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Wave Propagation in Elastic Half-space Under Surface Shear Loads

by

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Introduction

In many problems of earthquake engineering and dynamic soil-structure interaction, numerical modelling of wave propagation is necessary. The finite element method based on discretisation of the domain is one of the popular numerical techniques because of its effectiveness in handling spatial variation of mechanical properties of the domain and of complex geometries. Semi-analytical finite element approach was first proposed in finite element literature by Wilson (1965) to tackle axisymmetric problems under asymmetric loading. This formulation has been employed by Desai and Patil (1977), Randolph (1981) and Kuhlemeyer (1979) for analysing laterally loaded piles. Desai (1983) studied the behaviour of caisson foundations subjected to axial and lateral loads. Dynamic soil-structure interaction analysis in time domain using finite element method requires special boundary treatment to represent the semi-infinite extent of the soil medium in order that the stress waves are not reflected at the boundaries.

Many of the transmitting boundaries that could be used in direct time domain analysis have been outlined in Wolf (1988). Simplest and widely used transmitting boundary is the standard viscous boundary proposed by Lysmer and Kuhlemeyer (1969). When the role of the surface waves is not

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significant the viscous boundary approach is straightforward. However, where the effects of surface waves are to be considered the dashpot characterisation used in modelling the boundaries become frequency dependent thus rendering the direct time domain approach cumbersome. Extrapolation algorithm was proposed by Liao and Wong (1984) for the wave propagation study in a linear elastic half-space under plane strain idealisation subjected to vertical transient and harmonic loads on the surface of the half-space. Liao and Liu (1992) presented the numerical instabilities of this algorithm for one dimensional wave propagation analysis in a bar using frequency domain approach. Shridhar and Chandrasekaran (1992) studied the applicability of extrapolation algorithm for problem of one dimensional wave propagation in an infinite bar resting on spring bed and an elastic half-space under the action of suddenly applied vertical concentrated load.

In this paper the results of wave propagation studies in homogeneous and two layered medium subjected to uniformly distributed surface shear load over a circular area are presented using semi-analytical finite element formulation. The half-space is discretised by using eight noded quadrilateral elements and the artificial boundary conditions are implemented by using first order extrapolation algorithm. The surface shear loads considered in the present study are :

(1) Suddenly applied shear load at time $t = 0$ and thereafter maintained constant and

(2) Shear load varying sinusoidally with time

Semi-analytical Finite Element Formulation

Consider the axisymmetric idealisation of the half-space shown in Fig. 1. If the external loads are symmetrical about $\theta = 0$ plane, then the displacements may be expressed in the form of finite Fourier series. The radial, vertical and circumferential components of displacements for eight noded quadrilateral element may be represented as,

$$
\begin{aligned}
u(r,z,\theta) &= \sum_{n=1}^{L}\sum_{i=1}^{8} N_i \bar{u}_{ni} \cos n\theta \\
v(r,z,\theta) &= \sum_{n=1}^{L}\sum_{i=1}^{8} N_i \bar{v}_{ni} \cos n\theta \\
w(r,z,\theta) &= \sum_{n=1}^{L}\sum_{i=1}^{8} N_i \bar{w}_{ni} \sin n\theta
\end{aligned}
\tag{1}
$$

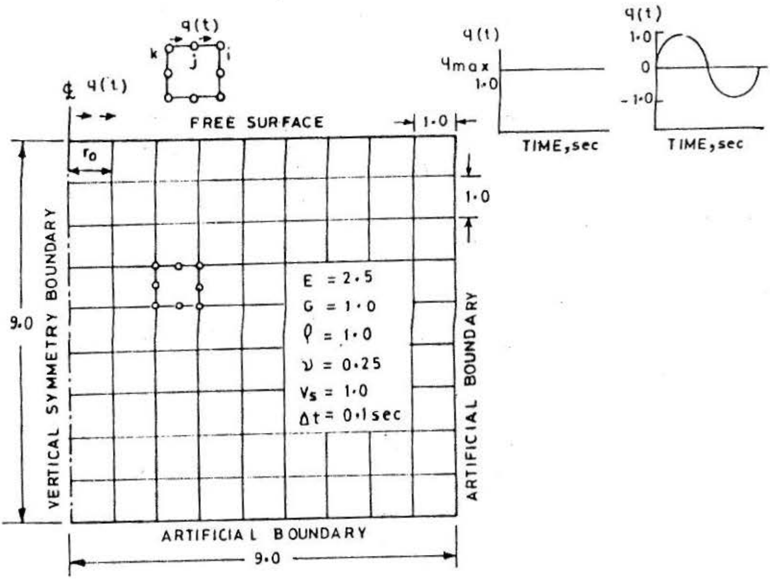

FIGURE 1
Homogeneous Elastic Half-space Under Surface Shear Load

where u, v and w are the displacements in r, z and θ directions respectively,

$\overline{u}_{ni}$, $\overline{v}_{ni}$ and $\overline{w}_{ni}$ are the amplitudes of displacements for the nth harmonic at node i,

N_i is the shape function for the node i, and

L is the total number of harmonics required to represent the load.

For uniform density and elastic properties in the circumferential direction, the orthogonality of trigonometric functions can be exploited to represent the general three dimensional problem into a series of uncoupled two dimensional problems. The complete solution is then simply the superposed solutions of all individual uncoupled two dimensional finite element solutions. For the present problem it would be sufficient to consider only n = 1 term in the formulation for the wave propagation study in an elastic half- space under the action of uniformly distributed surface shear load applied over a circular area. Following the standard procedure element stiffness matrix for n = 1 can be derived for eight noded element (Cook et al., 1989).

Lumped element mass matrix is derived based on special lumping scheme suggested by Hinton et al. (1976). The idea behind this procedure is to use only the diagonal terms of the consistent element mass matrix, but to scale them in such a way that the total mass of the element is preserved.

The lateral force Q acting on the free surface of the half-space may be replaced by a uniformly distributed shear stress q over a circular area of radius r_o. The components of stress on an element are in r, z and θ directions are,

$$\begin{Bmatrix} dq_r \\ dq_z \\ dq_\theta \end{Bmatrix} = q\ r\ dr\ d\theta \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix} \tag{2}$$

Equation (2) represents harmonic load term when $n = 1$. Using the principle of virtual work the equivalent nodal loads at nodes i, j and k (Fig. 1) may be obtained as,

$$\{Q_e\} = \frac{\pi q(r_i - r_k)}{6}\{r_i, 0, -r_i, 4r_j, 0, -4r_j, r_k, 0, -r_k\}^T \tag{3}$$

Transmitting Boundaries Using Extrapolation Algorithm

The method is based on the concept that the displacements on the artificial boundary at a given time step are extrapolated based on the displacements at earlier times along a line normal to the artificial boundary in the region's interior, thus allowing free transmission of waves across the boundary. Using quadratic interpolation in the time domain, the transmitting formula for the first order extrapolation may be written as (Liao and Liu, 1992).

$$\{U_J^{p+1}\} = [T_1]\{u_1\} \tag{4}$$

where $\{U_J^{p+1}\}$ = Displacement of the boundary node J at time

t = $(p + 1)\,\Delta t$, p is an integer and Δt is the time step

$\{u_1\}$ = Displacement vector along the line normal to the node J (Fig. 2)

$$\{u_1\} = \{u_J^p, u_{J-1}^p, u_{J-2}^p\}$$

$$[T_1] = [t_1, t_2, t_3] \tag{5}$$

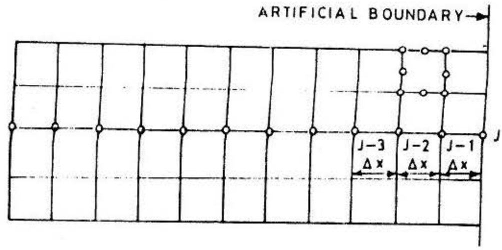

FIGURE 2
Artificial Boundary and Node Points Required for Extrapolation Algorithm

where $$t_1 = \frac{(2-s)(1-s)}{2} \; ; \; t_2 = s(2-s) \; ; \; t_3 = \frac{s(s-1)}{2} \tag{6}$$

and $$s = C_a \frac{\Delta t}{\Delta x} \tag{7}$$

The value of the apparent wave velocity C_a to be used in equation (7) is the S-wave velocity for problems of two dimensional elastic wave propagation. The significance of the apparent wave velocity lies in representing the phase velocities of waves impinging on the artificial boundary at various angles which is different from the physical velocity (P- and S-wave velocity) depending upon the material properties of the medium. Thus the artificial boundary condition, equation (4) can easily be implemented into dynamic finite element analysis.

Equation of Motion and its Solution in Direct Time Domain.

The equation of motion for the elastic half-space may be written as,

$$[M]\{\ddot{U}\} + [K]\{U\} = \{Q(t)\} \tag{8}$$

where $[M]$ is the mass matrix,

$[K]$ the stiffness matrix,

$\{\ddot{U}\}$ the acceleration vector,

$\{U\}$ the displacement vector and

$\{Q(t)\}$ the external load vector.

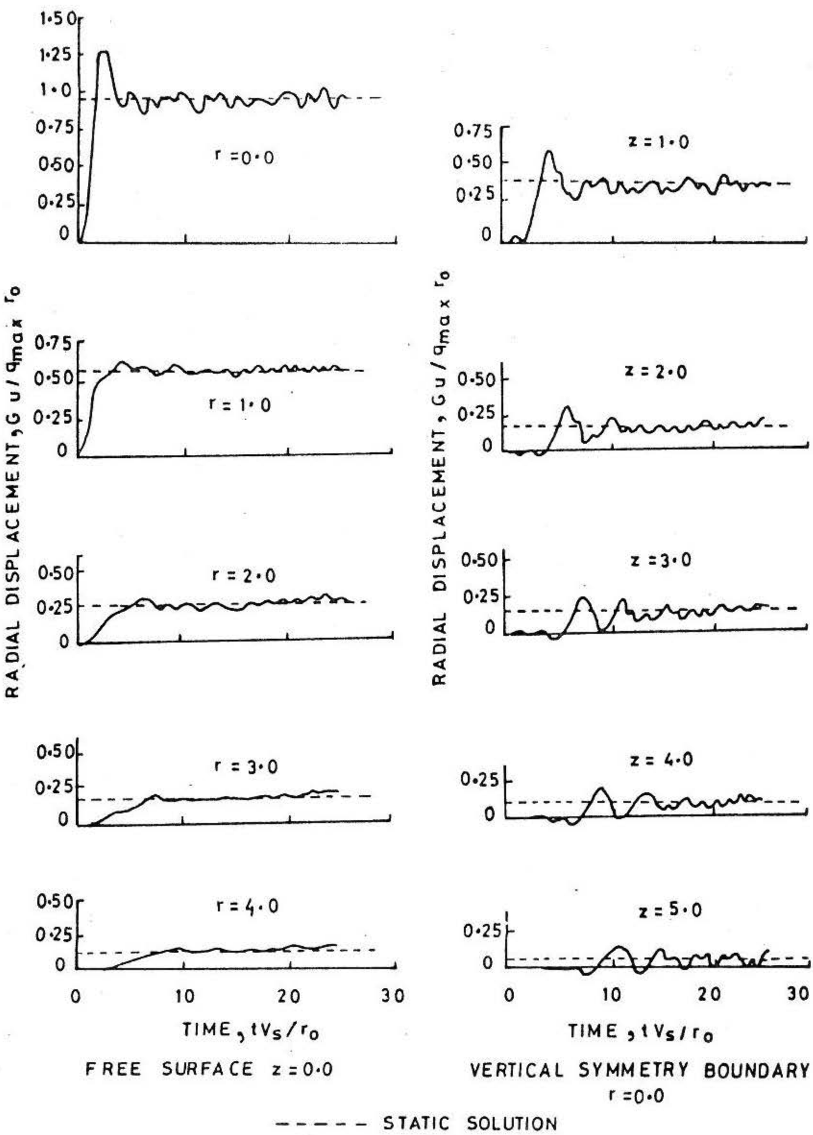

FIGURE 3

Response of Radial Displacements in Homogeneous Soil Medium Under Suddenly Applied Surface Shear Load

The solution of equation (8) can be obtained by carrying out the integration in direct time domain Different schemes for carrying out the integration of equation (8) have been outlined in Bathe (1982). In the present study the explicit method of time integration based on central difference approximation has been used. The stability and accuracy of the solution is based on the consideration that the time step Δt chosen is less than the critical time step $(\Delta t)_{critical}$ which depends upon the time required for the fastest wave to travel between two successive nodes. The time step Δt adopted in the present work is one third of $(\Delta t)_{critical}$.

Elastic Half-space Subjected to Suddenly Applied Surface Shear Load

Response of Homogeneous Elastic half-space

Consider the homogeneous elastic half-space discretised by using eight noded elements shown in Fig. 1 under the action of uniformly distributed suddenly applied surface shear load of unit intensity over a circular area of radius r_o. The response of the homogeneous elastic half-space is given in the form of variations of radial displacements with time at different locations on the vertical symmetry boundary and on the free surface (Fig. 3). The corresponding static solutions are also indicated in Fig. 3. The variations of radial displacements with time at different locations along the free surface $(z = 0)$, at varying radial distance $(r = 0.0,\ 1.0,\ 2.0,\ 3.0$ and $4.0)$ from Fig. 3, shows that the radial displacements attain peak amplitudes and later tend towards static solution. The peak amplitudes of the radial displacements tend to decrease with increase in the radial distances (Fig. 3). Similar behaviour of the responses of the radial displacements can be examined from Fig. 3, at various locations on the vertical symmetry boundary $(r = 0)$, at different depth$(z = 1.0,\ 2,0,\ 3,0,\ 4,0$ and $5,0)$. No reflections from the artificial boundaries can be detected in the foregoing analysis.

Response of Two Layered Medium

Consider the two layered medium under the surface shear loading as shown in Fig. 4 such that the S-wave velocity V_{s1} in the upper layer is half of the S-wave velocity V_{s2} in the lower layer. Waves travelling in such a medium will produce reflections and refractions at the interface of the medium and hence result in transmission of waves at a wide range of incident angles on the boundaries. Also multiple reflections within the medium can cause out of plane motions resulting in Love wave modes.

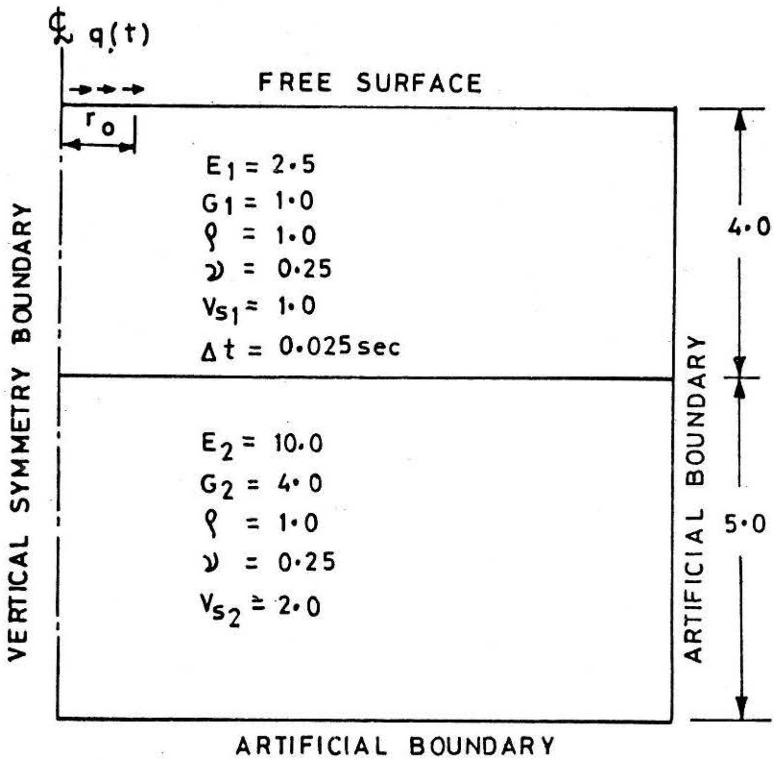

FIGURE 4
Two Layered Medium Under Surface Shear Load

Therefore the wave propagation studies in such a medium will be a severe test for the performance of the numerical model. Finite element idealisation adopted is same as in the case of the homogeneous soil medium. The response of the two layered medium subjected to suddenly applied surface shear load of unit intensity over a circular area of unit radius r_o is shown in Fig. 5 in the form of time histories of radial displacements at various locations on the free surface and on the vertical symmetry boundary. From Fig. 5, the variations of radial displacements with time at various locations on the vertical symmetry boundary ($r = 0$) with varying vertical distances ($z = 1.0$, 2.0, 3.0, 4.0 and 5.0), it can be observed that the radial displacements attain peak amplitudes and later tend towards the static solution and also the amplitudes of the radial displacements tend to decrease with increase in the vertical distances. Similar nature of the behaviour can be observed for the responses of radial displacements at various locations on the free surface (z=0) with varying radial distances ($r = 0.0$, 1.0, 2.0, 3.0 and 4.0).

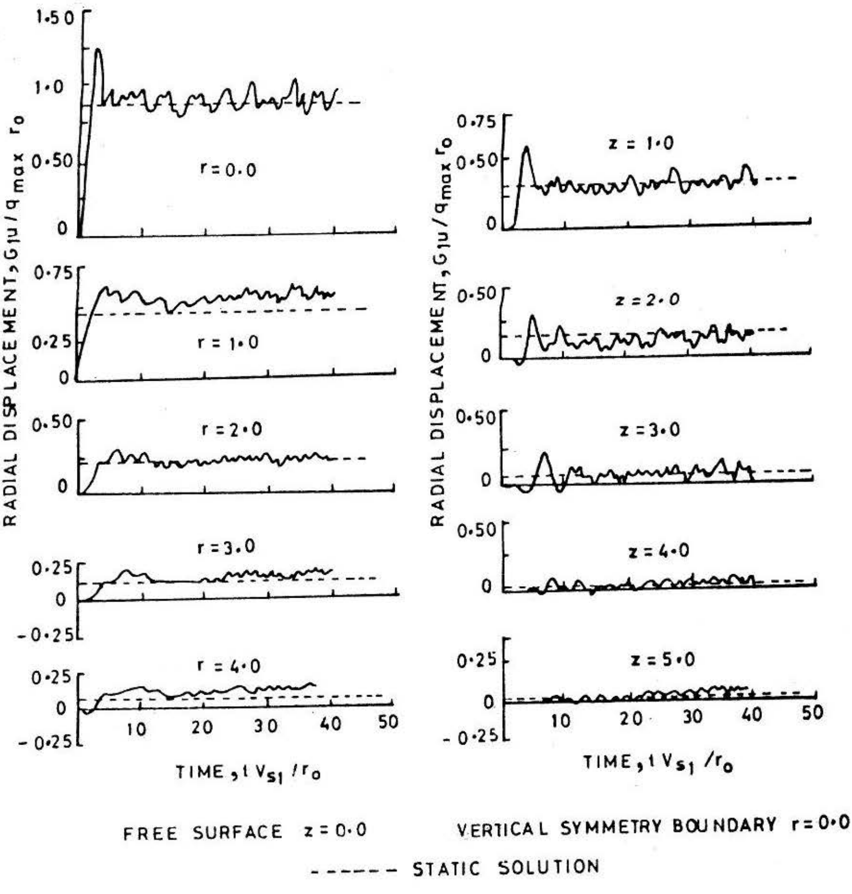

FIGURE 5
Response of Radial Displacements in Two Layered Medium Under Suddenly Applied Surface Shear Load

Elastic Half-space Subjected to Sinusoidal Surface Shear Load

Response of Homogeneous Elastic Half-space

Consider the homogeneous elastic half-space idealised by using eight noded rectangular elements shown in Fig. 1. The homogeneous soil medium is subjected to uniformly distributed surface sinusoidal shear load having unit amplitude with an exciting frequency of 1.25 rad/sec applied over a circular area of unit radius r_o. The results of the analysis are presented in

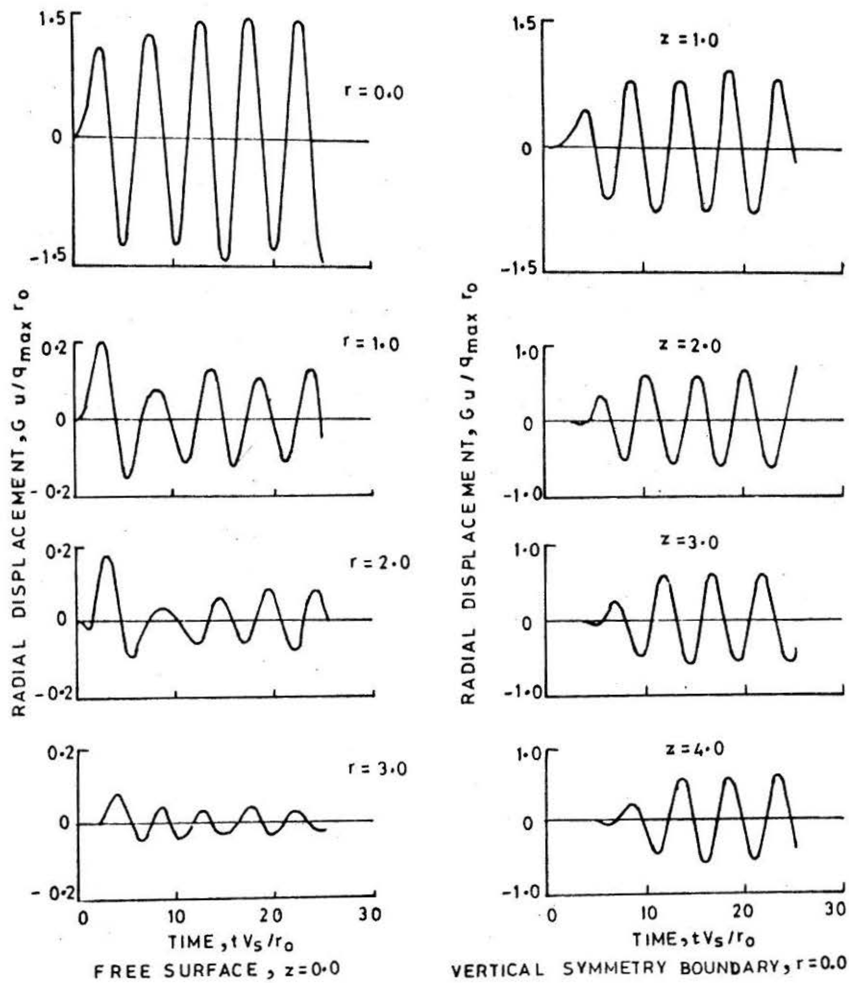

FIGURE 6

Response of Radial Displacements in Homogeneous Soil Medium Under Sinusoidal Surface Shear Load

the form of time histories of radial displacements (Fig. 6) and shear stresses computed at different locations in the medium (Fig. 7). It is observed from the analysis that at different locations ($z = 1.$, 2.0, 3.0 and 4.0) along the vertical symmetry boundary ($r = 0$), that the radial displacements become harmonic having stable amplitude once the initial phase is over. Also the amplitudes of the radial displacements tend to decrease with depth. Similar nature of the behaviour of the radial displacements can be seen at various locations on the free surface ($z = 0$) at different radial distances ($r = 0.0$, 1.0, 2.0 and 3.0) (Fig. 6). It can be observed from Fig. 7, that shear stress at a

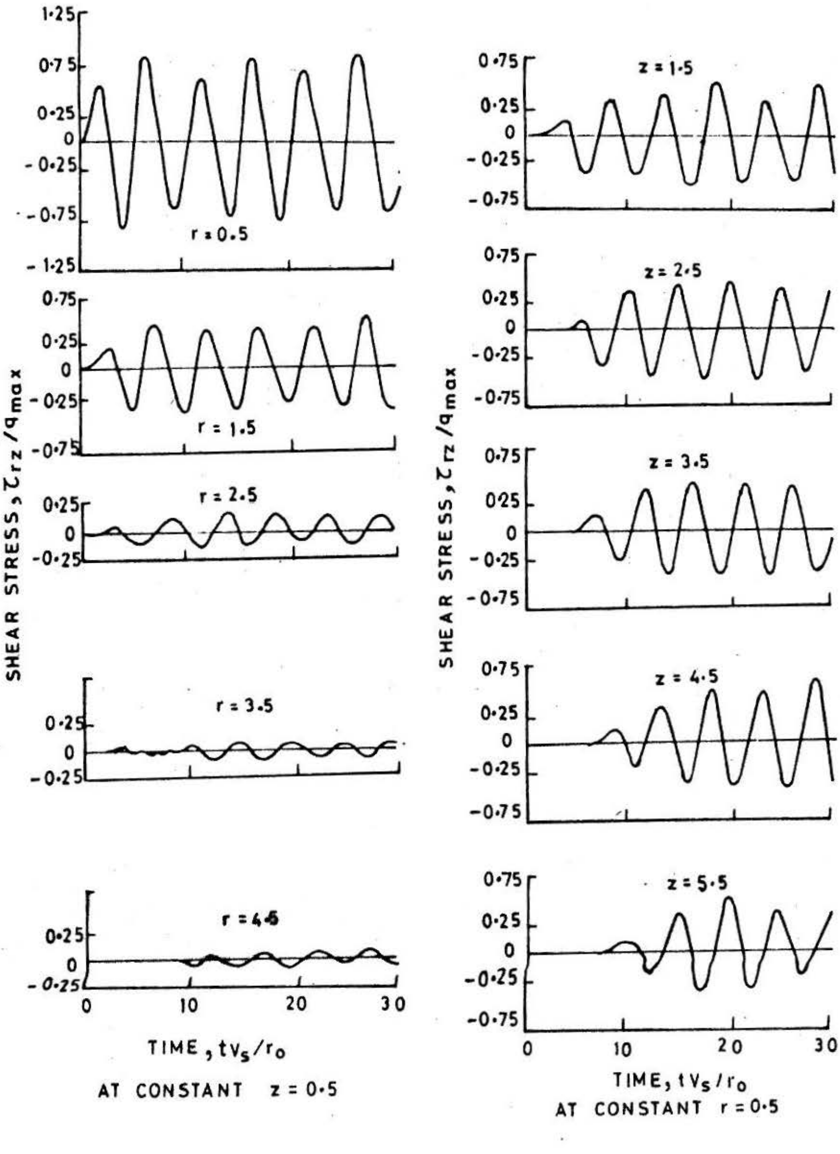

FIGURE 7
Response of Shear Stresses in Homogeneous Soil Medium Under Sinusoidal Surface Shear Load

given point also becomes harmonic having constant and stable amplitude. Also the amplitudes of the computed shear stresses tend to decrease with increase in the distance from the loaded area and no reflections from the artificial boundary can be detected (Fig. 7).

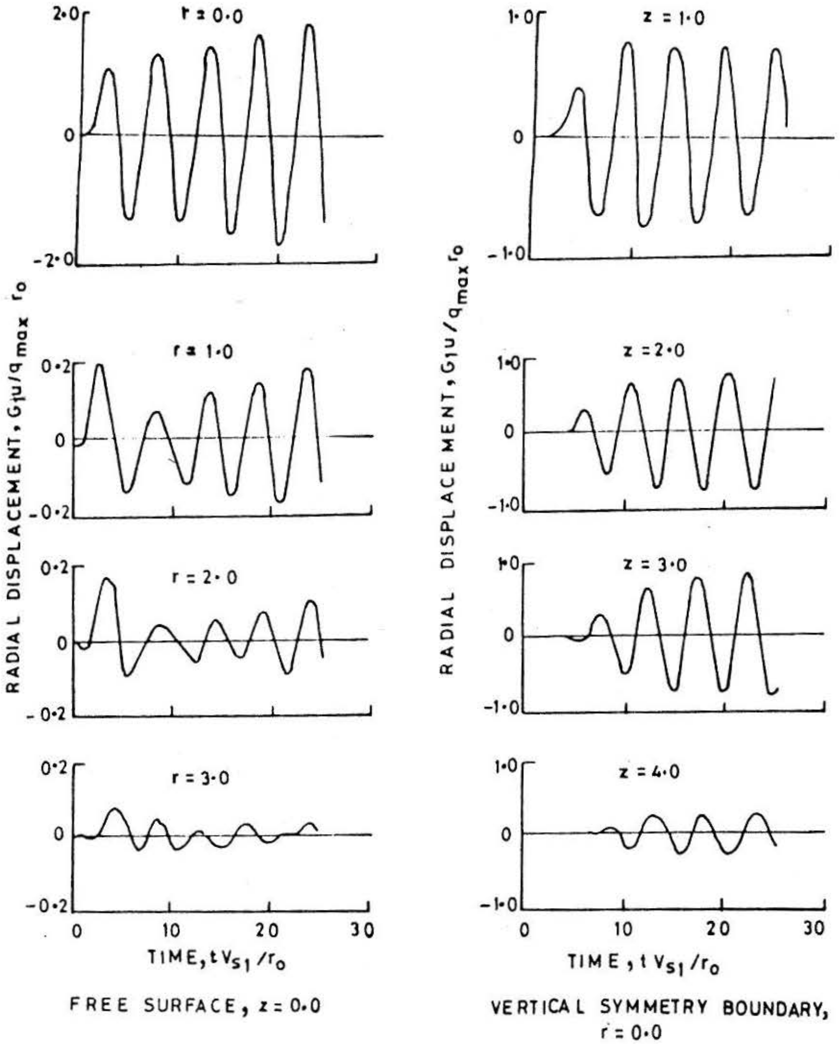

FIGURE 8
Response of Radial Displacements in Two Layered Medium Under Sinusoidal Surface Shear Load

Response of Two Layered Medium

Consider the two layered medium shown in Fig. 4 and subjected to uniformly distributed surface sinusoidal shear load having unit amplitude with an exciting frequency of 1.25 rad/sec applied over a circular area of unit radius r_o. The finite element discretisation used in this case is the same as in the previous problem of the homogeneous soil medium. It can be

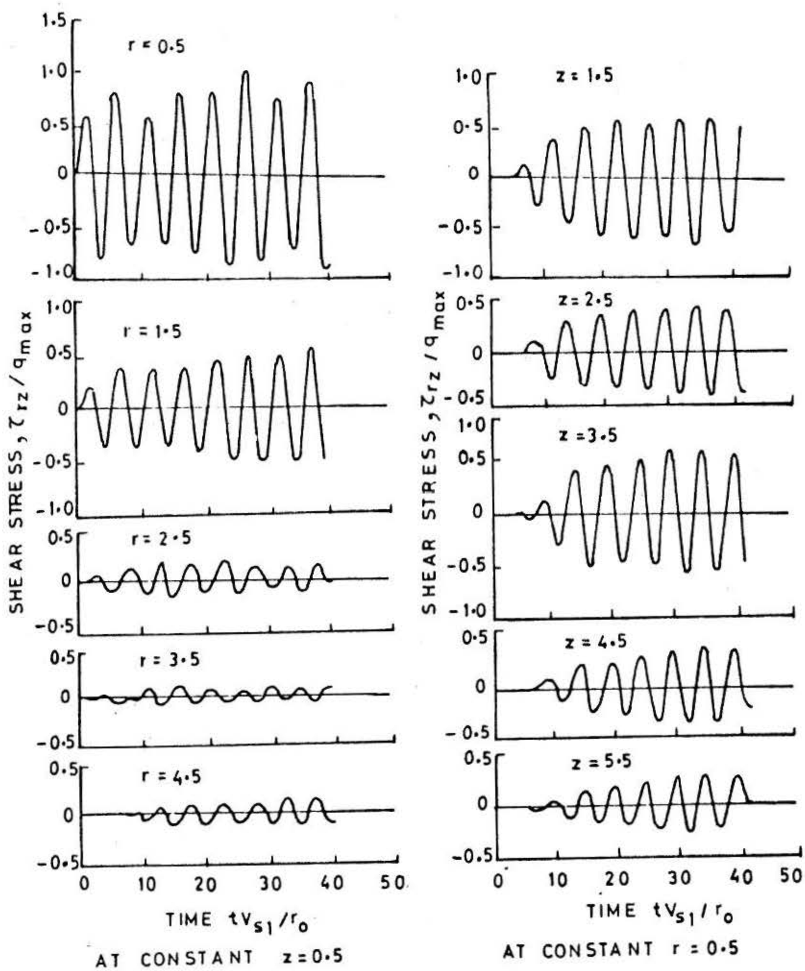

FIGURE 9
Response of Shear Stresses in Two Layered Medium Under Sinusoidal Surface Shear Load

observed from Fig. 8, that the responses of radial displacements at different locations along the vertical symmetry boundary ($r = 0$) with varying vertical distances ($z = 1.0$, 2.0,3.0 and 4.0) become harmonic having constant and stable amplitude. Responses of shear stresses are shown in Fig. 9 at various locations in the medium. the response of shear stresses also become harmonic wave having constant and stable amplitude. Also it is clearly seen from Figs. 8 and 9 that the peak amplitudes of radial displacements and shear stresses decrease with increase in the distances from the loaded area and reflection free finite element solutions are obtained.

Conclusions

Semi-analytical finite element formulation for wave propagation studies in homogeneous and layered medium subjected to surface shear loads based on time domain approach has been presented. The semi infinite extent of the soil medium has been modelled by using first order extrapolation algorithm. For the half-spaces subjected to suddenly applies surface shear load over a circular area the responses of displacements attain peak amplitudes and later tend towards the static solution. it is found that the extrapolation algorithm supports static load components for surface shear loads having non-vanishing time average. *The usefulness of the methods has* been demonstrated by considering surface shear loads varying harmonically with time in which the responses relating to displacements and stresses become harmonic waves. The amplitudes of displacements and stresses tend to decay with increasing distances from the loaded area and reflection free finite element solutions are obtained. The presented numerical procedure is extremely useful in the analysis of axisymmetric dynamic soil-structure interaction problems involving asymmetric loading.

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