Indian Geotechnical Journal, 23 (4), 1993

Estimation of Average Stress Increase due to Footing Loading

by

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Introduction

Several methods are in vogue for estimating the increased stress on an element of soil below a foundation. The simplest method which is comparable to Boussinesq's stress distribution, is to use a 2 : 1 slope. Some researchers prefer to use a stress distribution defined by a slope in the range of 30° to 45° with vertical as shown in Fig. 1.

The final consolidation settlement, S, of a clay stratum of thickness, H, can be predicted using Terzaghi's equation,

$$S = \frac{C_c}{1+e_o} H \log_{10}\left(\frac{p_o' + \Delta p}{p_o'}\right) \tag{1}$$

where C_c, e_o, P_o', and Δp are compression index, initial void ratio, existing effective overburden pressure at the centre of clay layer and average stress increase on clay layer due to footing loading, respectively.

Bowles (1982) presented a trial and error procedure to find the size of a square footing for a given settlement, assuming a 2 : 1 slope for stress dispersion. The average stress increase in the clay stratum when a square footing is loaded, is obtained by the following integration for any angle θ, of stress dispersion from vertical (Fig. 1),

$$\Delta p = \frac{1}{H} \int_{H_1}^{H_2} \frac{V}{(B + 2z \tan\theta)^2} \cdot dz \tag{2}$$

where H_1, and H_2 are the limits for clay thickness as shown in Fig. 2 and V is the vertical load. The average stress increase for a square footing is therefore

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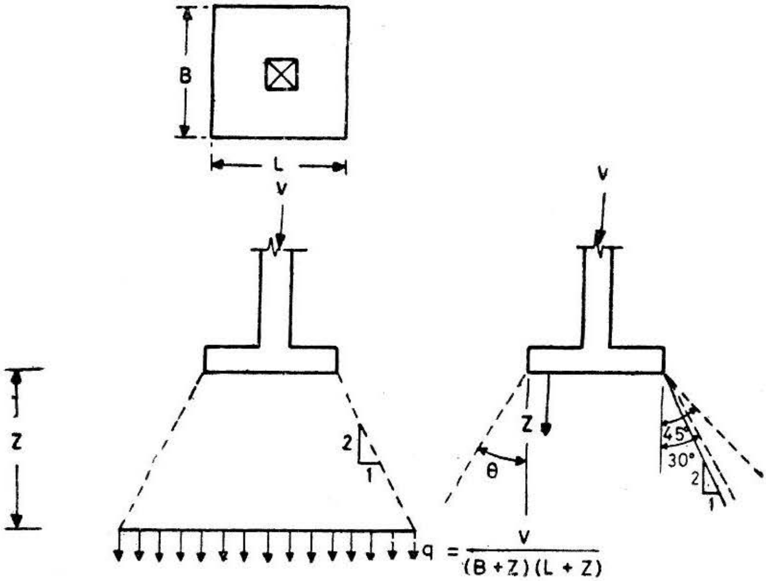

FIGURE 1

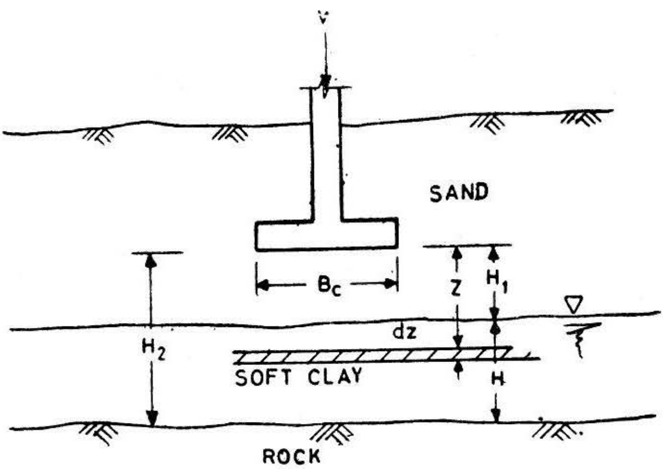

FIGURE 2

$$\Delta p = \frac{V}{(B+2H_1 \tan\theta)\,(B + 2H_2 \tan\theta)} \tag{3}$$

Similarly, the stress increase in the case of circular and rectangular footings are respectively as follows :

$$\Delta p = \frac{4V}{\pi(B_c + 2H_1 \tan\theta)\,(B_c+2H_2 \tan\theta)} \tag{4}$$

$$\Delta p = \frac{V}{2BH(\alpha-1)\tan\theta} \cdot \log_e \left[\frac{(\alpha B+2H_1\tan\theta) \quad (B+2H_2 \tan\theta)}{(B+\ 2H_1\tan\ ) \quad (\alpha B+2H_2\tan\theta)} \right] \tag{5}$$

A better method for estimating the average stress increase, Δp, is by double integration of the Boussinesq equation for point load. This method is developed in the present investigation. The results are compared with 2 : 1 stress dispersion method.

Boussinesq stress distribution is widely used for estimation of settlement in layered soils unless there is a significant difference, say, by a factor of five or more in the stress—strain modulus of the two materials (Morgan and Gerrard, 1971).

Formulation Based on Boussinesq Equation

Boussinesq's equation considers a point load, Q, on the surface of an infinitely large, homogeneous, isotropic, weightless elastic half-space. The vertical stress, q_v, at depth Z and at a horizontal distance, r, from load application is given by

$$q_v = \frac{3Q}{2\pi z^2} \frac{1}{\left[1 + \left(\frac{r}{z}\right)^2\right]^{5/2}} \tag{6}$$

Using the Eq. (6), expressions for average stress increase Δp, due to column loads on circular, square and rectangular shapes, are developed herein.

(a) Circular Footing

Using Eq. (6), an expression for q, the stress at depth, z, for a uniform loading of intensity, q_o, if spread on a circular area of radius, r, can be obtained on integration as;

$$q = q_o \left[1 - \frac{1}{\left\{1 + \left(\frac{r}{z}\right)^2\right\}^{3/2}}\right] \tag{7}$$

Eq. (7) has to be integrated once again to obtain average stress in case of circular footing (Fig. 2) in the clay layer which is situated between the depths H_1 and H_2 below the bearing level.

$$\Delta p = \frac{q_o}{H} \int_{H_1}^{H_2} \left[1 - \frac{1}{\left\{1 + \left(\frac{r}{z}\right)^2\right\}^{3/2}}\right] . dz \tag{8}$$

$$= \frac{q_o}{H} \left[z - \frac{2r^2 + z^2}{\sqrt{r^2 + z^2}} \right]_{H_1}^{H_2} \tag{9}$$

Let $r = \frac{B_c}{2}$ where B_c is diameter of a circular footing and $q_o = \frac{4V}{\pi B_c^2}$

Then

$$\Delta p = \frac{4V}{\pi B_c^2 H}\left\{H + \frac{B_c^2 + 2H_1^2}{\sqrt{B_c^2+4H_1^2}} - \frac{B_c^2 + 2H_2^2}{\sqrt{B_c^2+4H_2^2}}\right\} \quad (10)$$

(b) Rectangular Footing

Based on the work of Boussinesq for point load, Steinbrenner (1936), has developed the following expression for stress, q, at depth z under the centre of the rectangular footing of length $2a$ and width $2b$ when it is uniformly loaded with an intensity of q_o.

$$q = \frac{2q_o}{\pi}\left[\frac{abz\,(a^2+b^2+2z^2)}{(a^2+z^2)\,(b^2+z^2)\,\sqrt{a^2+b^2+z^2}} + \text{Sin}^{-1}\frac{ab}{\sqrt{(a^2+z^2)\,(b^2+z^2)}}\right] \quad (11)$$

If $a = \frac{L}{2}$ and $b = B/2$ where L and B are dimensions of rectangular footing.

$$q = \frac{2q_o}{\pi}\left[\frac{2LBz\,(L^2+B^2+8z^2)}{(L^2+4z^2)\,(B^2+4z^2)\,\sqrt{L^2+B^2+4z^2}} + \sin^{-1}\frac{LB}{\sqrt{(L^2+4z^2)(B^2+4z^2)}}\right]$$

Then the average stress increase over the compressible clay stratum can be expressed as

$$\Delta p = \frac{1}{H}\int_{H_1}^{H_2} q.dz = \frac{2q_o}{\pi H}\int_{H_1}^{H_2}\left[\frac{2LBz(L^2+B^2+8z^2)}{(L^2+4z^2)\,(B^2+z^2)\,\sqrt{L^2+B^2+4z^2}}\right.$$

$$\left. + \sin^{-1}\frac{LB}{\sqrt{(L^2+4z^2)\,(B^2+4z^2)}}\right].\,dz \quad (12)$$

$$\text{Let } I_1 = \int_{H_1}^{H_2}\left[\frac{2LBz\,(L^2+B^2+8z^2)}{(L^2+4z^2)\,(B^2+z^2\,\sqrt{L^2+B^2+4z^2}}\right].\,dz$$

and

$$I_2 = \int_{H_1}^{H_2}\sin^{-1}\left[\frac{LB}{(L^2+4z^2)\,(B^2+4z^2)}\right].\,dz$$

$$\text{Let } u = f(z) = \frac{LB}{(L^2+4z^2)\,(B^2+4z^2)} \quad (13)$$

$$\text{Then} \quad I_2 = \left[z\sin^{-1}f(z)\right]_{H_1}^{H_2} - \int_{H_1}^{H_2}\frac{z, f'(z)}{\sqrt{1-f^2(z)}}.\,dz \quad (14)$$

Taking log on both sides of Eq. (13)

$$\log_e f(z) = \log_e(LB) - \tfrac{1}{2}\log_e(L^2+4z^2) - \tfrac{1}{2}\log_e(B^2+4z^2) \quad (15)$$

$$\frac{f'(z)}{f(z)} = -\frac{4z}{L^2+4z^2} - \frac{4z}{B^2+4z^2}$$

Rearranging

$$z f'(z) = \frac{-4z^2 (L^2 + B^2 + 8z^2)}{(L^2 + 4z^2)(B^2 + 4z^2)} . f(z)$$

and $1 - f^2(z) = \dfrac{4z^2 (L^2 + B^2 + 4z^2)}{(L^2+4z^2)(B^2 + 4z^2)}$

Then Eq. (14) reduces to

$$I_2 = \left[z \sin^{-1} \frac{LB}{(L^2 + 4z^2)(B^2 + 4z^2)} \right]_{H_1}^{H_2}$$

$$- \int_{H_1}^{H_2} \left[\frac{-4z^2 (L^2+B^2+8z^2)}{(L^2+4z^2) B^2+4z^2)} f(z) \Bigg/ \sqrt{\frac{4z^2(L^2+B^2 + 4z^2)}{(L^2+4z^2)(B^2+4z^2)}} \right] . dz$$

substituting for $f(z)$ and rearranging

$$I_2 = \left[z \sin^{-1} \frac{LB}{\sqrt{(L^2+4z^2)(B^2+4z^2)}} \right]_{H_1}^{H_2}$$

$$+ \int_{H_1}^{H_2} \frac{2LBz(L^2+B^2+8z^2)}{(L^2+4z^2)(B^2+4z^2)\sqrt{(L^2+B^2+4z^2}} . dz$$

$$\therefore\ I_2 = \left[z \sin^{-1} \frac{LB}{\sqrt{(L^2+4z^2)(B^2+4z^2)}} \right]_{H_1}^{H_2} + I_1$$

However $I = I_1 + I_2$

$$\therefore\ I = 2I_1 + \left[z \sin^{-1} \frac{LB}{\sqrt{(L^2+4z^2)(B^2+4z^2)}} \right]_{H_1}^{H_2} \tag{16}$$

Consider $I_1 = \displaystyle\int_{H_1}^{H_2} \frac{2LBz (L^2+B^2+8z^2)}{(L^2+4z^2)(B^2+4z^2)\sqrt{L^2+B^2+4z^2}} . dz$

put $z^2 = w$, then $2z.dz = dw$

The lower and upper limits will change to H_1^2 and H_2^2 respectively.

$$\therefore\ I_1 = \int_{H_1^2}^{H_2^2} \frac{LB (L^2 + B^2 + 8w)}{(L^2+4w)(B^2 + 4w)\sqrt{(L^2 + B^2+ 4w)}} . dw \tag{17}$$

Let $\sqrt{L^2 + B^2 + 4w} = t$

Differentiating,

$$\frac{2}{\sqrt{L^2 + B^2 + 4w}} \cdot dw = dt$$

With the above substitution, the integration limits will change as follows:

$$\text{lower limit } t_1 = \sqrt{L^2+B^2+4H_1^2}$$

$$\text{upper limit } t_2 = \sqrt{L^2+B^2+4H_2^2}$$

Eq. (17) reduces to

$$I_1 = \int_{t_1}^{t_2} \frac{LB\,[L^2 + B^2\, 2(t^2-L^2-B^2)]}{2(t^2-B^2)\,(t^2-L^2)} \cdot dt$$

$$= \frac{LB}{2}\left[\frac{1}{2L} \log_e\left(\frac{t-L}{t+L}\right) + \frac{1}{2B} \log_e\left(\frac{t-B}{t+B}\right)\right]_{t_1}^{t_2}$$

$$2I_1 = \left[\log_e\left(\frac{t-B}{t+B}\right)^{\frac{L}{2}} + \log_e\left(\frac{t-L}{t+L}\right)^{\frac{B}{2}}\right]_{t_1}^{t_2}$$

$$= \tfrac{1}{2} \log_e\left[\left\{\frac{(t_2-B)\,(t_1+B)}{(t_2+B)\,(t_1-B)}\right\}^L \left\{\frac{(t_2-L)\,(t_1+L)}{(t_2+L)\,(t_1-L)}\right\}^B\right]$$

However $t_1 = \sqrt{L^2 + B^2+4\,H_1^2}$ and $t_2 = \sqrt{L^2+B^2+4H_2^2}$

$$\therefore\ 2I_1 = \tfrac{1}{2} \log_e\left[\left\{\frac{E_1}{E_2}\right\}^L \cdot \left\{\frac{E_3}{E_4}\right\}^B\right]$$

where $E_1 = \left(\sqrt{L^2+B^2+4H_2^2} - B\right) \cdot \left(\sqrt{L^2+B^2+4H_1^2} + B\right)$

$$E_2 = \left(\sqrt{L^2+B^2+4H_2^2} + B\right) \cdot \left(\sqrt{L^2+B^2+4H_1^2} - B\right)$$

$$E_3 = \left(\sqrt{L^2+B^2+4H_2^2} - L\right) \cdot \left(\sqrt{L^2+B^2+4H_2^2} + L\right)$$

$$E_4 = \left(\sqrt{L^2+B^2+4H_2^2} + L\right) \cdot \left(\sqrt{L^2+B^2+4H_1^2} - L\right)$$

Eq. (16) becomes

$$I = \tfrac{1}{2} \log_e \left\{\left\{\frac{E_1}{E_2}\right\}^L \cdot \left\{\frac{E_3}{E_4}\right\}^B\right\} + \left[z \sin^{-1} \frac{LB}{\sqrt{(L^2+4z^2)\,(B^2+4z^2)}}\right]_{H_1}^{H_2}$$

Substituting in Eq. (12)

$$\Delta p = \frac{2q_o}{\pi H}\left[\frac{1}{2}\log_e\left\{\left\{\frac{E_1}{E_2}\right\}^L \cdot \left\{\frac{E_3}{E_4}\right\}^B\right\} + H_2 \sin^{-1}\left(\frac{LB}{\sqrt{(L^2+4H_2^2)(B^2+4H_2^2)}}\right)\right.$$

$$\left. - H_1 \sin^{-1}\left(\frac{LB}{\sqrt{(L^2+4H_1^2)(B^2+4H_1^2)}}\right)\right] \quad (17)$$

Let $L = \alpha B$ where α is a constant

Area of rectangular footing $= L \times B = \alpha B^2$

and $q_o = \frac{V}{LB} = \frac{V}{\alpha B^2}$

Then substituting and rearranging Eq. (17)

$$\Delta = \frac{2V}{\alpha \pi B^2 H}\left[\frac{B}{2} \cdot \log_e\left\{\left\{\frac{F_1}{F_2}\right\}^\alpha \cdot \left\{\frac{F_3}{F_4}\right\}\right\}\right.$$

$$+ H_2 \sin^{-1} \frac{\alpha B^2}{\sqrt{(\alpha^2 B^2+4H_2^2)(B^2+4H_2^2)}}$$

$$\left. - H_1 \sin^{-1} \frac{\alpha B^2}{\sqrt{(\alpha^2 B^2+4H_1^2)(B^2+4H_1^2)}}\right] \quad (18)$$

where $F_1 = \left(\sqrt{(\alpha^2+1)B^2+4H_2^2} - B\right)\left(\sqrt{(\alpha^2+1)B^2+4H_1^2} + B\right)$

$$F_2 = \left(\sqrt{(\alpha^2+1)B^2+4H_2^2} + B\right)\left(\sqrt{(\alpha^2+1)B^2+4H_1^2} - B\right)$$

$$F_3 = \left(\sqrt{(\alpha^2+1)B^2+4H_2^2} - \alpha B\right)\left(\sqrt{(\alpha^2+1)B^2+4H_1^2} + \alpha B\right)$$

$$F_4 = \left(\sqrt{(\alpha^2+1)B^2+4H_2^2} + \alpha B\right)\left(\sqrt{(\alpha^2+1)B^2+4H_1^2} - \alpha B\right)$$

(*c*) *Square Footing*

For square footing, substitution of $\alpha = \frac{L}{B} = 1$ in Eq. (18), gives

$$\Delta p = \frac{2V}{\pi B^2 H}\left[B \log_e\left\{\frac{F_5}{F_6}\right\} + H_2 \sin^{-1}\left\{\frac{B^2}{B^2+4H_2^2}\right\}\right.$$

$$\left. - H_1 \sin^{-1}\left\{\frac{B^2}{B^2+4H_1^2}\right\}\right] \quad (19)$$

$$\text{where } F_5 = \left(\sqrt{2B^2 + 4H_2^2} - B\right)\left(\sqrt{2B^2 + 4H_1^2} + B\right)$$

$$F_6 = \left(\sqrt{2B^2 + 4H_2^2} + B\right)\left(\sqrt{2B^2 + 4H_1^2} - B\right)$$

Making use of Eqs. 10, 18 and 19, the average stress increase for circular, square and rectangular shapes can be obtained. The present practice for obtaining the average stress increase, Δp, is to divide the clay layer into number of sub layers and evaluate the stresses at the centre of each such sub-layers and find the weighted average of all of them. This method can be replaced by using the expressions developed in this investigation. An illustrative example is presented herein.

Illustration

An isolated footing resting on sandy soil is shown in Fig (2). A clay layer of 4.50m thick is present at a depth of 3.0m from the bottom of footing. The bulk densities of sand and clay deposits are $18.0 kN/m^3$ and $19.0\ kN/m^3$ respectively. The clay stratum is divided into 8, 16, 50, 100, 500 and 1000 sub-layers respectively and the stresses at the middle of each sub-layer are calculated corresponding to square, rectangular and circular isolated footings (Tables. 1 to 3). The average stresses are obtained by summing up the calculated stresses at the centre of the sub-layers and dividing by the number of sub-layers. The angles of dispersion for stress are assumed as 26.565° (2 : 1) 30°, 35° and 45° with the vertical. These values are compared with the values obtained using Eqs. (3), (5) and (4). The average stress is seen to correspond to the results obtained for 500 sub-layers.

In order to compare these simple stress dispersion methods with Boussinesq equations for circular and rectangular loadings, the average stresses for the various sub-layers werec alculated using Eqs. (7) and (11) by finding the stresses at mid points of sub-layers and then calculating the average stresses for 8, 16, 50, 100, 500 and 1000 sub-layers. These values are compared with the Eqs. (10), (18) and that are developed in this investigation. The results agree with the summation methods for 500 sub-layers.

Therefore average stress increase in clay layer can be determined using either equations 4, 3 and 5 due to simple stress dispersion or equations 10, 19 and 18 using Boussinesq equation. These expressions are for circular, square and rectangular footings respectively.

Conclusion

Expressions for finding the average stresses in clay layer due to isolated footing loadings are developed in this investigation for simple and Boussinesq stress dispersion methods. The advantage in using these expressions is

TABLE 1

Estimation of Average Stress Increase Using Various Methods For Square Footing

B = 2.790 m

Sl No	No. of sub-layers	Average Stress Increase Due to External Loading In kN/m²									
		2:1 Stress dispersion		$\theta=30°$ Dispersion		$\theta=35°$ Dispersion		$\theta=45°$ Dispersion		Boussinesq Eqn.	
		Summation method	Using eq(3)	Summation method	Using eq(3)	Summation method	Using eq(3)	Summation method	Using eq(3)	Summation method	Using eq(19)
1	8	6.3687		5.2977		4.0811		2.4243		6.8044	
2	16	6.3757		5.3042		4.0869		2.4286		6.8185	
3	50	6.3778		5.3062		4.0887		2.4299		6.8227	
4	100	6.3780		5.3064		4.0888		2.4300		6.8231	
5	500	6.3781		5.3064		4.0888		2.4301		6.8232	
6	1000	6.3781	6.3781	5.3064	5.3064	4.0888	4.0888	2.4301	2.4301	6.8232	6.8232

TABLE 2

Estimation Of Average Stress Increase Using Various Methods For Rectangular Footing

$B = 2.173\text{m}$ $\qquad \frac{L}{B} = 1.6$

Average Stress Increase Due To External Loading In kN/m^2

Sl No	No. of sub-layers	2:1 Stress dispersion		$\theta=30^\circ$ Dispersion		$\theta=35^\circ$ Dispersion		$\theta=45^\circ$ Dispersion		Boussinesq eqn.	
		Summation method	Using eq(5)	Summation method	Using eq(5)	Summation Method	Using eq(5)	Summation method	Using eq(5)	Summation method	Using eq(18)
1	8	6.7561		5.5902		4.2778		2.5136		6.7366	
2	16	6.7641		5.5976		4.2842		2.5183		6.7503	
3	50	6.7665		5.5998		4.2862		2.5197		6.7544	
4	100	6.7667		5.6000		4.2863		2.5198		6.7548	
5	500	6.7668		5.6001		4.2864		2.5198		6.7549	
6	1000	6.7668	6.7668	5.6001	5.6001	4.2864	4.2864	2.5198	2.5198	6.7549	6.7549

TABLE 3

Estimation of Average Stress Increase Using Various Methods For Circular Footing

Diameter $B = 3.751$m

Sl. No.	No. of sub-layers	2:1 Stress dispersion		$\theta=30°$ Dispersion		$\theta=35°$ Dispersion		$\theta=45°$ Dispersion		Boussinesq eqn.	
		Average Stress Increase Due To External Loading In kN/m^2									
		Summation method	Using eq(4)	Summation method	Using eq(4)	Summation method	Using eq(4)	Summation method	Using eq(4)	Summation method	Using eq.(10)
1	8	6.3627		5.3960		4.2619		2.6410		6.4519	
2	16	6.3687		5.4013		4.2663		2.6449		6.4632	
3	50	6.3697		5.4028		4.2687		2.6460		6.4666	
4	100	6.3699		5.4030		4.2683		2.6462		6.4669	
5	500	6.3699		5.4030		4.2684		2.6462		6.4670	
6	1000	6.3699	6.3699	5.4030	5.4030	4.2684	4.2684	2.6462	2.6462	6.4670	6.4670

that the entire compressible layer can be treated as one single layer and for finding the average stress in clay layer, the stresses at several layers need not be calculated and summed up.

REFERENCES

BOWLES, JOSEPH. E (198?), "Foundation Analysis and Design", McGraw Hill Inc, New York.

MORGAN, J.R. and GERRAD (1971), "Behaviour of sands under Surface loads", *Journal of Soil Mechanics and Foundation Division*, ASCE, Vol. 97, SMR, Dec. pp. 1675-1699.

STEINBRENNER, W. (1936), "Tafein Zur Setzungs Berechnung" Strasse. Vol. 1.