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Unconstrained Minimization Techniques for Passive Earth Pressure - A Comparative Study

by

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Introduction

Choosing an appropriate method for either minimizing or maximizing a function subjected to inequality constraints arising out of analysis of stability problems is an art. Fox (1971) has provided guidelines for such an exercise. In such problems, it is extremely difficult to obtain an initial feasible design vector and, as such, the interior penalty function method (Fiacco and McCormick, 1968) can not be used. In such cases the problem has to be solved by using either the exterior penalty function method or obtaining an initial feasible design vector following a procedure outlined by Fox and using the interior penalty function method. Even when the interior penalty function method is used, owing to the 'long step' nature of the unconstrained optimization algorithms the path may be diverted into infeasible regions. In such cases the function is set to an arbitrary high value and the minimization procedure is left to correct the situation on its own. Sometimes this approach presents numerical difficulties. Therefore, Basudhar (1976) has suggested the use of the extended penalty function method enunciated by Kavlie and Moe (1971). This readily accepts infeasible design points and needs no special treatment. Another reason for the choice made is the availability of well established unconstrained minimization techniques in the literature (Fox, 1971; Rao, 1984). As exact gradient of the function is not available in such an analysis. Basudhar was guided by the

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suggestion made by Fox and used the nongradient technique viz. Powell's method of conjugate direction for unconstrained minimization with Quadratic fit for linear minimization. But when the design variables and the constraints arè too many arising out of consideration of more number of elements, adoption of a gradient based technique may be necessary. In this connection the Fletcher–Reeve's (FLRV) method of conjugate gradient and Davidon–Fletcher–Powell (DFP) variable metric method with finite difference approximation of the gradient are worth mentioning. However, the use of Conjugate gradient method has been discouraged by Fox unless the problem is very large and well conditioned. For ill conditioned problems variable metric method is likely to work better as compared to other gradient and nongradient methods. Like all other numerical methods the efficiency of these optimization algorithms is problem oriented. As such, it is necessary to make a critical appraisal of these algorithms in order to pick up the most suitable one. In the following sections such a study has been reported with reference to the stability analysis of a retaining wall previously studied by Lysmer (1970). Basudhar (1976) and Basudhar et al. (1979).

Another objective of the paper is to demonstrate that the stress field is extensible which is one of the primary requirements for the solution to be a true lower bound. None of the studies as mentioned above include such investigations.

Statement of the Problem

Fig. 1 shows a rough retaining wall of 3.05 m height. The backfill consists of dry sand. The unit weight of sand (γ) is 16.018 kN/m^3. The objective is to find the most efficient unconstrained minimization method for isolating the maximum passive earth pressure acting on the wall. The different values of angle of wall friction ($\delta = 20°$, 26.56°, 14°) and the effective angle of shearing resistance of the sand ($\delta = 40°$ and 34°) have been considered. For a direct comparison of the results of the present study with those of Lysmer (1970) the identical problem previously solved by him using linear programming has been adopted here. Lysmer used FPS units; the decimal places appearing in the height of the wall and unit weight of sand are due to the conversion of the units.

General Method of Analysis

The soil mass under consideration is discretized into a finite number of triangular elements. All nodal points and elements are then numbered in

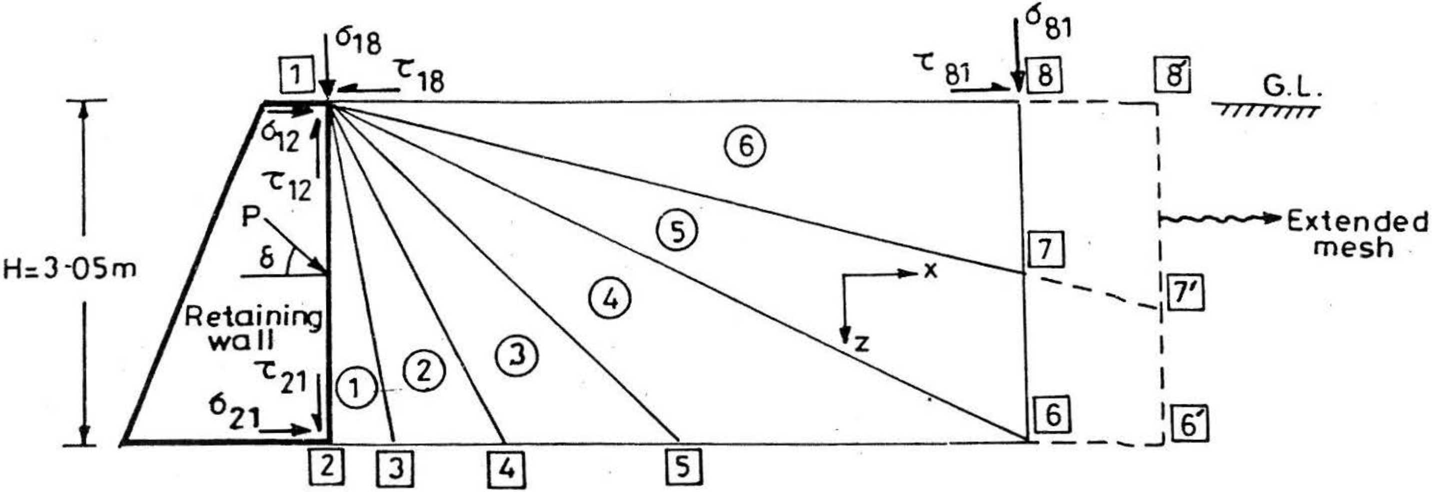

FIGURE 1 Mesh pattern for earth pressure problem.

some arbitrary order as shown in Fig. 1. The encircled digits and the digits in the square refer to the element numbers and the node numbers respectively. The primed node numbers refer to the extended mesh. The material properties are specified for these elements. This is followed by choosing a linear stress field ensuring the element equilibrium (Lysmer, 1970). It has been demonstrated by Singh and Basudhar (1993) and Lysmer (1993) that satisfying the no-yield constraints only at the corners of each triangular element is the sufficient condition for the solution to be a true lower bound. Interface equilibrium is satisfied by matching the normal and shear stresses at the element interfaces. This leads to a set of linear equality constraints. The boundary conditions also yield a set of linear equality constraints. The no-yield Mohr–Coulomb condition is incorporated directly in the analysis. In addition, no tension constraints on the principal unknowns (normal stresses on the element sides), σ_j, are imposed. Compressive stresses have been taken to be positive in the analysis. As such the no-tension constraints are of the type $-\sigma_j \leq 0$.

In general, as many stress fields will satisfy the condition of static admissibility, the isolation of the stress field which optimizes the objective function (in the present case, the bearing capacity) is imported. It is a nonlinear programming problem which can be stated in a standard form as follows :

Find **Dm** such that;

$$F(\mathbf{Dm}) = \sum a_j \sigma_j \quad \text{is minimum} \tag{1}$$

Subject to $g_j(\mathbf{Dm}) \leq 0$ (Nonlinear no-yield condition and no-tension constraints) (2)

and, $l_j(\mathbf{Dm}) = 0$ (Interface shear stress and boundary shear stress equality constraints) (3)

where, **Dm** is the design vector whose elements are picked out of the principal unknowns, σ_j.

The system of linear equations $[\, l_j(\mathbf{Dm})]$ is not column regular and a non trivial solution can be obtained in terms of some free parameters. The number of such free parameters treated as design variables can be determined from the total number of principal unknowns and the rank of the coefficients matrix. By expressing the design variables in terms of the basic variables the linear equality constraints are eliminated. The problem, thus turns out to be one of nonlinear programming subjected to inequality constraints only.

Following the suggestions made by Basudhar et al. (1979) the constrained optimization problem is converted to an unconstrained one using the extended penalty function method (Kavlie and Moe, 1971). Sequential unconstrained minimization of the composite function so developed is carried out for a decreasing sequence of the penalty parameter (r_k) to isolate the optimal solution. The adopted techniques are Powell's conjugate direction method (POWELL), Fletcher–Reeves conjugate gradient method (FLRV) and Davidon–Fletcher–Powell variable metric method (DFP). For linear minimization Quadratic interpolation (QFIT) and Cubic interpolation (CFIT) techniques have been used. These methods are available in any standard text book on Optimization (Fox, 1971; Rao, 1984). The composite function $\psi(\mathbf{D}, r_k)$ is developed by blending the objective function and constraints as follows :

$$\psi(\mathbf{D}, r_k) = F(\mathbf{D}) + r_k \sum_{j=1}^{M} H\left[g_j(\mathbf{D})\right] \tag{4}$$

The function $H\left[g_j(\mathbf{D})\right]$ is chosen as :

$$H\left[g_j(\mathbf{D})\right] = \begin{cases} 1/g_j(\mathbf{D}) & ; g_j(\mathbf{D}) \le 0 \\ \left[2 \cdot \lambda - g_j(\mathbf{D})\right]/\lambda^2 & ; g_j(\mathbf{D}) > \lambda \end{cases} \tag{5}$$

where $\lambda = -r_k / \delta t$. (6)

The penalty parameter (r_k) is decreased sequentially at (k+1)th minimization cycle as follows :

$$(r_{k+1}) = 0.1(r_k) \tag{7}$$

In this approach infeasible starting points are readily acceptable to the minimization algorithm, which makes it a powerful technique for solving various engineering problems even if an initial feasible design vector is difficult to guess.

Boundary Conditions and the Objective Functions

Boundary conditions :

As shown in Fig. 1 the following boundary conditions are imposed.

$$\begin{aligned}
&1)\quad \sigma_{18} = \sigma_{81} = \tau_{18} = \tau_{81} = 0 \\
&2)\quad \tau_{12} - \sigma_{12} \tan\delta \le 0 \\
&3)\quad \tau_{21} - \sigma_{21} \tan\delta \le 0
\end{aligned} \tag{8}$$

Objective function:

At the limiting state the passive earth pressure on the wall will be the minimum value of $(\sigma_{12} + \sigma_{21})$. So as per the present formulation it is equivalent to the minimization of $-(\sigma_{12} + \sigma_{21})$. Detailed analysis procedure of finding the passive earth pressure on retaining walls subjected to the boundary condition as described above has been presented in details by Basudhar et al. (1979). As such, for the sake of brevity this is not reported here.

Results and Conclusions

The results are obtained on HP-9000/850s computer system. In Figs. 2(a) and 2(b) the performance of the various algorithms used in analysis has been studied in terms of the number of function evaluations required to achieve the optimal objective function values (normalized with γH) and the penalty parameter (r_k) for the case when $\phi = 40°$ and $\delta = 20°$. The effect of initial design vector on the final solution and efficiency of the technique has also been studied and presented in the figures. Two initial design vectors designated by 1 and 2, in the figure have been used. It can be seen from Fig. 2(a) that for starting point 1, only POWELL algorithm converges to a solution which is close to 8.97 (Lysmer, 1970) at about 2500 number of function evaluations and a corresponding r_k being equal to 1E-08. It can be seen that DFP and FLRV converge to feasible solutions quite far off from the Lysmers' values.

It is seen from Fig. 2(b) that the variations of penalty term has a great influence on POWELL as compared to DFP and FLRV beyond $r_k = 0.1$. A near optimal solution has been used as a starting point 2 for DFP and FLRV to check whether there is any improvement or not. The study as depicted in Figs. 2(a) and 2(b) show that DFP with QFIT and CFIT converges almost to the same results as given by POWELL where as FLRV diverges.

Another initial design vector 3 was also considered. This resulted in the initial objective function value, 8.97, obtained by Lysmer. As depicted

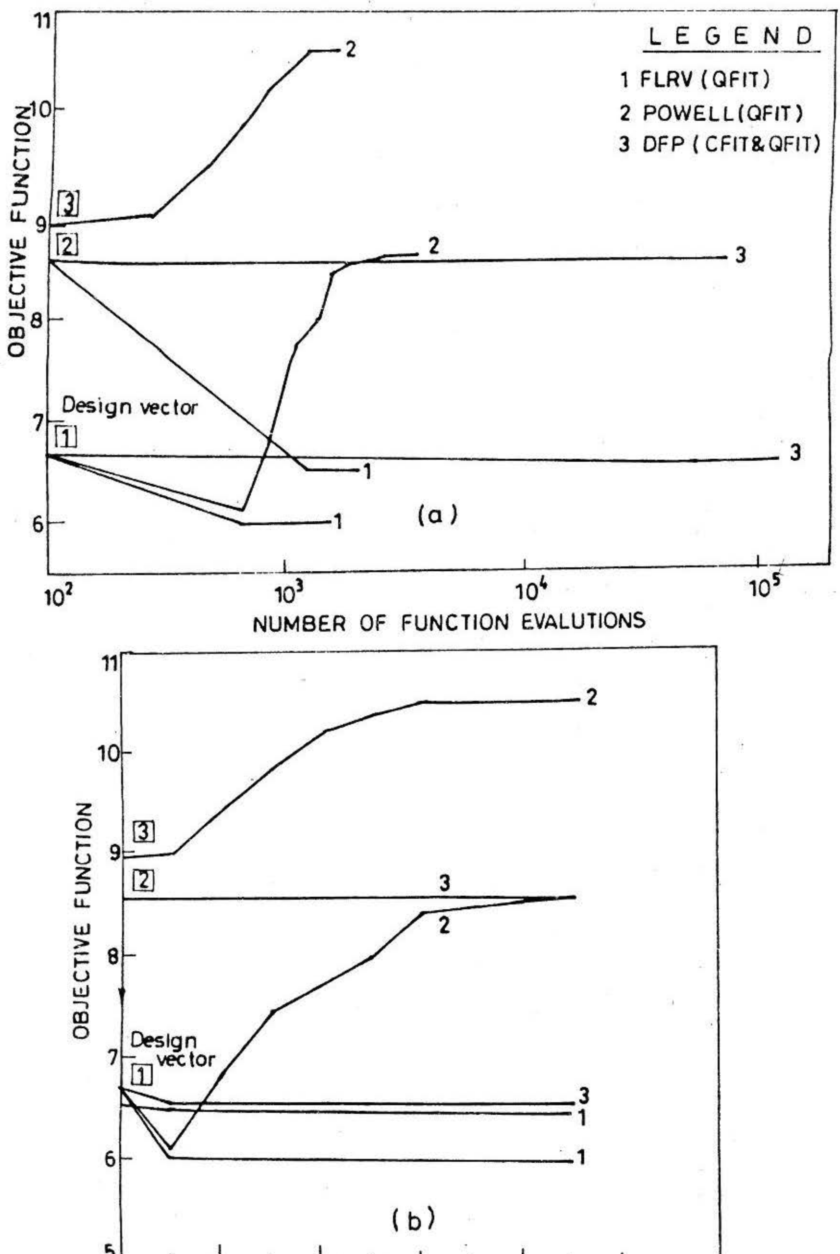

FIGURE 2

(a) Variation of objective function with number of functijon evaluations.
(b) Variation of objective functijon with penalty parameter.

TABLE 1

Non-dimensional Verticle Stress ($\sigma_z/\gamma H$) Values for the Elements at the Corresponding Nodes for $\phi = 40°$ and $\delta = 20°$

Element Number	Nodal Point	Lysmer (1970)	Present Study			
			Powel	DFP		FLRV
				QFIT	CFIT	
1	1	0.000	0.000	0.000	0.000	0.000
	2	3.886	3.901	5.632 (3.881)	5.626 (3.894)	4.583 (2.665)
	3	3.200	3.227	4.463 (3.210)	4.458 (3.222)	3.523 (2.314)
2	1	0.000	0.000	0.000	0.000	0.000
	3	3.200	3.075	1.589 (3.062)	1.613 (3.077)	1.535 (2.209)
	4	2.333	2.385	1.465 (2.377)	1.454 (2.392)	1.413 (1.821)
3	1	0.000	0.000	0.000	0.000	0.000
	4	2.333	2.126	1.425 (2.157)	1.415 (2.182)	1.376 (1.712)
	5	1.410	1.535	1.366 (1.509)	1.357 (1.512)	1.224 (1.367)
4	1	0.000	0.000	0.000	0.000	0.000
	5	1.410	1.261	1.102 (1.254)	1.107 (1.232)	1.144 (1.173)
	6	0.937	0.964	1.007 (0.970)	1.038 (1.003)	0.959 (0.979)
5	1	0.000	0.000	0.000	0.000	0.000
	6	1.020	1.024	1.027 (1.024)	1.032 (1.028)	1.038 (1.035)
	7	0.459	0.490	0.493 (0.490)	0.489 (0.488)	0.502 (0.503)
6	1	0.000	0.000	0.000	0.000	0.000
	7	0.500	0.500	0.500 (0.500)	0.500 (0.500)	0.500 (0.500)
	8	0.000	0.000	0.000	0.000	0.000

Note : Values in the parantheses correspond to Design Vector 2.

TABLE 2
Non-dimensional Verticle Stress ($\sigma_x/\gamma H$) Values for the Elements at the Corresponding Nodes for $\phi = 40°$ and $\delta = 20°$

Element Number	Nodal Point	Lysmer (1970)	Present Study			
			Powel	DFP		FLRV
				QFIT	CFIT	
1	1	0.000	0.000	0.000	0.000	0.000
	2	8.970	8.590	6.674	6.671	6.016
				(8.566)	(8.553)	(6.555)
	3	8.320	7.970	6.188	6.185	5.617
				(7.943)	(7.927)	(6.159)
2	1	0.000	0.000	0.000	0.000	0.000
	3	8.320	7.970	6.074	6.071	5.538
				(7.937)	(7.923)	(6.155)
	4	7.340	7.060	5.755	5.748	5.237
				(7.033)	(7.016)	(5.579)
3	1	0.000	0.000	0.000	0.000	0.000
	4	7.340	6.995	5.745	5.738	5.228
				(6.978)	(6.964)	(5.552)
	5	5.860	5.752	5.271	5.270	4.757
				(5.705)	(5.677)	(4.722)
4	1	0.000	0.000	0.000	0.000	0.000
	5	5.860	5.478	5.007	5.020	4.678
				(5.449)	(5.397)	(4.528)
	6	4.260	4.426	4.609	4.631	4.017
				(4.426)	(4.492)	(3.753)
5	1	0.000	0.000	0.000	0.000	0.000
	6	4.750	4.667	4.691	4.604	4.335
				(4.642)	(4.595)	(3.973)
	7	2.070	2.141	2.165	2.108	1.858
				(2.129)	(2.103)	(1.708)
6	1	0.000	0.000	0.000	0.000	0.000
	7	2.300	2.297	2.273	2.281	1.825
				(2.282)	(2.291)	(1.659)
	8	0.000	0.000	0.075	0.117	0.000
				(0.001)	(0.066)	(0.000)

Note : Values in the parantheses correspond to Design Vector 2.

TABLE 3
Non-dimensional Verticle Stress ($\tau_{zz}/\gamma H$) Values for the Elements at the Corresponding Nodes for $\phi = 40°$ and $\delta = 20°$

Element Number	Nodal Point	Lysmer (1970)	Present Study			
			Powel	DFP		FLRV
				QFIT	CFIT	
1	1	0.000	0.000	0.000	0.000	0.000
	2	3.250	3.128	2.429 (3.115)	2.429 (3.122)	1.994 (1.979)
	3	2.680	2.547	1.503 (2.539)	1.504 (2.543)	1.277 (1.646)
2	1	0.000	0.000	0.000	0.000	0.000
	3	2.680	2.517	0.928 (2.509)	0.935 (2.514)	0.879 (1.626)
	4	1.840	1.656	0.726 (1.754)	0.719 (1.754)	0.695 (1.185)
3	1	0.000	0.000	0.000	0.000	0.000
	4	1.840	1.627	0.706 (1.643)	0.699 (1.649)	0.676 (1.131)
	5	0.715	0.768	0.464 (0.742)	0.463 (0.723)	0.412 (0.602)
4	1	0.000	0.000	0.000	0.000	0.000
	5	0.715	0.494	0.200 (1.486)	0.214 (0.443)	0.332 (0.408)
	6	0.169	0.064	0.004 (0.051)	0.038 (0.018)	0.004 (0.041)
5	1	0.000	0.000	0.000	0.000	0.000
	6	0.040	0.057	0.044 (0.057)	0.024 (0.034)	0.163 (0.151)
	7	0.110	0.039	0.046 (0.038)	0.073 (0.063)	0.001 (0.012)
6	1	0.000	0.000	0.000	0.000	0.000
	7	0.000	0.000	0.019 (0.000)	0.029 (0.016)	0.000 (0.000)
	8	0.000	0.000	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)

Note : Values in the parantheses correspond to Design Vector 2.

TABLE 4
Stress Field and the Stress-Strength Ratios at the Nodal Points with Starting Point 3 using POWELL ($\phi = 40°$ and $\delta = 20°$)

Element Number	Nodal Point	($\sigma_x/\gamma H$)	($\sigma_z/\gamma H$)	($\tau_{zx}/\gamma H$)	Stress-Srength Ratio
1	1	0.0000	0.0000	0.0000	–
	2	10.5900	17.4337	8.3286	0.9994
	3	8.9243	11.2393	5.0418	0.6372
2	1	0.0000	0.0000	0.0000	–
	3	8.6921	5.4349	3.8809	0.8593
	4	7.1690	3.1313	2.0897	0.7703
3	1	0.0000	0.0000	0.0000	–
	4	6.9106	2.0955	1.5718	0.9867
	5	5.6865	1.4383	0.6955	0.9527
4	1	0.0000	0.0000	0.0000	–
	5	5.5541	1.2932	0.5504	0.9978
	6	4.3716	0.9670	0.0690	0.9859
5	1	0.0000	0.0000	0.0000	–
	6	4.5355	1.0079	0.0129	0.9800
	7	2.2042	0.4953	0.0188	0.9704
6	1	0.0000	0.0000	0.0000	–
	7	2.2792	0.5000	0.0001	0.9919
	8	0.0004	0.0000	0.0001	–

TABLE 5
Comparison of the Present Study with the Values Reported by Lysmer (1970)

f	d	Objective Function Value		% Difference
		Lysmer (1970)	Present Study	
40	20.00	8.97	10.59	18.06
34	26.56	6.94	7.11	2.45
34	14.00	5.37	7.11	32.40

in Fig. 2 the starting point 3 converges to an optimal objective function value equal to 10.59 corresponding to 1184 number of function evaluations and r_k being equal to 1E-06. The predicted earth pressure value is 18.06% higher than the value obtained by Lysmer (1970).

As lower bound analysis involves the generation of statically admissible stress field, it is of interest to study the state of stress at the limiting state in the soil medium (corresponding to the optimal solution). Stress values at different nodal points are also presented and compared with those reported by Lysmer (with $\phi = 40°$ and $\delta = 20°$) in Tables 1, 2 and 3 just for the sake of completeness. The study supports the conclusions as drawn regarding the relative efficiency of these methods.

The complete stress field along with the stress–strength ratio at the nodal points obtained by using POWELL is shown in Table 4 for the starting points 3 ($\phi = 0.40°$ and $\delta = 20°$). The stress-strength ratio is defined as :

$$\left[(\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2\right] \Big/ \left[(\sigma_z + \sigma_x)\sin\phi\right]^2 \tag{9}$$

where, σ_x and σ_z are the normal stresses on the plane through a nodal point in x and z directions respectively.

τ_{zx} is the shear stress acting on zx plane through a nodal point.

The ratio close to unity signifies the limiting equilibrium state. From this consideration it may be observed that for the chosen mesh pattern most of the nodal points are very near to the limiting state signifying that the obtained solution is excellent.

To demonstrate that predicted earth pressure values are true lower bounds the mesh pattern of Fig. 1 has been extended as shown in the figure. For starting point 3 the extended mesh pattern yields a value 10.35 which is only 2.26% lower from the value 10.59 obtained earlier. For all practical purposes this deviation may be considered as negligible and hence the stress field is extensible and statically admissible throughout the medium.

Table 5 presents a comparative study of the present solutions with those reported by Lysmer (1970). It is observed from the table that there is an improvement over the lower bound solutions as obtained and reported by Lysmer; the order of magnitude of the percent difference between the solutions ranges from 2.45 to 32.40%.

Conclusions

The following conclusions can be drawn based on the results and discussions presented above :

1) The comparative study of the different unconstrained *minimization* algorithms reveals that POWELL (conjugate direction) method is the most suitable technique for isolation of the optimal value of passive earth pressure.

2) Davidson–Fletcher–Powell variable *metric method and Fletcher–Reeves* conjugate gradient method are not recommended for solving such problems where exclusive analytical expressions for gradients are not available.

3) For such problems non-gradient methods are superior and quite efficient is isolating the optimal lower bound than the gradient based techniques which may even diverge and terminate at points which are quite far off from the optimal value.

4) The predicted earth pressure values are much better than the values reported by Lysmer (1970). The improvement in the results ranges from 2.45 to 32.40%.

References

Basudhar, P.K. (1976) : "Some Applications of Mathematical Programming Techniques to Stability Problems in Geotechnical Engineering", *Ph.D. Thesis,* I.I.T. Kanpur, India.

Basudhar, R.K., Madhav, M.R. and Valsangkar, A.J. (1979) : "Optimal Lower Bound of Passive Earth Pressure using Finite Elements and Nonlinear Programming", *Int. J. Numer. Analyt. Meth. in Geomech,* Vol. 3, pp. 367-379.

Flacco, A.V. and McCormick, G.P. (1968) : *"Nonlinear Programming : Sequential Unconstrained Minimization Techniques"*, John Wiley, NY.

Fox, R.L. (1971) : *"Optimization Methods for Engineering Design"*, Addison-Wesley Publishing company.

Kavlie, D. and Moe, J. (1971) : "Automated Design of Frame Structures," *Jour, of Str. Div.*, ASCE, Vol. 97, ST1, pp. 33-61

Lysmer, J. (1970) : "Limit Analysis of Plane Problems in Soil Mechanics", *Jour. of Soil Mech. and Foundations Div.*, ASCE, Vol. 96, SM4, pp. 1311-1334.

Lysmer, J. (1993) : *"Proof That It is Sufficient to Satisfy the No-yield Condition at the Ccorner of the Elements"*, Personal Communication.

Rao, S.S. (1984) : "*Optimization Theory and Application*", Wiley- Eastern Limited.

Singh, D.N. and Basudhar, P.K. (1993) : "Determination of the Optimal Lower Bound Bearing Capacity of Reinformed Soil Retaining Walls by using Finite Elements and Non-linear Programming", *Geotextiles and Geomembrances*, Vol. 12, pp. 665-686.

Notations

a_j = Coefficient to σ_j in linear function to be optimized

D = Design vector

F(**D**) = Objective function.

g_j = Inequality constraints.

H = Height of the retaining wall.

l_j = Equality constraints.

M = Total number of inequality constraints

r_k = Penalty parameter

δ_t = Transition term between two penalty terms

δ = Angle of wall friction.

γ = Unit weight of the soil.

σ_j = principal unknown

σ_{ij} , τ_{ij} = Normal and shear stresses at nodal point i on side connecting nodal points i and j.

ϕ = Internal friction angle of soil.