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Bearing Capacity of Footings on Anisotropic Sands

by

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and

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Introduction

Sands in general are assumed to be isotropic with respect to shear strength. Sokolovski (1965) has used the methods of characteristics to solve several problems in geotechnical engineering when soils are isotropic with respect to shear strength. Several angular sands exhibit anisotropy in angle of internal friction. The object of the present paper is to study the effect of anisotropy on ultimate bearing capacity. The studies of Oda and Koishikawa (1979) and Tatsuoka et al. (1989) indicate friction, ϕ. Based on the studies, Oda and Nakayama (1989) proposed the variation of ϕ for Toyoura sand as given in Fig. 1. The usual methods like limit equilibrium and limit analysis cannot be used for determining ultimate bearing capacity for such cases as the variation of direction of major principal stress in the soil is not known. The method of characteristics can be used for these cases. The ultimate bearing capacity of a strip footing placed on this type of soil is arrived at herein using the method of characteristic approach. In the method of characteristics the failure surfaces are arrived at and the equations of equilibrium are fully satisfied. In Fig. 1, ψ is the angle between bedding plane and direction of major principal stress.

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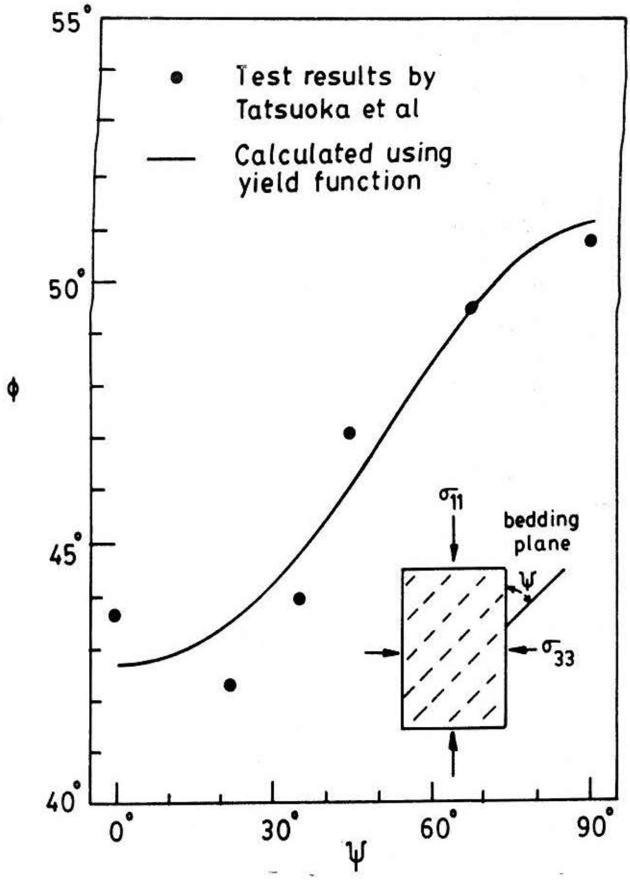

FIGURE 1 : Strength Anisotropy of Toyoura Sand

Assumptions

$\pi/2$

(i) The footing is a two dimensional one.

(ii) The footing is smooth.

(iii) The sand is anisotropic with respect to angle of internal friction and is given in Fig. 1 calculated using yield function.

(iv) Mohr-Coulomb failure condition is valid for the soil.

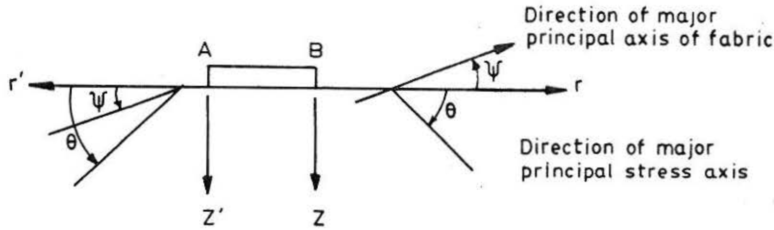

FIGURE 2a : Angle Between the Direction of Major Principal Stress and the Direction of Major Fabric (Bending Plane)

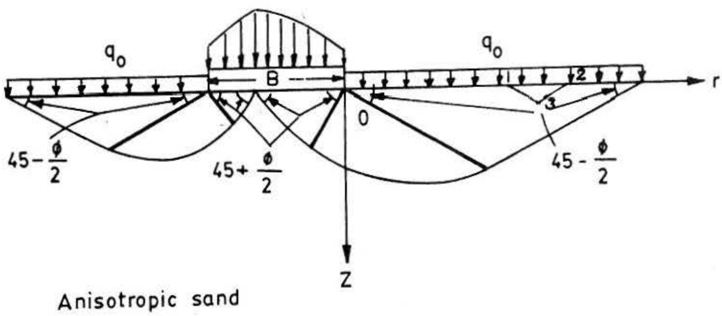

FIGURE 2b : Failure Mechanism for Anisotropic Sand

Failure mechanism used

Figure 2a shows a strip footing placed on an anisotropic sand with direction of major fabric of sand inclined at an angle ψ with horizontal *r - axis (Points* A *and* B are at the two edges of the footing). Let θ be an angle between major principal axis and horizontal r-axis. Since the soil is at failure and the shear stress along Br and Ar′ is zero, $\theta = 0$ along Ar′ and Br. As the footing AB is smooth, $\theta = \pi/2$ along AB. Considering two points which are similarly placed in zones zBr and z′Ar′, the values of r and r ; z′ and z *will* be the same. For an isotropic soil, the value of θ at these two points will be the same. However, for an anisotropic soil, the values of ϕ at these two points will be different since the angle between the direction of major principal stress and major fabric will be different for the two points. Referring to Fig. 2a for rBZ system, the angle between direction of major principal stress and direction of major fabric is $(\theta+\psi)$. For r′Az′ system, the angle is Abs$(\theta-\psi)$. For both the cases, θ increases from 0° along Br and Ar′ to 90° along base of footing AB. Hence, the pressure distribution for $\psi = 0$ or 90 will not be symmetrical with the centre line of the footing. Taking this into account, and as the footing is smooth the failure mechanism shown in Fig. 2b is used to arrive at the ultimate bearing capacity of the footing. Similar failure mechanism was assumed by Mandel (vide Harr, 1966) in arriving at bearing capacity of a footing with structures on either side of the footing by using the method of characteristics.

Equations along the characteristics

The equations of equilibrium for plane strain problem are

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} = 0 \tag{1}$$

and

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} = \gamma \tag{2}$$

where γ = body force per unit volume of soil in vertical z axis direction

σ_r = normal stress in horizontal r-direction

$$= \sigma(1 + \sin\phi \cos 2\theta) \tag{3}$$

σ_z = normal stress in z direction

$$= \sigma(1 - \sin\phi \cos 2\theta) \tag{4}$$

τ_{rz} = shear stress

$$= \sigma \sin\phi \sin 2\theta \tag{5}$$

$$\sigma = \frac{\sigma_r + \sigma_z}{2} \tag{6}$$

and ϕ = angle of internal friction of soil which depends on θ, the angle between directions of major principal stress and r axis.

Substituting the expressions for σ_r, σ_z, and τ_{rz} in the equations of equilibrium after modification, and using the total differentials of σ and θ, the equations along the characteristics are obtained. Along $(\theta - \mu)$ family,

$$\frac{dz}{dr} = \tan(\theta - \mu) \tag{7}$$

and

$$d\sigma - 2\sigma \tan\phi \, d\theta - \gamma(dz - \tan\phi \, dr) - \sigma\left(\frac{\partial \phi}{\partial r}dz - \frac{\partial \phi}{\partial z}dr\right) = 0 \tag{8}$$

Along $(\theta+\mu)$ family,

$$\frac{dz}{dr} = \tan(\theta+\mu) \tag{9}$$

and

$$d\sigma + 2\sigma\tan\phi\, d\theta - \gamma(dz + \tan\phi\, dr) + \sigma\left(\frac{\partial\phi}{\partial r}dz - \frac{\partial\phi}{\partial z}dr\right) = 0 \tag{10}$$

where $\mu = 45° - \phi/2$

The above equations are non dimensionalized for ultimate bearing capacity problems by dividing the lengths by B (= width of footing) and dividing the stresses by the characteristic stress γB. For nondimensional variables symbols, the respective dimensional symbols with prime have been used. For example non-dimensional z is z′.

Boundary conditions

1. Shear stress $\tau_{rz} = 0$ along Br and Ar′ (Fig. 2a)
2. Normal stress along Br and Ar′ = q_o
3. Shear stress along AB = 0.

From condition 1, $\theta = 0$ along Br and Ar′. From condition 2, σ along Br and Ar′ = $\frac{q_o}{(1-\sin\phi)}$. From condition 3, θ along AB = $\pi/2$

Analysis at singular points A and B (Fig. 2a)

At points A and B, the $(\theta-\mu)$ characteristic shrinks to a point and along it we have

$$d\sigma - 2\sigma\tan\phi\, d\theta = 0 \tag{11}$$

Integrating Eqn. 11 and using the values of q_o and θ along Br, the value of σ for any given value of θ at the singular point can be obtained from

$$\ln\sigma' = \ln\left\{\frac{q'_o}{(1-\sin\phi_{\theta=0})}\right\} + \int_0^\theta 2\tan\phi_\theta \, d\theta \tag{12}$$

where σ' = *non dimensional* σ

q'_o = non dimensional q_o

ϕ_θ = value of ϕ at θ

Analysis along characteristics

Knowing the values along Br and Ar′ the values of r, z, σ and θ at the points of intersection of the $(\theta+\mu)$ and $(\theta-\mu)$ characteristics are obtained by expressing the equations along the characteristics in finite difference form and solving them. Considering points 1, 2 and 3 (Fig. 2b) such that the values of r, z, σ and θ are known at points 1 and 2 and the values at point 3, which is the point of intersection of the characteristics passing through points 1 and 2 are to be determined. Representing the values of the variables at points 1, 2 and 3 by the variable with the corresponding subscript (for example, value of z at point 3 is z_3), and expressing the four characteristic equations in terms of these and solving them, by direct substitution, the following expressions are obtained for values at point 3. By solving Eqns. 7 and 9,

$$r_3 = \frac{r_2\tan(\theta_2-\mu) - r_1\tan(\theta_1+\mu) - z_2 + z_1}{\tan(\theta_2-\mu) - \tan(\theta_1+\mu)} \tag{13}$$

and

$$z_3 = z_2 + (r_3 - r_2)\tan(\theta_2-\mu)$$

Solving for θ and σ from Eqns. 8 and 10,

$$\theta_3 = \frac{-y_5 \pm \sqrt{y_5^2 - 4y_6(\sigma_2-\sigma_1)\tan^2\phi_3}}{2(\sigma_2-\sigma_1)\tan^2\phi_3} \tag{15}$$

and

$$\sigma_3 = \frac{y_1 - \sigma_1\theta_3\tan\phi_3}{y_2 + \theta_3\tan\phi_3} = \frac{y_3 + \sigma_2\theta_3\tan\phi_3}{y_4 - \theta_3\tan\phi_3}$$

where
$$y_1 = \sigma_1(1+\theta_1\tan\phi_3 - B_1) + A_1$$
$$y_2 = 1-\theta_1\tan\phi_3 + B_1$$
$$y_3 = \sigma_2(1-\theta_2\tan\phi_3 + B_2) + A_3$$
$$y_4 = 1+\theta_2\tan\phi_3 - B_2$$
$$y_5 = (y_3 + y_2\sigma_2 + y_4\sigma_1 + y_1)\tan\phi_3$$
$$y_6 = y_2 y_3 - y_1 y_4$$
$$A_1 = z_3 - z_1 + (r_3 - r_1)\tan\phi_3$$
$$A_2 = \cos\phi_3(r_3 - r_1) + (z_3 - z_1)(1-\sin\phi_3)$$
$$A_3 = z_3 - z_2 - (r_3 - r_2)\tan\phi_3$$
$$A_4 = \cos\phi_3(r_3 - r_2) - (z_3 - z_2)(1-\sin\phi_3)$$
$$B_1 = \frac{1}{2}\left\{\left.\frac{\partial\phi}{\partial r}\right|_3 (z_3 - z_1) - \left.\frac{\partial\phi}{\partial z}\right|_3 (r_3 - r_1)\right\}$$
$$B_2 = \frac{1}{2}\left\{\left.\frac{\partial\phi}{\partial r}\right|_3 (z_3 - z_2) - \left.\frac{\partial\phi}{\partial z}\right|_3 (r_3 - r_2)\right\}$$

When $\sigma_1 = \sigma_2$,

$$\theta_3 = -\frac{y_6}{y_5}$$

To obtain the values of σ at the bottom of the smooth footing, in addition to using the equations along the $(\theta-\mu)$ characteristic, the conditions $\theta = \pi/2$ and $z = 0$ are used. Knowing the values of σ and θ at the bottom of the footing the normal pressure on it is obtained. For obtaining better accuracy of values at the intersections of the characteristics, the procedure by Larkin (1968) has been used.

Determination of Bearing Capacity Factors N_γ and N_q

The average bearing pressure q is expressed as

$$q = 0.5\gamma B N_\gamma + q_o N_q \quad (17)$$

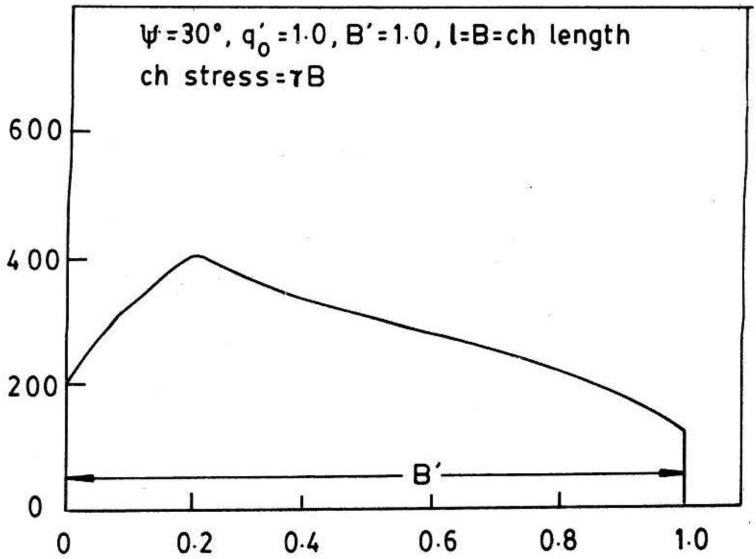

FIGURE 3 : Typical Nondimensional Normal Pressure Distribution

where B = width of the footing

N_γ and N_q = bearing capacity factors.

Dividing Eqn. 17 by the characteristic stress B,

$$q' = 0.5\gamma N_\gamma + q'_o N_q \tag{18}$$

The failure mechanism shown in Fig. 2b, which gives minimum total force on the footing, is used to find q'. Taking $q'_o = 1.0$, the nondimensional normal pressure along the footing are determined to get q' of Eqn. 18. Next the stresses along the footing are determined once again but with $\gamma = 0$, which gives N_q as

$$q' = N_q \tag{19}$$

Knowing N_q from Eqn. 19, N_γ is determined using Eqn. 18.

Results and Discussion

Figure 3 shows a typical nondimensional normal pressure distribution on the footing. It is seen that when $\psi \neq 0°$ and $90°$, the maximum pressure

TABLE 1
Bearing Capacity Factors N_γ and N_q

Sl. No.	ψ	N_γ	N_q	$0.5 N_\gamma + N_q$	Location of Max. Pr.
1	0°	294	192	337	0.5
2	15°	280	181	321	0.69
3	30°	254	166	293	0.795
4	45°	228	151	265	0.864
5	60°	224	156	268	0.822
6	75°	238	172	291	0.705
7	90°	243	183	305	0.5

Prandtl : $N_q = \left(\frac{1+\sin\phi}{1-\sin\phi}\right) e^{\pi \cdot \tan\phi}$

for $\phi = 51°$, $N_q = 386$

for $\phi = 42.65°$, $N_q = 93.96$ (Average $N_q = 240$)

for $\phi = 45°$, $N_{\gamma(\text{Meyerhof})} = 390$

is away from the centre line of the footing. Table 1 gives the values of N_q, N_γ, the average nondimensional bearing pressure and the location of the maximum pressure on the footing. It also gives the values of Prandtl (1920) and Meyerhof (1951). It is seen from the table that the bearing factor N_γ varies by about 25% and N_q varies by about 21% when ψ changes from 0° to 90°. The average nondimensional bearing pressure changes by 21% and the location of maximum pressure shifts from the centre line by a maximum of 0.346 B when ψ is changed from 0° to 90°. The values of N_γ and N_q given by the present analyses are less than the values of Prandtl and Meyerhof.

Conclusion

The results of the study show that the ultimate bearing capacity can vary by 25% and the maximum pressure on the footing can be away from the centre line by 0.364 B due to anisotropy in angle of internal friction of Toyoura sand.

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Nomenclature

The following symbols are used in this paper :

B	=	width of footing
q	=	average bearing pressure
q'	=	nondimensional average bearing pressure
q_o	=	surcharge
q'_o	=	nondimensional surcharge
N_γ, N_q	=	bearing capacity factors
r	=	radial coordinate
r_1, r_2, r_3	=	radial coordinates at points 1, 2 and 3 respectively

z	=	z-coordinate
z_1, z_2, z_3	=	z-coordinates at points 1, 2 and 3, respectively
γ	=	body force per unit volume of soil
θ	=	angle between r axis and direction of major principal axis
μ	=	$45° - \phi/2$
σ	=	$(\sigma_r + \sigma_z)/2$
σ_1, σ_2, σ_3	=	values of σ at points 1, 2 and 3, respectively
σ_n	=	normal stress along footing base
σ'_n	=	nondimensional normal stress along base of footing
σ_r, σ_z	=	normal stresses in r and z directions, respectively
τ_{rz}	=	shear stress
ϕ	=	angle of internal friction of soil; and
ψ	=	angle between direction of major principal stress and bedding plane in Fig. 1
	=	angle between bedding plane and r axis in Fig. 2