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Combined Well Shell Pile Foundation as Innovative Deep Foundation

by

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Introduction

The innovative design of combined well shell pile (WSP) foundation is derived from the stilt root system of plants alongwith the apron concept derived from the root system of palm tree. Conventional well foundation under distress is protected from excessive scour by dumping boulders filled in wire cages around the well which act as a protective apron. In this innovative design, a rigid protective apron in the form of two toriconical shells has been proposed which spans radially from the well steining to a ring beam encompassing a group of piles arranged concentric to the well. The innovative modification of well foundation is analysed and improvements in vertical and lateral resistance are ascertained by using numerical modelling by finite elements for vertical loads and approximate behaviour for lateral actions.

Combined WSP System

A modified structural form comprising of conventional well, shell and pile elements has been proposed as an efficient form of deep foundation, the arrangement of individual elements being analogous to the stilt root system as in maize plants. The well acts as a primary anchoring stem connected to a group of piles through a ring beam and two toriconical shells spanning the gap between the pile group and the well, Fig. 1. The piles, arranged in a circle of diameter 2 to 2.5 times the well diameter concentric to the well, are analogous to secondary plant roots, share vertical loads and contribute significantly to lateral and torsional stability of the system. To strengthen the pile connection, radial beams spanning from the well steining to the pile head may be provided in the gap between the upper and lower toriconical shells.

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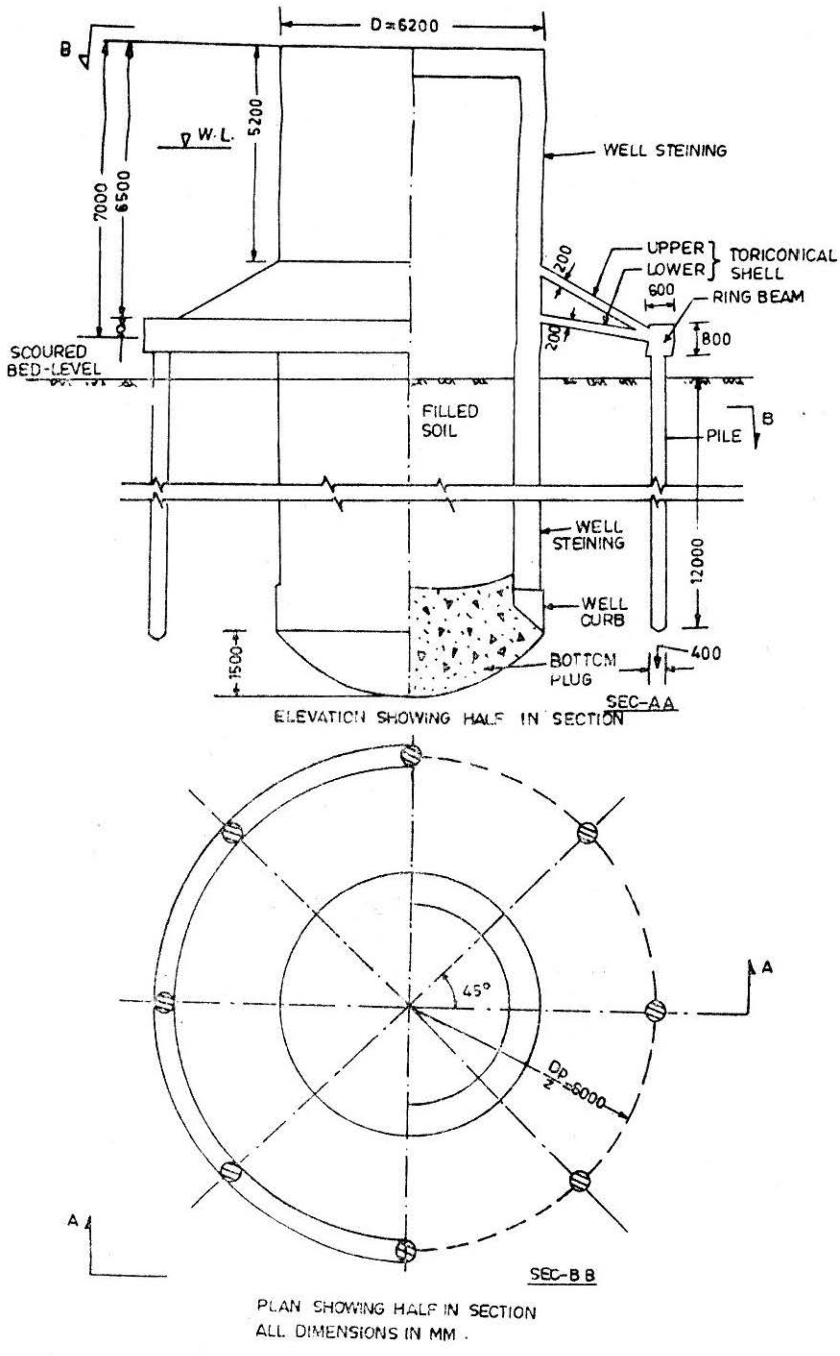

FIGURE 1 General Arrangement of WSP Foundation

The toriconical shells act like an apron and mechanically cut off water current causing scouring action around the well. Scour is locally confined around the piles. This combined system gives increased resistance against scour and improved load carrying capacity. As a consequence, the total length of well can be reduced.

Resistance Against Scour

A State-of-the-Art report on the subject of local scour around cylindrical pier has been presented at length by Breussers *et al.* (1977). A set of design suggestions together with possibilities for protection against scour have also been reported. Various investigators have reported considerable reduction in local scour on placement of cut-off sheet pile or additional piles upstream of the well pier.

Shen and Schneider (1970) investigated a caisson system surrounded by a vertical lip (cut-off sheet pile) and reported reduction in scour by half. Tests conducted by Chabert and Engeldinger (1956) on a system of small piles placed upstream ahead of the pier are reported to have yielded scour reductions as high as fifty percent.

In the WSP system, the toriconical shells act as a mechanical apron around the well by breaking the incident current and weakening the vortex causing erosion. It physically restricts the vortex causing erosion to come near the well surface and mechanically cuts off the down-ward water current causing scour around the well by reverse deflection of water current as depicted in Fig. 2. The pile group, arranged in a circle of diameter twice that of outer diameter of the well, also reduces local scour around the well by protective action caused by the upstream piles.

In the WSP system, local scour would occur around the piles, the extent being limited to 1.5 times the pile diameter as predicted by Beussers' relation and supported by experimental results obtained from experimental investigations carried out at U.P. Irrigation Research Institute on models.

For design considerations, total scour for conventional well is worked out as twice the Lacey's depth to include effects of local scour. With the present system, the local scour around the well would be drastically reduced, thereby increasing the grip length. Hence the design concept for well, primarily based on scour considerations, would be revised to give reduced depth of well and result in economy.

Finite Element Study

The foundation configuration exhibits radial symmetry and vertical loads are axisymmetric. Hence linear soil-structure axisymmetric analysis assuming elastic soil has been adopted.

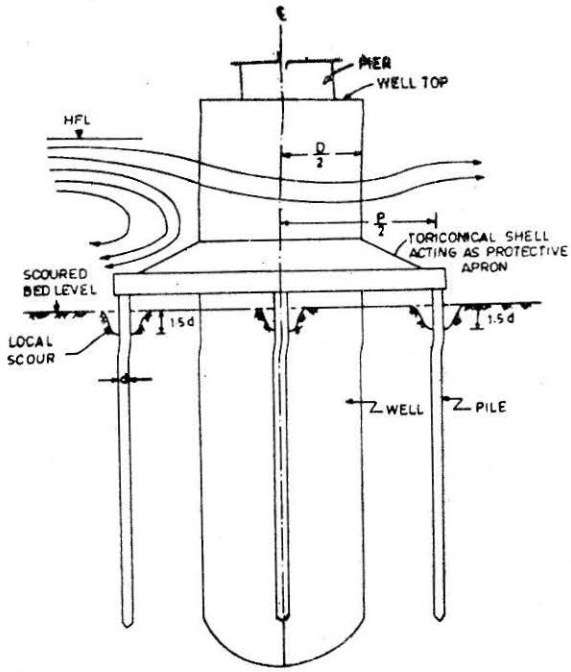

FIGURE 2 Socur Consideratiors for WSP Foundation

Structural Idealization

The piles arranged radially around the well have been idealized by an equivalent thin cylinder, concentric with the circular well the centre line radius of this cylinder being same as the radius of the circular periphery on which piles are arranged. This cylinder is assumed to be integrally connected with the radial beam to which the two toriconical shells are connected.

To maintain the same relative stiffness with regard to soil, the same material properties of the pile material are retained for the equivalent pile cylinder. The thickness of the equivalent pile cylinder is worked out by equating the cross-sectional areas of the actual piles to that of the equivalent pile cylinder.

The other approach to simulate the actual load distribution with the equivalent pile cylinder is to equate the total frictional resistance force due to soil in the two equivalent structures. This can not be achieved by making the external surface area of the equivalent pile cylinder equal to the total external surface area of all the piles, as the radius of the equivalent cylinder is already fixed. To overcome this difficulty, softer soil element with modulus of elasticity E_{sp} smaller than the general soil modulus E_s, surrounding the pile cylinder elements have been assumed. Lower values of E_{sp} imply lower value of frictional resistance, thereby resulting in comparable values of frictional resistance, force in the idealized structure vis-a-vis the original structure configuration. The mesh adopted for axisymmetric analysis is shown in Fig. 3.

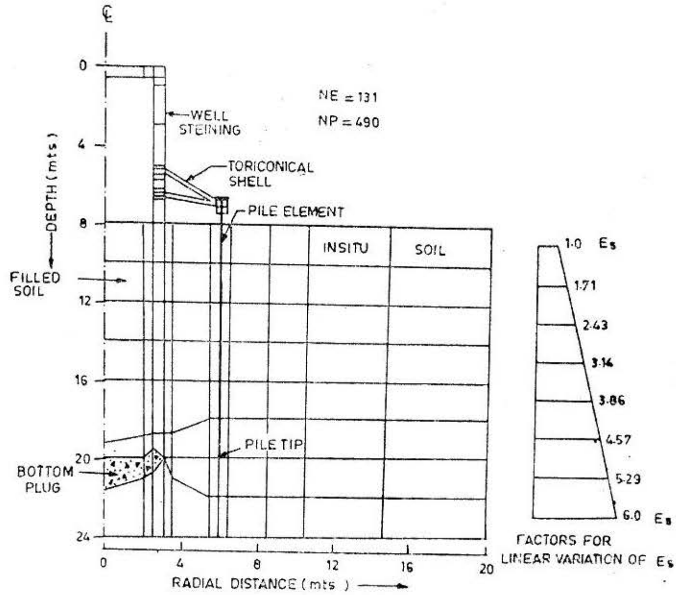

FIGURE 3 Finite Element Mesh Configuration of WSP Foundation with Soil Elements

Soil Variation

Parametric study of WSP foundation has been carried out for two cases unit. To simulate cohesionless soil in field which exhibit near linear variation of E_s value with depth, analysis is carried out for various sets of E_s values varying linearly with depth. The soil mass along the pile length is assumed to consist of layers of finite elements with increasing E_s values as determined by the linear variation of E_s from 1 to 6 E_s over a depth of 16 m as shown in Fig. 3.

Vertical Load Shared by Pile

The variation of percentage vertical load shared by piles of the total superimposed load on well top is parabolic in nature (Fig. 4.), the pile group sharing more load in the case of soft soil than hard soil. In case of soil with linearly varying E_s value, the load taken by pile group varies from 14.1 percent for hard soil (m = 300 to 50) to 37.6 percent for soft soil (m = 3600 to 600). The percentage load carried by the pile group for the same soil element surrounding the pile element $\left(\frac{E_{sp}}{E_s} = 1\right)$ is greater by 10 percent than the corresponding case of softer soil element surrounding the

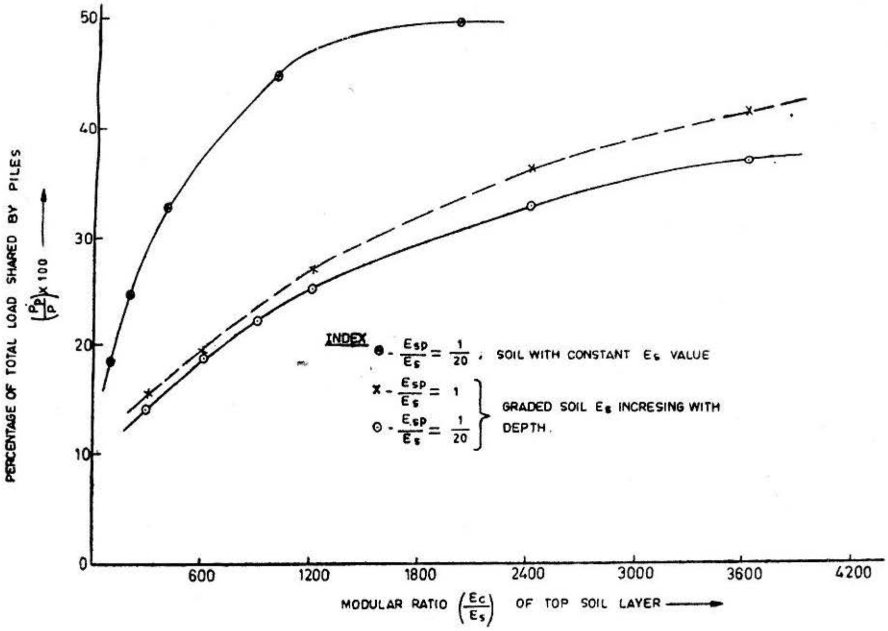

FIGURE 4 Vertical Load Shared by Pile Group for WSP Foundation

pile element $\left(\frac{E_{sp}}{E_s} = \frac{1}{20} \right)$ reflected on account of increased frictional force mobilised at the normal soil and pile element interface in comparison to that at the softer soil pile element interface. The ratio of load shared by the pile group to the total vertical load varies parabolically with increase in cross sectional area of the pile group (Fig. 5).

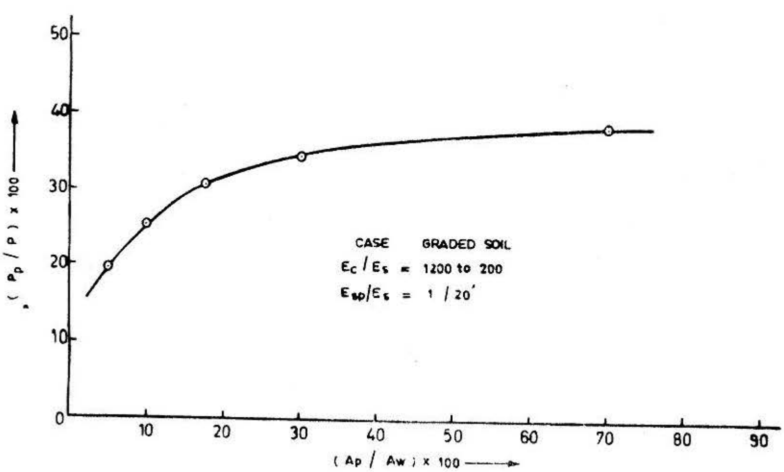

FIGURE 5 Ratio of Cross Sectional Area of Equivalent Pile Cylinder to that of Well Steining.

Case Study

For an appraisal of stress resultants in constituent elements or WSP foundation, a detailed case study is reported for a soil with linear E_s variation with depth, viz. m = 1200 to 200, $\frac{E_{sp}}{E_s} = 1/20$ and $t = 0.026\ m$. Finite element analysis shows the ratio of the vertical load shared by the equivalent pile cylinder to the total load is 25.35 per cent.

Along the length of the well, the variation of stress resultants τ_{rz} acting on the outer surface of the well, the vertical stress in the well and the moment M_s in steining are represented in the Fig. 6. Frictional forces given by τ_{rz} on the equivalent pile cylinder both on the inside and outside forces are plotted along the pile length. From the plot of τ_{rz}, it is observed that the frictional force mobilized on the outer surface is approximately double the average value on the inner surface.

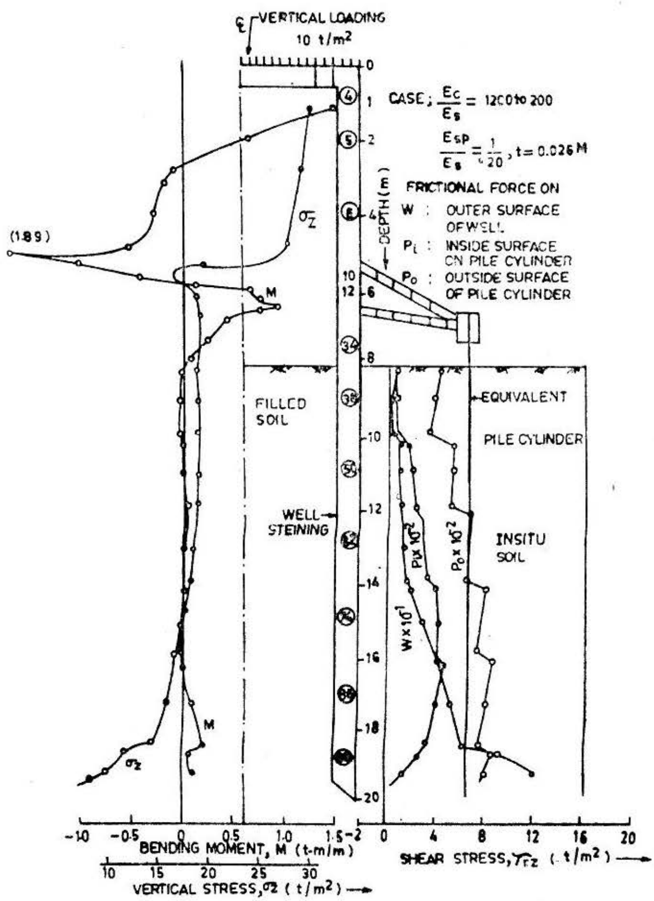

FIGURE 6 Stress Resultants in Well and Pile of WSP Foundation

The variation of stress resultants N_s, N_θ, M_s for the upper toriconical shell is shown in Fig. 7. Axial force N_s is compressive in the entire upper shell and decreases gradually towards the pile junction. Hoop stress resultant N_θ for the shell changes from compression to tension while moving from the well towards the ring beam. In the lower shell, N_s is tensile in nature and the hoop stress N_θ is also tensile throughout, Fig. 7.

From the study, it is evident that the load distribution between the well and pile group is affected by the relative well settlement with reference to the pile group causing the toriconical shell to come into play. The

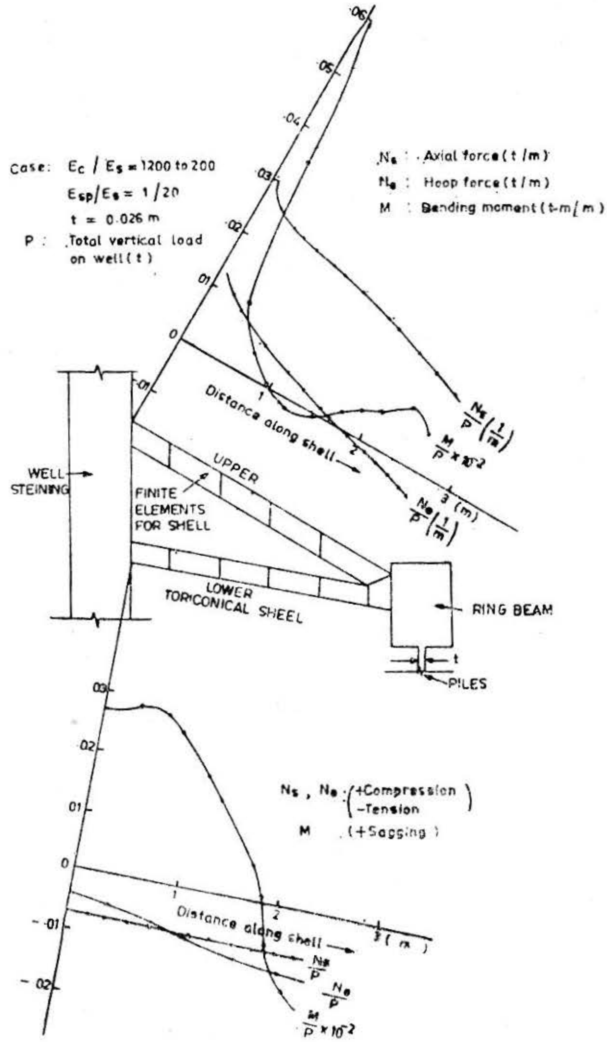

FIGURE 7 Bending Moment, Hoop and Axial Forces in Upper and Lower Toriconical Shell

toriconical shell exhibits truss action for load transfer to the pile group as the upper shell is in compression and the lower is in tension. The shell transfers load primarily by axial forces.

Settlement Variation with Load Shared by Piles

It is observed that the relative settlement of the well with reference to the pile group in the combined WSP foundation is directly proportional to the percentage of the total vertical load shared by the pile group, Fig 8.

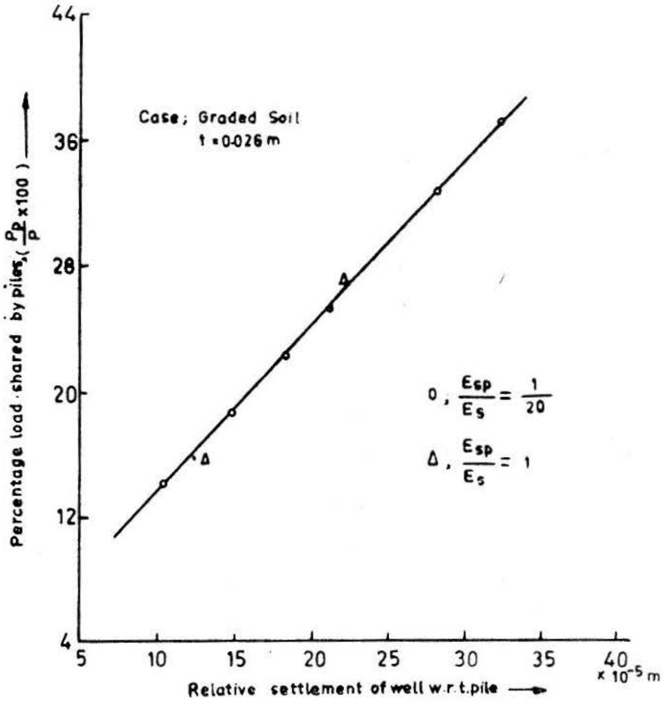

FIGURE 8 Relation of Load Taken by Piles and Relative Settlement

Hoop Force

To consider the equilibrium or forces along r and z axes at the centre of the ring beam, the junction of the toriconical shell and the pile group, hoop force T borne by the ring beam as per FE analysis is determined using σ_θ values at Gaussian points of the ring beam elements. The outward horizontal force $\left(H = \frac{2T}{2R} \right)$ resisted by the ring beam expressed as ratio of the total vertical load P varies with E_c/E_s soil values as shown in Fig. 9. Value of H increases with softer soil which also corresponds to large settlement of the well with reference to the pile group and a higher percentage of vertical load shared by the pile group.

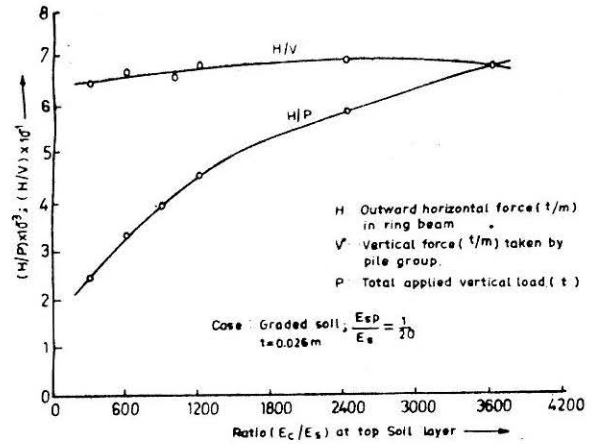

FIGURE 9 Variation of (H/V) and (H/P) with Soil

Approximate Theory

Elastic Settlement Analysis

For design purpose, the vertical load distribution between the well and pile group can be determined based on elastic settlement analysis which presumes that forces resisted are proportional to deformation, and the deformations suffered by the well or pile group are within elastic limits.

Stiffness relation of the from $(P = K\,\delta)$ individually for the well and the pile group are combined together for the combined WSP system. Load carried by the WSP system is the sum of the loads carried by the well and the pile group :

$$P = P_w + P_p = K_w\,\delta_w + K_p\delta_p \qquad \ldots(1)$$

Settlement of pile group and well are related as

$$\delta_p = \alpha\,\delta_w \qquad \ldots(2)$$

where α is function of the rigidity of the toriconical shells and properties of supporting soil media. Based on *FE* studies, variation of α values is given in Table 1

In terms of α, the total load shared by the WSP system is

$$P = K_w\,\delta_w + \alpha\,K_p\,\delta_w \qquad \ldots(3)$$

or

$$P = \left(1 + \alpha\frac{K_p}{K_w}\right) K_w\,\delta_w \qquad \ldots(4)$$

TABLE 1

Variation of α Values

Graded soil E_c/E_s $\left(\frac{E_{sp}}{E_s} = \frac{1}{20}\right)$	300 to 50	600 to 100	900 to 150	1200 to 200	2400 to 403	3600 to 600
α	0.750	0.778	0.798	0.813	0.857	0.882

or

$$\frac{P}{P_w} = \left(1 + \alpha \frac{K_p}{K_w}\right) \qquad \ldots(5)$$

This ratio, termed as augment factor, gives the increased carrying capacity of the combined WSP system as compared to the ordinary well foundation for the same level of settlements in the well.

The ratio of load shared by the pile group to the total load resisted is given as

$$\frac{P_p}{P} = 1 - \frac{P_w}{P} = 1 - \frac{1}{1 + \alpha \dfrac{K_p}{K_w}} \qquad \ldots(6)$$

This relation can be used for practical design approach. Elastic spring constants K_p and K_w can be approximated from actual load settlement curves available for the well and the piles separately at proposed site of foundation, whereas reasonable value of α may be adopted from *FE* studies.

Correlation of this approach has been carried out with *FE* analysis. Values of elastic spring constants are determined separately by *FE* analysis which are substituted in Eq. (8) to arrive at the ratio P_p/P. The values so calculated are quite close to the values obtained by finite element analysis of the combined WSP foundation (Table 2.)

Lateral Moment Resistance

Response of the WSP system subjected to moment is determined in a simplified manner assuming linear stress distribution across the axis of symmetry (Fig. 10). The moment is resisted by axial forces generated in the piles and well. Moment resisted by base of well is only accounted for, though lateral pressure over well sides also resist moment.

TABLE 2

Elastic Settlement Constant and Load Distribution Factors

Graded soil $\frac{E_e}{E_s}$	K_{well} from FE analysis of well only (t/m)10^3	K_p from FE analysis of pile only (t/m)10^3	$\alpha = \frac{\delta_p}{\delta_w}$	Augment factor $\left(k+\alpha \frac{K_p}{K_w}\right)$	$1-\frac{1}{1+\alpha \frac{K_p}{K_w}}$ (%)	$\left(\frac{p_p}{p}\right)$ from FE analysis of WSP (%)
600 to 100	385.2	163.0	0.77	1.326	24,57	18.73
1200 to 200	213.5	117.8	0.81	1.448	30,96	25.35
2400 to 400	112.9	79.2	0.85	1.600	37.52	32.97
3600 to 600	76.8	59.8	0.88	1.687	40.73	37.58

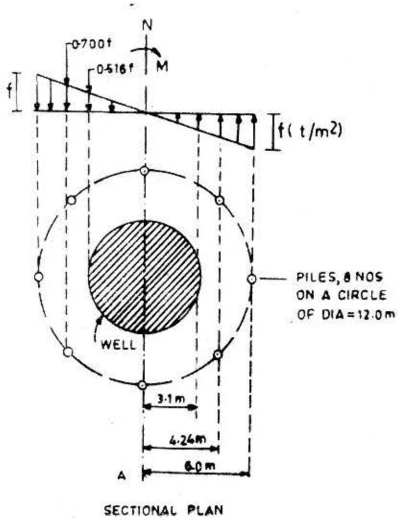

FIGURE 10 Stress Distribution Across WSP Foundation

Moment resisted by eight piles located on a circle of diameter D_p is given by

$$M_p = A_p f (D_p \div 2) (0.706 \, D_p \, \text{Cos } 45°) \qquad \ldots(7)$$

Moment resisted by well base of diameter D is

$$M_w = 2 \left[2 \int_0^a \sqrt{(D/2)^2 - Z^2} \; \frac{2Z^2}{D} \; 0.516 \, f dZ \right]$$

$$= \frac{5\pi}{24} \left(\frac{D}{2} \right)^3 \times 0.516 \, f \qquad \ldots(8)$$

Ratio of moment shared by the pile group

$$\frac{M_p}{M} = \frac{M_p}{M_p + M_w} = \frac{2A_p . D_p}{2A_p . \ D_p + \dfrac{5\pi \times 0.516 \ D^3}{24 \times 8}} \quad (9)$$

Action under horizontal load

Under the action of horizontal load at the junction of the pile group and the well, horizontal translation of the junction takes place. The connecting toriconical shells are assumed to be rigid, thereby the well and pile group suffer the same horizontal deformation. Lateral load deflection characteristics of the two systems are worked out separately and then combined together to obtain the load deflection characteristics of combined WSP system.

Load-deflection curve of a single pile, assuming no interaction as placed far apart is determined by elasto-plastic method. Poulos method (Poulos and Davis, 1980) is adopted assuming cohesionless soil with $E_s = N_h$. Z increasing linearly with depth and curve plotted for proposed size of pile assuming it to be a fixed heat type (Fig. 11).

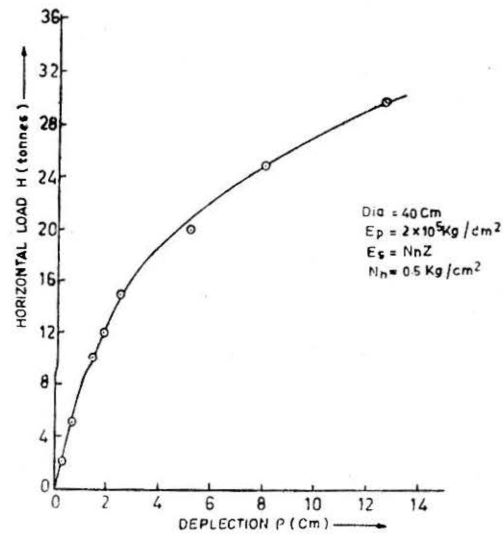

FIGURE 11 Lateral Load Deflection Curve for a Pile for Soil with Linearly Varying Es: (Poulos Method)

The well being a rigid bulkhead, its behaviour is indealized by short and rigid free head plates as given by Dougles and Davis (1964).

$$\rho = \frac{H}{E_s \ D} \ I_{\rho h}/F_\rho \ ; \ H_u = 0.25 \ p_u \ dL$$

Load-deflection characteristics for

$D = 6.2\ m$ well with soil parameters $E_s = \frac{1}{2} N_h\ L = 300\ kg/m^2$, $\phi = 34°$ is summarised in Table 3.

TABLE 3

Load-Deflection Characteristics for Well

H (tonnes)	$\frac{H}{H_u}$	F_ρ	ρ (cm)
100	0.09	1.0	0.473
200	0.185	1.0	0.946
300	0.278	1.0	1.419
400	0.371	1.0	1.892
500	0.464	1.0	2.365

$H_u = 1077$ tonnes

Sharing of horizontal load by the pile group and the well is depicted in Table 4. Load resisted by pile group is simply assumed to be the product of the number of piles and the effect of an individual pile.

TABLE 4

Sharing of Horizontal Loads

Lateral displacement	Load taken by well H_w(t)	Load taken by pile group H_p(t)	Total load taken by WSP system $H = H_w + H_p$ (t)	Percent load shared by pile $\frac{H_p}{H} \times 100$
0.473	100	32	132	24.24
0.946	200	56	256	21.87
1.419	300	82	382	21.46
1.892	400	99	499	19.83
2.365	500	117	617	18.93

Design Approach

Primarily, to arrive at design forces for piles and shell elements, load distribution of both vertical and lateral loads between the well and pile needs to be worked out.

Pile group

The percentage of total vertical load resisted by the pile group can be assessed for practical design purpose by use of Eq. (6). Elastic spring constant values can be determined from the actual load-settlement curves for the well and test piles, separately obtained at the site of proposed foundation.

Alternatively, values from the existing load settlement curves for similar soil condition may also be used in the absence of an experimental settlement curve at foundation site. Suitable value of α may be assumed in the range from 0.75 to 0.85 depending on the soil parameter and the toriconical shell flexibility. In the absence of such data, another approach can be to directly use *FE* results (Fig. 4), by assigning suitable E_s value of the soil elements. In the case of soft graded soil, the percentage load shared by the piles approaches 40 percent. Hence, designing the pile group for 40 percent of total vertical load would be on the safer side.

Axial forces generated in piles to resist lateral moment, as determined by Fig. 6, are also to be added to vertical forces reisted by piles. As piles are separated, no interaction occurs ; hence, for vertical forces, the pile group is designed as a group of individual piles.

Shell and Beam

The two toriconical shells transfer loads from the well steining to the pile group by Truss or axial action, as substantiated by *FE* analysis. The upper toriconical shell primarily bears axial compression while the lower bears axial tension. The bending moment is nominal except at junction points with the well where it increases sharply due to edge effects. Hence these shells have to be primarily designed for membrane forces alongwith suitable strengthening at junctions for bending moments.

At the ring beam centroid, the junction of the shell and the pile, there exists statical equilibrium of axial stress resultants (Fig. 12).

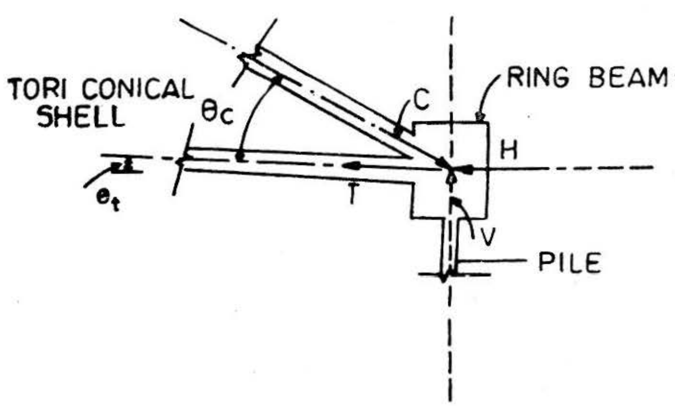

FIGURE 12 Equilibrium of Forces at Shell Pile Junction

The governing statical equations are

$$C \text{ Sin } \theta_c - T \text{ Sin } \theta_1 = V \quad ...(10)$$

$$-C \text{ Cos } \theta_c + T \text{ Cos } \theta_1 = -H \quad ...(11)$$

For practical design problem, value of V, the vertical load per unit length shared by the equivalent pile cylinder, can be known from the percentage load shared by the pile group. For values of H, the outward horizontal force resisted by the ring beam, it has been shown by *FE* analysis that the ratio of H/V does not vary significantly and that remains almost constant with soil E_s values (Fig 9.). Values of C and T worked out by the above relation using values of V and H in a typical case compares well within practical design limits with the actual values as determined by *FE* analysis.

The ring beam needs to be designed as a curved beam supported on piles with vertical loading V per unit length and outward horizontal force H per unit length.

The stress resultants generated in the toriconical shells when subjected to lateral loads are also to be accounted for in design values. No rigorous method exists for direct determination of stress resultants except by 3-*D* *FE* analysis. An approximate alternative is to consider the well-shell-pile connection as a 2-*D* problem and distribute imposed lateral loads and moments by moment distribution and then work out stresses in the shell element.

Summary

The pile group augments the vertical and lateral load resisting capacity of the well. The load distribution between the well and pile group for vertical loads has been thoroughly investigated by *FE* studies. Pertinent observations are summarised below.

Vertical Resistance

The finite element model adopted for various soil parameters indicates the following behaviour of the WSP system :

(*i*) For soil with lineary varying E_s corresponding to cohesive soils and normal clays in field, the ratio of vertical load shared by the pile group to the total vertical load varies from 14.1 per cent for hard soil (m = 300 to 50) to 37.6 percent for soft soil (m = 3600 to 600).

(*ii*) In case of soils with constant E_s values with depth corresponding to over-consolidated clays in field, the percentage vertical load shared by the pile group varies from 18.4 percent for hard soil (m = 100) to 49.9 percent for soft soil (m = 2000).

(*iii*) With the increase in area of cross-section of the pile group, for constant E_s soil case, the increase in percentage of the total vertical loads carried by the piles is quite less ; for a ten fold increase in the former, the latter increases by twice only;

Settlement Variation vs Load Shared by Piles

For vertical loads, the well settles more than the pile group. The vertical load shared by the pile is directly proportional to the relative settlement of the well with respect to the piles. This is turn depends on soil parameters ; the relative settlement increases with decreasing values of E_s.

Stress Resultants in Toriconical Shells

FE analysis of WSP system indicates, in general, that

(*i*) the upper toriconical shell is subjected to compression,

(*ii*) the lower toriconical shell is under tension,

(*iii*) membranee forces in the upper and lower toriconical shells can be determined by the vertical load shared by the piles and the hoop taken up by the ring beam, considering equilibrium of forces at centroid of the ring beam,

(*iv*) the magnitude of membrane forces in the toriconical shells is directly related to the vertical load shared by the pile group, which in turn is directly proportional to the relative settlement of the well with reference to the pile.

The relative settlement is dependent on soil parameters. For softer soil, the relative settlement is more and hence the magnitude of shell forces is higher compared to that for hard soil case.

Conclusions

The proposed WSP system, due to increased vertical and lateral load capacities, can be adopted as a viable, efficient alternative to conventional foundation system in new construction and also to strengthen existing well foundation under distress.

The vertical load capacity of a conventional well system can be augmented ranging from 1.3 to 1.7 times, as per *FE* analysis, by suitable configuration of shell and pile which are relatively cheapter constitutents of the WSP system.

The increase in lateral load resistance of the system compared to conventional well lies in the range from 1.2 to 1.4 times as determined by lateral load *FE* analysis.

On taking into account the increase in scour resistance, further economy can be effected. Due to protection by the shell apron and pile group, local scour at the well location is considerably reduced. For design purposes the general scour depth, given by Lacey's depth d_o, is doubled to account for local scour around the well. As substantiated by model studies the total scour around the well is reduced to an extent of fifty percent as local scour is considerably limited in the well vicinity. Hence, for WSP foundation, the maximum depth of scour can be taken as 1.2 d_o to 1.5 d_o, including a safety margin, instead of $2d_o$. This would lead to reduction in the overall length of the well and consequently the diameter of the well, leading to economy.

During construction, the loading of piles can be controlled by sequence of connection between the well and shell-pile system. Initially, the entire dead load can be transferred to the well installed first of all, and then piles can be connected to augment the vertical and lateral resistance. Piles can also be viewed as anchor points for the shell apron which drastically reduces scour around well, instead of an element sharing vertical load. In such cases, the depth of piles can be reduced appropriately. Such a configuration would be intended primarily to reduce scour in the vicinity of the well, and to augment the lateral load resisting capacity due to enlarged base.

Several scour studies of the WSP system have been recently carried out. A considerable reduction in scour takes place depending upon various factors. A modification in the arrangement of apron has been further suggested by providing two rings of piles as shown in Fig. 13. In this system, the eddies are formed over the apron to further dissipate the hydraulic energy causing scour.

Notations

E_s, E_c = Modulus of elasticity of soil and concrete respectively

E_{sp} = Modulus of elasticity of soil element surrounding pile element

t = Thickness of equivalent pile cylinder

m = $\frac{E_c}{E_s}$ = Modular ratio

σ_θ = Hoop stress

τ_{rz} = Shear stress along Z axis

N_s = Axial stress resultant for straight element

N_θ = Hoop stress resultant

M_s = Moment per unit length

T = Total hoop force

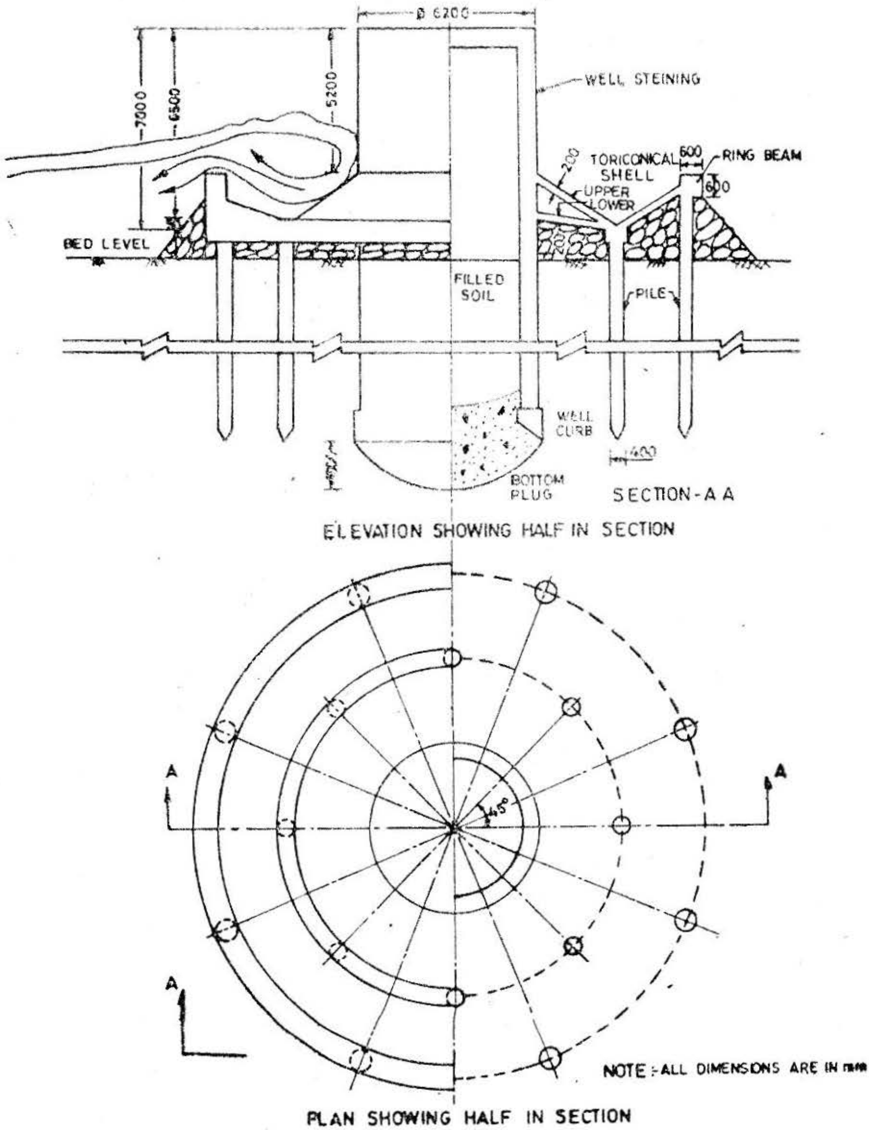

FIGURE 13 Modification of WSP Foundation with Two Rows of Piles

H = Outward horizontal force per unit length

P = Total vertical load carried by WSP foundation

P_w, P_p = Vertical load carried by well and piles respectively

K_w, K_p = Elastic spring constants for well and piles

δ_w, δ_p = Vertical deflections of well and pile respectively

$\alpha = \dfrac{\delta_p}{\delta_w}$

A_p = Cross sectional area of a pile

f = Stress value

D_p = Diameter of circle on which piles are located

D = Diameter of well

M = Lateral moment resisted by WSP foundation.

$M_w M_p$ = Lateral moment resisted by well and piles respectively

N_h = Rate of increase of E_s with depth

Z = Distance along Z axis

ρ = Lateral deflection

H = Lateral load

d = Pile diameter

I_{ph} = Elastic influence factors for displacement caused by horizontal load

F_ρ = Yield displacement factors for short and rigid free head plates

L = Depth of well

H_u = Ultimate load carried by short and rigid free head plates

P_u = Ultimate soil pressure (uniform with depth)

H_w, H_p = Lateral load resisted by well and pile respectively

V = Vertical axial force per unit length

C = Axial compression per unit length

T = Axial tension per unit length

θ_c, θ_t = Inclination of upper and lower toriconical shell respectively with the horizontal

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