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Finite Element Consolidation Model for Subsidence Problems Based on Biot's Three Dimensional Theory

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Introduction

Subsidence is a phenomenon which is due to extraction of fluid or solid from underground that can cause ground surface deformation. In this study, subsidence was studied only due to groundwater withdrawal. In recent years, it has been confirmed that the land subsidence in Kerman province is due to consolidation of soil layers, caused by extensive groundwater withdrawal for agricultural development which introduced large land settlement and earth fissures. Up to date no one studied the prediction and simulation of settlement in the region, but some uncompleted reports are available which measured to some extent the settlement of ground and water table declination. In order to simulate and predict such phenomenon in a given aquifer, the authors developed an axisymmetric fully coupled finite element model, to show that it is possible to obtain satisfactory results. Formulation of finite element was based on Biot's three dimensional consolidation theory. As excess effective stress due to water withdrawal in whole scale is small, behaviour of soil skeleton was assumed to be elastic, but it should be noted that pore water pressure variation is still function of time, depth and other properties and boundary conditions.

Finite Element Formulation

The basic formulation presented here is based on Biot's consolidation theory. In the theory of Biot, the soil skeleton is treated as a porous elastic solid and the laminar pore fluid are coupled by the conditions of compressibility and of continuity.

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In computations cylindrical coordinates were assumed, and when water is pumped out from the aquifer through wells, both radial and axial flow can take place, which are symmetric. In order to simulate this condition by finite element the exact behaviour should be achieved by actual mathematical equations. For each reason Biot's governing equation was selected, which is:

$$c_r\left(\frac{\partial^2 u_c}{\partial r^2}+\frac{1}{r}\frac{\partial u_c}{\partial r}\right)+c_z\frac{\partial^2 u_c}{\partial z^2} = \frac{\partial u_c}{\partial t}-\frac{\partial p}{\partial t} \tag{1}$$

where
- u_e = excess pore water pressure,
- p = mean total stress,
- z and r = axial and radial directions,
- t = time, and
- c_r, c_z = coefficient of consolidation in radial and axial directions, respectively.

The equilibrium equation with assumption of zero volumetric force, can be written as follows :

$$\begin{aligned}\frac{\partial \sigma'_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\partial u_e}{\partial r} &= 0\\ \frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma'_z}{\partial z}+\frac{\partial u_e}{\partial z} &= 0\end{aligned} \tag{2}$$

the stress-strain relations for such condition can be written as follows :

$$\begin{Bmatrix}\sigma'_r\\ \sigma'_z\\ \tau_{rz}\\ \sigma'_\theta\end{Bmatrix} = \frac{E(1-\nu)}{(1-\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{(1-\nu)} & 0 & \frac{\nu}{(1-\nu)}\\ \frac{\nu}{(1-\nu)} & 1 & 0 & \frac{\nu}{(1-\nu)}\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0\\ \frac{\nu}{(1-\nu)} & \frac{\nu}{(1-\nu)} & 0 & 1\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \gamma_{rz}\\ \varepsilon_\theta\end{Bmatrix} \tag{3}$$

where
- E = modules of elasticity,
- v = Poisson's ratio,
- σ' = effective stress,
- ε = strain, and

$$\begin{Bmatrix} q_r \\ q_z \end{Bmatrix} = \frac{1}{\gamma_w} \begin{bmatrix} k_r & 0 \\ 0 & k_z \end{bmatrix} \begin{Bmatrix} \frac{\partial u_e}{\partial r} \\ \frac{\partial u_e}{\partial z} \end{Bmatrix} \tag{4}$$

where q_r, q_z = volumetric flow rates per unit area into and out of the element,

k_r, k_z = coefficient of permeability in redial and axial *directions, respectively.*

For fully saturated soil and incompressible fluid condition, outflow from an element of soil equals the reduction in volume of element. Hence :

$$\frac{\partial q_r}{\partial r} + \frac{\partial q_z}{\partial z} = \frac{d}{dt}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) \tag{5}$$

where u and v = displacements in r and z directions, respectively.

Combining Eqns. (4) and (5) :

$$\frac{k_r}{\gamma_w}\frac{\partial^2 u_e}{\partial r^2} + \frac{k_z}{\gamma_w}\frac{\partial^2 u_e}{\partial z^2} + \frac{d}{dt}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) = 0 \tag{6}$$

As usual in a displacement method σ, ε are eliminated in terms of u and v so that the final coupled variables are u, v, u_e.

These are now discretized in the normal way :

$$\begin{aligned} u &= \mathbf{N\,u} \\ v &= \mathbf{N\,v} \\ u_e &= \mathbf{N\,u_e} \end{aligned} \tag{7}$$

where N is the vector of shape function.

When discretization and the Galerkin process are completed, Eqns. (2) and (6) lead to the pair of equilibrium and continuity equations, which are:

$$\begin{aligned} \mathbf{KMr} + \mathbf{Cu_e} &= \mathbf{F} \\ \mathbf{C}^T \frac{d\mathbf{r}}{dt} - \mathbf{KPu_e} &= 0 \end{aligned} \tag{8}$$

where, for a four-nodded element,

$$\mathbf{r} = \{u_1, \; v_1, \; u_2, \; v_2, \; u_3, \; v_3, \; u_4, \; v_4\}^T$$
$$\mathbf{u_e} = [u_{e1}, \; u_{e2}, \; u_{e3}, \; u_{e4}]^T \tag{9}$$

KM is the elastic stiffness matrix and is

$$\mathbf{KM} = \iint \mathbf{B}^T \mathbf{D} \mathbf{B} \, r \, dr \, dz \tag{10}$$

where, B = AN,
N = vector of shape function, and

$$\mathbf{A} = \begin{Bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{Bmatrix} \tag{11}$$

KP is the fluid stiffness matrix is

$$\mathbf{KP} = \iint \left(c_r \frac{\partial \mathbf{N}_i}{\partial r} \frac{\partial \mathbf{N}_j}{\partial r} + c_z \frac{\partial \mathbf{N}_i}{\partial z} \frac{\partial \mathbf{N}_j}{\partial z} \right) r \, dr \, dz \tag{12}$$

C is a rectangular coupling matrix, can be written as follows :

$$\mathbf{C} = \iint \mathbf{N}_i \frac{\partial \mathbf{N}_j}{\partial r} \, dr \, dz \tag{13}$$

and F is the external loading vector.

Equation (8) must be integrated in time. To integrate Eqn. (8) with respect to time, there are many methods available, but we consider only the simplest linear interpolation in time using finite differences, thus :

$$\theta \mathbf{K} \mathbf{M} \mathbf{r}_1 + \theta \mathbf{C} \mathbf{u}_{e1} = (\theta - 1) \mathbf{K} \mathbf{M} \mathbf{r}_0 + (\theta - 1) \mathbf{C} \mathbf{u}_{e0} + \mathbf{F}$$
$$\theta \mathbf{C}^T \mathbf{r}_1 - \theta^2 \Delta t \mathbf{K} \mathbf{P} \mathbf{u}_{e1} = \theta \mathbf{C}^T \mathbf{r}_0 - \theta(\theta - 1) \Delta t \mathbf{K} \mathbf{P} \mathbf{u}_{e0} \tag{14}$$

In above equations, if $\theta \geq 0.5$ the system will be stable without any condition, in the Crank-Nicolson type of approximation, θ is made equal to 0.5, or in the Galerkin approximation θ is equal to 0.67. By using $\theta = 0.5$

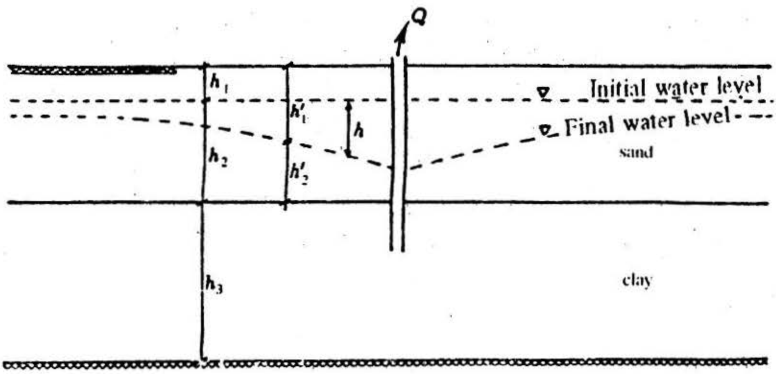

FIGURE 1 : Water Level Decline due to Ground Water Withdrawal

in Crank-Nicolson method, Eqn. (14) can be written as follows :

$$\begin{bmatrix} \mathbf{KM} & \mathbf{C} \\ \mathbf{C}^T & -\frac{\Delta t}{2}\mathbf{KP} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n+1} \\ \mathbf{u}_{e_{n+1}} \end{bmatrix} = \begin{bmatrix} -\mathbf{KM} & -\mathbf{C} \\ \mathbf{C}^T & \frac{\Delta t}{2}\mathbf{KP} \end{bmatrix} \begin{bmatrix} \mathbf{r}_n \\ \mathbf{u}_{e_n} \end{bmatrix} + \begin{bmatrix} 2\,\mathbf{F} \\ 0 \end{bmatrix} \qquad (15)$$

Therefore values of unknown can be calculated at time $t = t_{n+1}$ based on known parameters at time $t = t_n$. For initial conditions at time $t = 0$ all values are known.

After finding governing matrix equations for a single element, the assembled matrices for total elements can be obtained and boundary conditions can be introduced. Solving such equations at any time, horizontal and vertical deformations (u,v) at various nodal points can be found and finally stress and strain values for each element can be calculated.

Estimation of Vertical Load

The equivalent external load due to water table decline can be computed from Fig. 1. If water table drops to be equal h, then :

$$\begin{aligned} h &= h_1' - h_1 = h_2 - h_2' \\ \sigma_{v0}' &= \gamma_{sand}\, h_1 + (\gamma_{satsand} - \gamma_w)\, h_2 \\ \sigma_{v1}' &= \gamma_{sand}\, h_1' + (\gamma_{satsand} - \gamma_w)\, h_2' \\ &= \gamma_{sand}(h + h_1') + (\gamma_{satsand} - \gamma_w)(h_2 - h_1) \qquad (16) \\ \Delta\sigma_v' &= \sigma_{v1}' - \sigma_{v0}' = \left[\gamma_{satsand} - (\gamma_{satsand} - \gamma_w)\right] h \end{aligned}$$

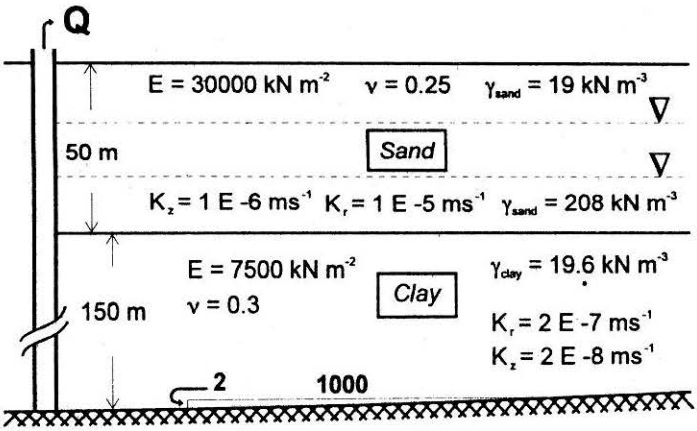

FIGURE 2 : Soil Profile of Rafsanjan Aquifer

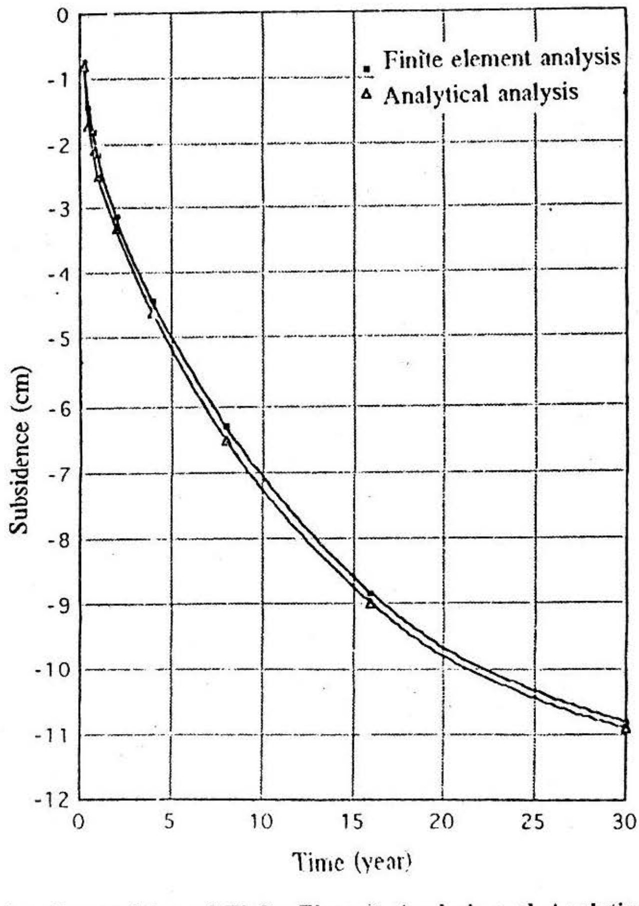

FIGURE 3 : Comparision of Finite Element Analysis and Analytical Analysis

where σ'_{vo} = initial vertical effective stress,
σ'_{v1} = final vertical effective stress, and
$\Delta\sigma'_{v}$ = estimated vertical load at top layer of clay.

Numerical Results

Formulation of finite element analysis for subsidence problem was discussed in previous section. A computer program was developed to predict and examine various soil behaviour and conditions. In order to verify the computer model, analysis for simple behaviour such as one dimensional consolidation was performed. As an example for examination of model, properties of Rafsanjan aquifer were considered which is given in Fig. 2. It should be noted that values of E and other material properties can be varied in depth or other directions. Values of c_r, c_z are functions of k, E, ν, γ and Δt was chosen from 30 minutes to one day depend on required accuracy and the problem.

For complete study four stages of analysis were performed in this research. It is assumed at first stage of analysis that water table suddenly drops by about one meter. A section with height of 200 meters and width of 1000 meters was discretized to 160 rectangular elements with 189 nodes. Comparison of results based on finite element (assuming 1-D flow) and classical analysis are shown in Fig. 3. It can be seen that a good correlation exists between the results. It should be noted that for analysis based on classical consolidation theory water flows only in vertical direction.

At second stage of this study, it is assumed that water table suddenly drops by about one meter similar to previous stage but, water flows in axial and radial directions under axi-symmetric conditions. This simulation is very close to actual field conditions under pumping of groundwater through wells. At this condition water flows in three dimensions, and consolidation process and water drainage occur faster than single drainage. The analysis for this case is shown in Fig. 4 accompanied by previous condition.

It can be seen from Fig. 4 that subsidence rate at early period of consolidation is higher for three dimensional than one dimensional consolidation. But at later period of consolidation the final results are quite similar. This is because in three dimensional condition, pore water pressure can dissipate quicker in different directions.

By using Biot's three dimensional equation of consolidation it is also possible to estimate differential settlement at various distances from wells. Figure 5 shows subsidence versus distance after one year for sudden water table drop of one meter. It can be seen that higher settlement occurred at areas surrounding the wells. This can be explained on the basis that at areas

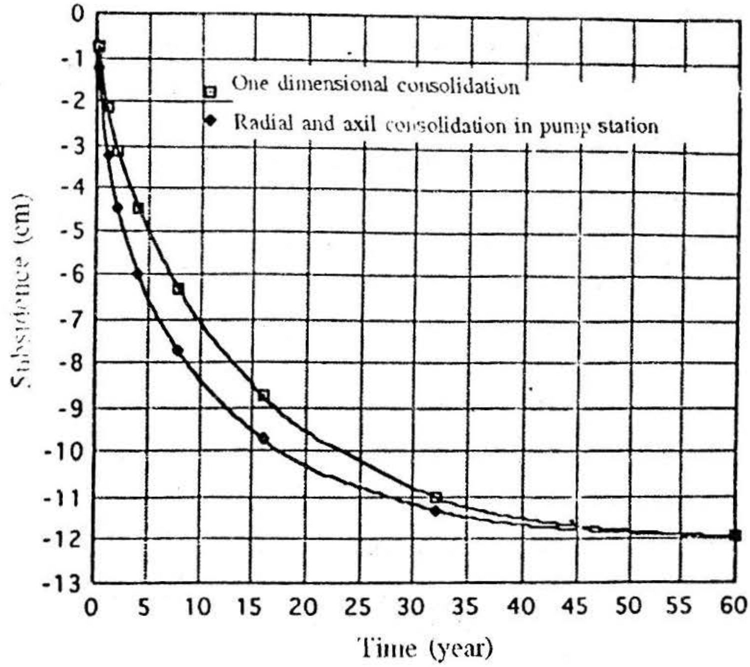

FIGURE 4 : Relationship of Subsidence-Time due to 1 meter Water Table Decline

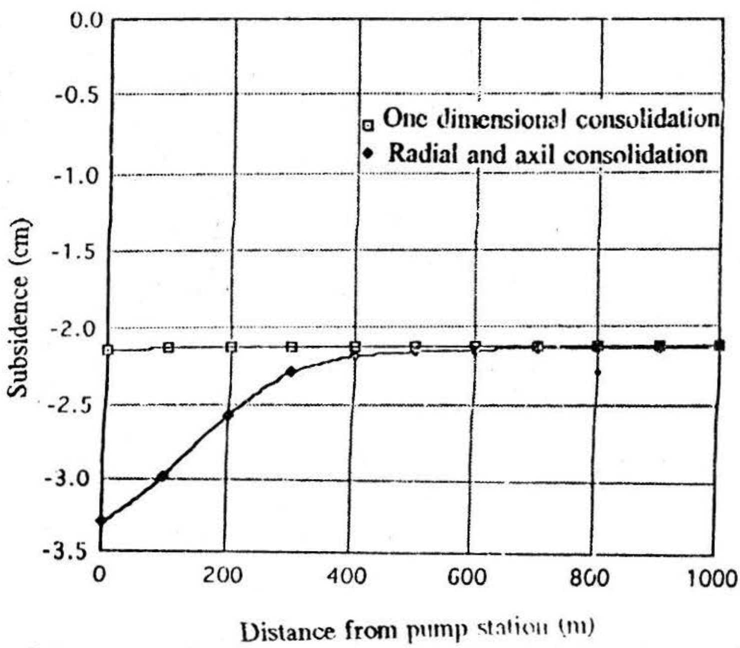

FIGURE 5 : Subsidence due to 1 meter Water Table Decline after 1 year

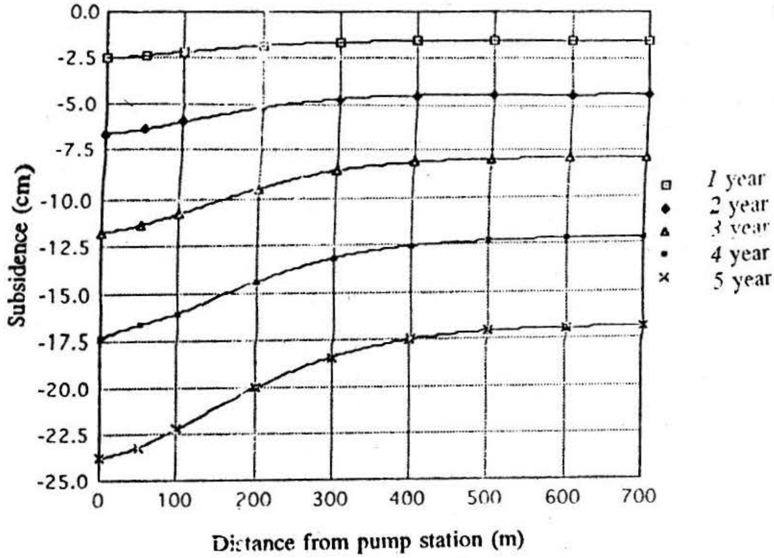

FIGURE 6 : Relationship of Subsidwnce and Distance from Pump Station at Various Time

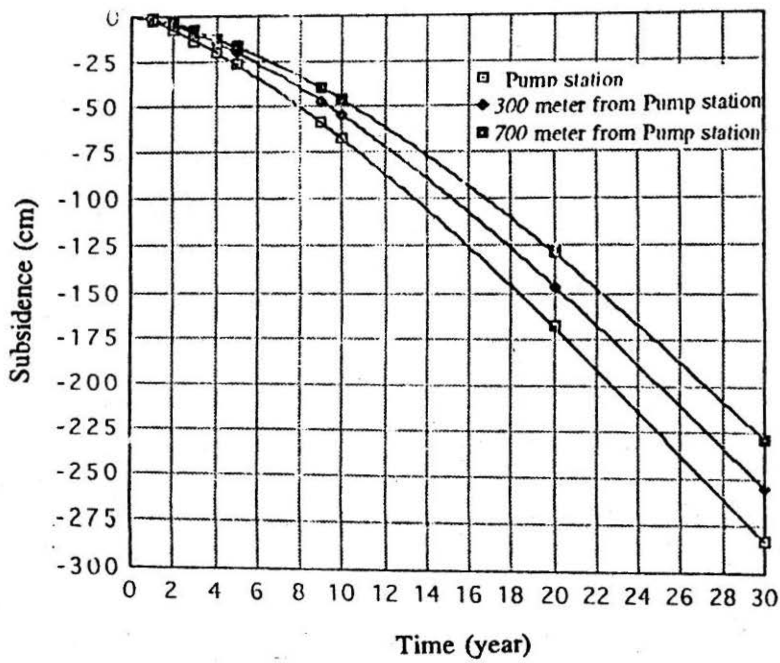

FIGURE 7 : Relationship between Subsidence and Time at Various Distance from Pump Station

close to well, radial drainage causes dissipation of pore water pressure which resulted in faster settlement.

It should noted that the final settlement at different distances from wells will be similar for a given load or water table drop.

At the third stage, it is assumed that water table drops at the rate of one meter per year which is equivalent to pistachio farming area at Rafsanjan aquifer. One of the main advantages of developed computer program is that can consider the actual slow drop of water table level. In other words, the subsidence problem considered here is time dependent in terms of pore water pressure dissipation and also time dependent in terms of load application. For this conditions, the water also time dependent in terms of load application. For this conditions, the water continuously drops at rate of 1 meter/year for five years and relationships between subsidence and distance from wells for different times are shown in Fig. 6.

From these results it can be concluded that rate of differential settlement increases with passage of time. This is because after each year, the water declination increases cumulatively and results in higher settlements and higher rates of differential settlement.

At fourth stage of this study, it is assumed that water table drops at rate of one meter per year but with inclination (gradient) of 0.0001. This can simulate field condition where many wells are located at aquifer basin which is very similar to case studied in this research. In order to create the actual field condition, the bed rock also has a slope of 0.002.

Figure 7 shows that the final results of settlement prediction at Rafsanjan aquifer. It can be predicted that if water table drops continuously at a rate of 1 m/year, after 30 years, the maximum subsidence at pump station is about 280 cm and the difference between subsidence at pump station and 700 m from pump station is about 50 cm which can easily cause shear zones and earth fissures at site.

The other verification of this research is based on site data regarding well casing growth at field for proposed example about 11 cm which confirms the finite element simulation of the sites.

Conclusions

The developed computer program based on Biot's three dimensional consolidation theory gave a satisfactory results. First, the proposed method was examined with classical and one dimensional consolidation theory and they extended to more complicated causes which still confirmed field data, and finally based on that the prediction of future settlement can be obtained. The limitation of this study is that aquifer was assumed as a confined one. This study first was developed for considering only one well, but it would easily extend for groundwater withdrawal in a regional problem similar to assumption which was made in first stage of this study or consider the actual variation of water table level in the field as an input data in finite element analysis.

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