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Lateral Displacements Due to Vertical Surface Loads

by

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Introduction

The classic solution for a point load acting on the surface of semi-infinite, homogeneous, isotropic and elastic medium (Boussinesq, 1985) has been the source of many applications in geotechnical engineering. It has been utilised for the computation of vertical stresses, strains, settlements, etc. beneath loaded areas. A compendium of most available solutions is available (Poulos and Davis, 1975). In this note, the solution for the horizontal displacement due to a point load on the surface of a semi-infinite medium, is integrated to obtain the lateral displacements beneath uniformly loaded circular, square, and rectangular areas. Based on these charts, the optimum depth of reinforcement for reinforced foundation beds is suggested.

Theory

The lateral (radial) displacement at any point, A, (Fig. 1*a*) due to point load on the surface of a semi-infinite medium, is given as

$$\rho_r = \frac{P\,(1+\nu_s)}{2\pi\, E_s\, R}\left\{\frac{r.\,z}{R^2} - \frac{(1-2\nu_s)\, r}{R+z)}\right\} \tag{1}$$

where P is the point load, E_s and V_s the elastic deformation parameters for the soil, r and z are the radial and vertical coordinates of point, A, with respect to the point load, and $R = \sqrt{r^2+z^2}$. Eq. (1) is rewritten for convenience as

$$\rho_r = \frac{P}{(E_s\, z)}\, I_h \tag{2}$$

where

$$I_h = \frac{(1+\nu_s)}{2\pi\sqrt{1+\bar{r}^2}} \cdot \left\{\frac{\bar{r}}{1+\bar{r}^2} - \frac{(1-2\nu_s)\,\bar{r}}{1+\sqrt{1+\bar{r}^2}}\right\}$$

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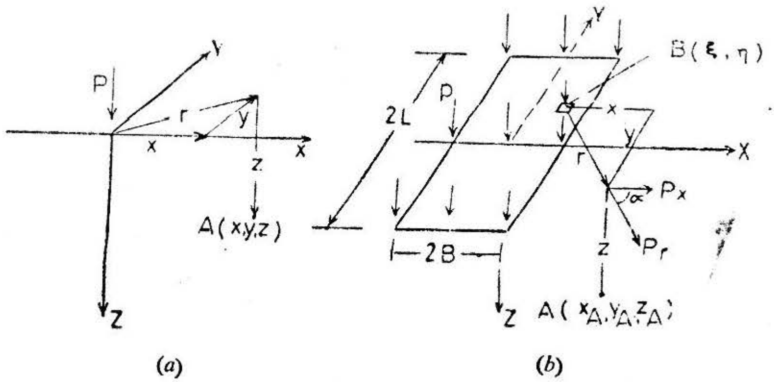


FIGURE 1 Definition Sketch (a) Point load (b) Uniformly loaded rectangle

and $\bar{r} = r/z$.

Ahlvin and Ulery (1162) have tabulated the results for stresses, strains, and displacements for points beneath a uniformly loaded circular area. The radial displacement is given as

$$\rho_r = p \; \frac{(1+\nu_s)}{E_s} \; (-r) \, \{(1-2\nu_s) \, E_1 - D\} \tag{3}$$

where p-is the intensity of uniform load, and D and E_1—non-dimensional coefficients dependent on the ratios r/a and z/a, and a-the radius of the loaded area. Values of D and E_1 are available in poulos and Davis (1975).

Eq. 3 can be expressed as

$$\rho_r = \frac{p \, . \, a}{Gs} \; I_r \tag{4}$$

where $I_r = 0.5(-r/a) \; \{(1-2 \; \nu_s) \; E_1 - D\}$ a dimensionless coefficient which signifies radial displacements.

The lateral displacement p_x of a point, A, beneath a uniformly loaded rectangular area (Fig. 1b) can be obtained by integrating Boussinesq solution for the point load with appropriate modification for the direction. That is, the radial displacement $\Delta\rho_r$ at A due to stresses on an elemental area ΔA, is

$$\Delta\rho_r \;=\; \frac{p\Delta A \; (1+\nu_s)}{2\pi \, E_s \, R} \left\{ \frac{rz}{R^2} - \frac{(1-2\nu_s) \, r}{(R+z)} \right\} \tag{5}$$

The lateral displacement $\Delta\rho_x$ along the x-direction is

$$\Delta\rho_x = \Delta\rho_r . \; \cos \alpha \tag{6}$$

where α is the angle between the radial line from the elemental area to point A

and the x-direction. The total lateral displacement, ρ_x, of point A is

$$\rho_x = \int_A \Delta\rho_x = \int_A \Delta\rho_r \cos\alpha \tag{7}$$

Eq. (6) is evaluated numerically as

$$\rho_x = \sum_i \sum_j \Delta\rho_r (i, j) \cos\alpha_{ij} \tag{8}$$

where i and j are the variables corresponding to the point and the loaded the sub-area. For points beneath the mid plane, using symmetry

$$\rho_x = 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta\rho_r (i, j) \cos\alpha_{ij} \tag{9}$$

wherein only half the loaded area need be subdivided into smaller elemental areas ($m \times n$). The radial distance, r, and the angle α of point A with respect to point B (ξ, η) corresponding to the sub area on which the applied stress is acting, are evaluated. It is very easy to program these steps and carry out the numerical integration on a computer. The final result, the lateral displacement, $\rho_{x'}$ is expressed as

$$\rho_x = \frac{p.B}{G_s} \cdot I_x \tag{10}$$

where G_s is the shear modulus of the soil and I_x is a dimensionless coefficient that depends on L/B—the aspect ratio of the loaded area, the depth z/B and distance x/B of the point at which displacement is being evaluated and Poisson's ratio, ν_s.

Results

The variation of lateral displacement influence coefficient, I_h, with distance, r/z for a point load on the surface is presented in Fig. 2 for Poisson's ratios of 0, 0.3 and 0.5. The coefficient increases with the ratio, r/z, and reaches its maximum value at r/z equal to about 0.5, 0.6 and 0.7 for Poisson's ratios of 0, 0.3 and 0.5 respectively. With further increase in r/z values, the influence coefficient decreases rapidly with r/z till r/z values of 2.0 to 2.5, and very gradually beyond these values. The lateral displacement coefficients are sensitive to the values of Poisson's ratio. The maximum value for the undrained condition ($\nu_s = 0.5$) is about 0.092 while for $\nu_s = 0.3$, it is 0.059. The coefficient is positive for all values of r/z for $\nu_s = 0.5$ indicating outward lateral movements of all points. For $\nu s = 0.3$ the lateral movements are outwards at any depth till r/z values of 3.1, beyond which they are inward. For smaller values of ν_s (< 0.3), the zone of outward movement is much smaller, and larger inward movements occur over a large range of r/z values ($r/z > 1.25$).

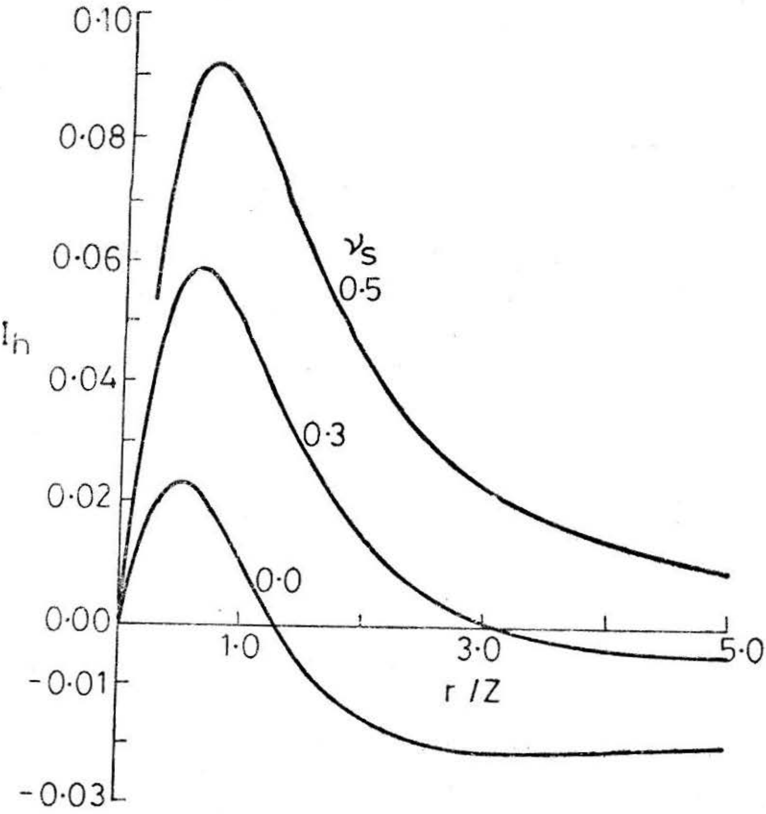
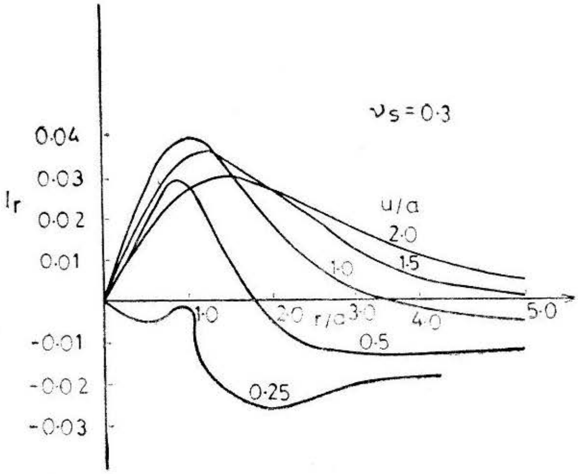
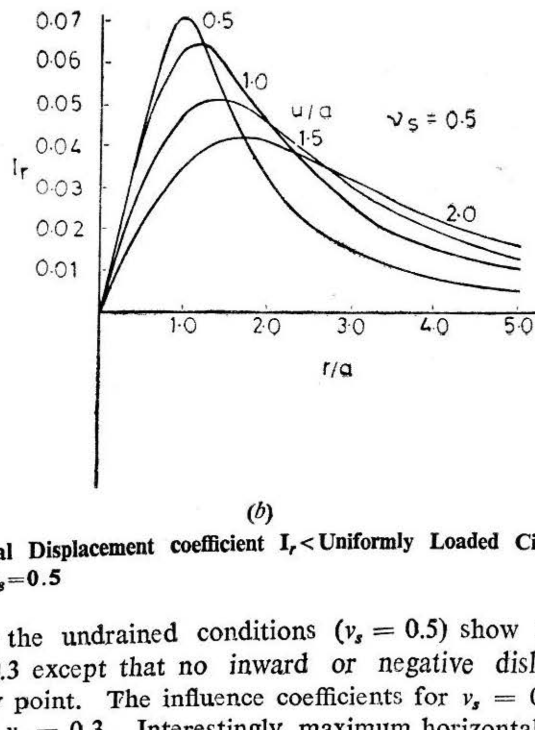


FIGURE 2 Lateral Displacement Coefficient I_h—Point Load

Lateral displacement influence coefficients I_r for points at different depths beneath a uniformly loaded circular area for ν_s = 0.3 and 0.5 are depicted in Fig. 3. These curves are plotted from results of Ahlvin and Ulery (1962). At a shallow depth, z/a equal to 0.25, the lateral displacements are inward at all points. The inward movement is negligible beneath the edge but is significantly larger out side the loaded area ($r/a > 1.0$). The displacements are positive and outward for depth ratio z/a equal to 0.5 or more. The coefficients increase with radial distance, reach their maximum values at points beneath the edge of the load. Maximum displacement of 0.04 occurs at a depth '*a*' and radial distance of 'a'. The lateral displacements decrease with distance out side the loaded area and become negative *i.e.* inward movements at shallower depths. This effect is noted clearly beneath the point load. The distances beyond which the horizontal displacements are inward are 1.75*a* for z/a = 0.5 and 3.2*a* or z/a = 1.0. Increasing the length of reinforcement strip beyond these distances would be ineffective. Horizontal displacements are more uniform and persist over larger distances for points at depths of 1.5*a* and 2*a*.



(a)



(b)

FIGURE 3 Radial Displacement coefficient I_r< Uniformly Loaded Circle (a) $\gamma_s = 0.3$ (b) $v_s = 0.5$

Results for the undrained conditions ($v_s = 0.5$) show similar trends as for $v_s = 0.3$ except that no inward or negative dislacements are noticed at any point. The influence coefficients for $v_s = 0.5$ are higher than those for $v_s = 0.3$. Interestingly, maximum horizontal displacement occurs at $r/a = 1.0$, for $z/a = 0.5$. The decrease in I_r with r/a values is rapid for shallow depths but gradual at greater depths.

The variation of horizontal displacement coefficient, I_λ, with normalised distance, x/B, for square ($L/B=1$) and rectangular ($L/B=2$, 5, and 10) foot ings is presented in Fig. 4 (*i*) (*ii*) (*a, b, c* and *d*), for a Poisson's ratio of 0.3.

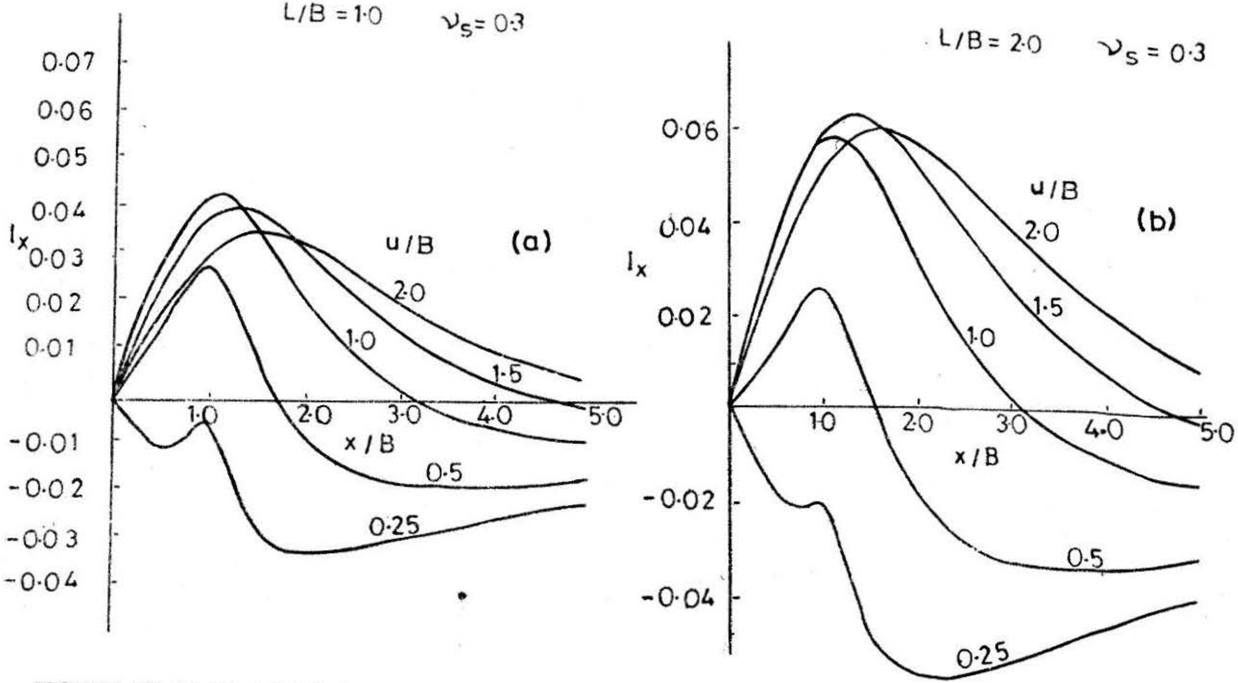


FIGURE 4(i) Horizontal Displacement Coefficient I_x—Uniformly Loaded rectangle (a) L/B=1.0 (b) L/B=2.0

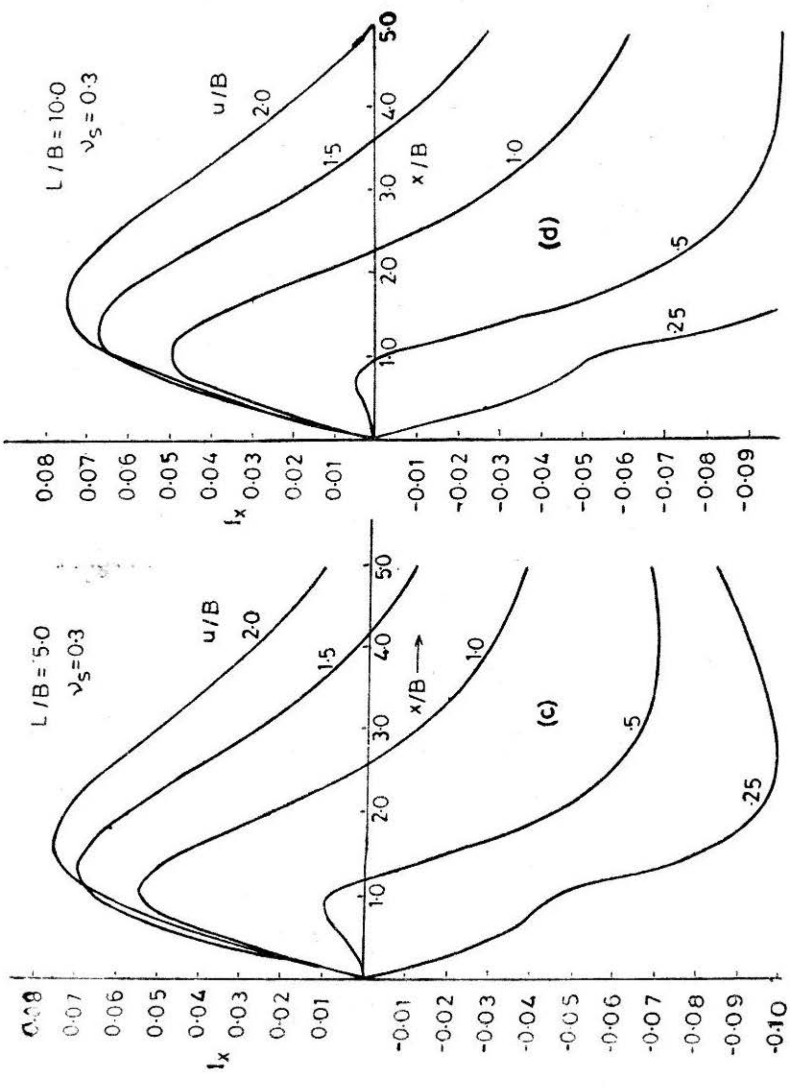


FIGURE 4(ii) (c) L/B=5.0

(d) L/B=10.0

The results for a square footing follow closely those corresponding to a uniformly loaded circle. Beneath a square footing and a shallow depths, $u/B = 0.25$, the displacements are negative indicating that the soil moves inward. However for $u/B > 0.5$, the lateral displacements are positive and increase with depth. The distance at which the horizontal displacement is a maximum increases with depth. The maximum displacement influence coefficients are 0.029, 0.044, and 0.041 at values of x/B equal to 0.95, 1.1 and 1.3 for u/B equal to 0.5, 1.0, and 1.5 respectively. It is noted that the displacement coefficients are a maximum for u/B equal to 1.0 over a distance x/B less than 1.4. The influence coefficients become negative at large distances from the centre.

The horizontal displacement influence coefficients beneath rectangular footings ($L/B = 2.5$, and 10) show similar trends (Fig. 4 *b c*, and *d*) as for points beneath a square footing. At shallow depths, the coefficients are negative but become positive with increasing depths. The depth upto which I_x values are negative increases with L/B ratio. For u/B equal to 0.5, the maximum values of I_x are 0.025, 0.011 and 0.004 for L/B values 2, 5, and 10 respectively. For footings with $L/B \geqslant 10$, the I_x—values are negative for depths upto 0.5B. The maximum values of I_x at depths of 1.0B. 1.5B and 2.0B are respectively 0.057, 0.062 and 0.0585 for L/B=2.0, 0.0555, 0.705 and 0.076 for $L/B = 5$, and 0.0485, 0.067 and 0.074 for L/B =10. While beneath a square footing, the I_x values are maximum at $u/B = 1.0$ they reach their maximum values at larger u/B ratios beneath rectangular loads. I_x values are large at depths u/B=1-2 beneath longer rectangles ($L/B \geqslant$, 5) than for a square or shorter rectangles ($L/B \leqslant 2.0$).

Fig. 5 depicts the variation $I_{x\ max}$ (the maximum values of I_x) at any depth with depth for rectangular loads. Beneath a square footing, the values range between 0.025 to 0.045. For long rectangles, the $I_{x\ max}$

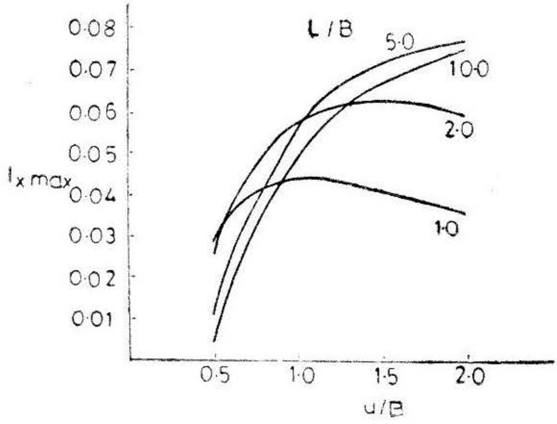


FIGURE 5 Variation of $I_{x\ max}$ with normalised depth

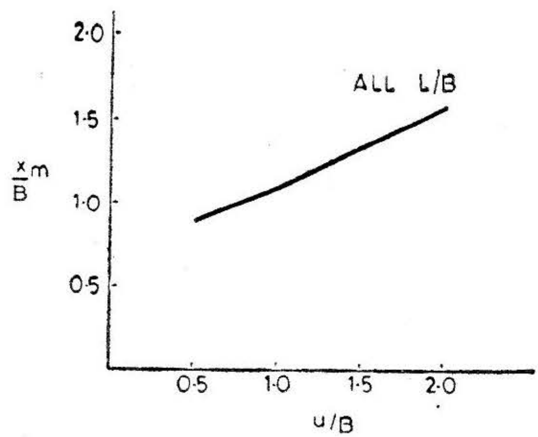


FIGURE 6 Variation of X_m/B with Normalised Depth.

values increase with depth upto depths of 2B. The horizontal distance, x_m, values increase with depth upto depths of 2B. The horizontal distance, at which I_x reaches its maximum value, increases with u/B ratio (Fig 6). This relation is independent of the size of the rectangle.

Reinforcement—Soil Interaction

Reinforcement in the form of strips, grids, sheets, and cells are increaingly being provided to imporve the load carrying capacity and to reduce the settlements of footings founded on the soil containing the reinforcement. The reinforcement achieves its function by counteracting with the tendency of the soil to move laterally under applied loads, at the location where reinfrocement is embedded. The interaction between the soil and the reinforcing element occurs in the form of shear stresses at the top and bottom interfaces. Thus if the lateral displacements of points at different depths and different distances due to applied surface loads are estimated, an indication is obtained where they are positive i.e. outward, and large. By providing reinforcement at these locations to counter lateral displacements, the soil is strengthened and stiffened. The shear stresses are directed inward to prevent outward lateral displacements and cause heave of the surface. In other words, there would be a reduction of the settlement of the footing located therein.

Reinforcement Location

Basset and Last (1978) studied possible locations for reinforcement in terms of tensile strains based on which they have recommended ideal and practical reinforcement patters. However, since the interaction between soil and reinforcement is governed by displacements, the following recommendations are made based on the study presented (Fig. 7).

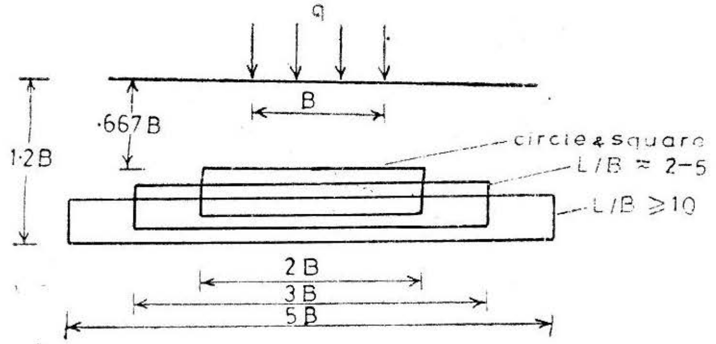


FIGURE 7 Recommended Locations for reinforcement

Lateral (horizontal) displacements are inward at shallow depths. Reinforcement provided at this location can be ineffective and in practice would be counter-productive. For circular and square uniform loads, lateral movements are a maximum at depths corresponding to the radius and half width of the footing respectively. Thus the reinforcement if placed close to this depth will be most effective. For longer rectangles, it is preferable to provide reinforcement at depths in the range half to one times the width of the footing. Similarly, the length of the reinforcement should be in the range 4B to 6B for long rectangles ($L/B \approx 5$) and strips ($L/B \geqslant 10$).

Example Problem

Consider a square footing of width 2m transmitting a uniform load of 10 kN/m^2. It is desired to calculate the horizontal displacements at a depth of 1m along the x-direction. From Fig. 4a, I_x values can be obtained for $u/B = 1.0$ ($2B = 2$m and $u = 1$m). For x values of 1m, 2m, 3m, 4m and 5m from the centre at a depth of 1m, the corresponding I_x values are 0.042, 0.02, 0.003, —0.007 and —0.01. The horizontal displacements at these points can be calculated from eq. 10. Assuming G_s of soil to be 10^4 $kN/$m^2, the ρ_x values at the above mentioned points are 0.042 mm, 0.02 mm, 0.003 mm, —0.007 mm and —0.01 mm respectively. Beyond an x value of $3m$, the displacements are negative, indicating inward movement of soil. This indicates a reinforcement of length $3m$ would be best suited at this depth. Any increase in its length would be ineffective.

Similarly for a strip footing of 2m width a reinforcement of length 4m to 5m placed at a depth of $2m$ ($u/B = 2.0$) would be most effective.

Conclusions

Boussinesq's solution for a point load on the surface of a semi-infinite medium is analysed for lateral displacements at different depths and different radial distances. This solution is integrated for uniformly loaded circular and rectangular areas. Radial or lateral horizontal displacements are evaluated at different depths and for various distances from the centre. The lateral displacements are negative, i.e. inward movement at shallow depths ($u/B < 0.5$). The points in the soil move outward in the depth range of $1.0B$ to $2.0B$. Thus reinforcement to be most effective should be provided only in the zones mentioned above. The length of reinforcement should be in the range $4B$ to $6B$, where B is half width of the footing, the larger lengths to be provided at greater depths and beneath larger rectangles.

Acknowledgement

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