterdam.

SPENCER, E. (1973) : "The Thrust Line Criterion in Embankment Stability Analysis", *Geotechnique*, 23, 85-101.

YAMAGAMI, T. and UETA, Y. (1987) : "Optimisation Techniques for Back-Analysis of Failed Slopes", *Proc. of the 8th Asian Regional Conference*, Kyoto, 1, 513-516.