Elasto-Plastic Finite Element Analysis of Circular Tunnels

by

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Introduction

Underground openings are extremely complex structures. With the increasing use of underground openings for a variety of purposes (e.g. hydroelectric projects, highway and sewage disposal networks, oil and gas storage and nuclear waste disposals), some of the easiest sites have already been exploited. Those under planning now, include some of the most difficult and complex sites posing unforeseen challenges to Geotechnical engineers.

For the design of an underground opening (tunnel), briefly, the requirement is to know the insitu stress field, induced stress field and related deformations when an opening is made, the characteristics of support system and interaction behaviour of support and ground. Since geological conditions affect the overall behaviour the most, an accurate interpretation of geological conditions is an essential prerequisite for any rational design.

One of the first requirements for the design of an underground opening is the knowledge of redistribution of stresses and displacements in the surrounding rocks due to excavation for the existing geological conditions. The stresses and displacements developed around an underground opening depend on a variety of factors, e.g. geological conditions, shape and size of opening, construction procedure, mechanical properties, initial stresses of the media, the length of period during which the opening is left unsupported and characteristics of the support system. In cases where large stress changes take place due to excavation, the rock behaviour is in the plastic range in many cases. The sequence of excavation has significant influence over the final stresses and displacements, if the rock behaviour is in the plastic range.

The methods of analysis and design generally used by practising engineers are empirical formulae, standardised codes and closed form solutions. These methods assume grossly simplified models of a situation which is really complex. For example, analytical solutions are available for elasto-plastic analysis of circular tunnels, excavated in single stage (full

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face) in isotropic and homogeneous geological media with insitu stress ratio K_o (horizontal stress, σ_h /vertical stress, σ_v) = 1 condition (Obert and Duvall 1967, Goodman 1980, Hoek and Brown 1980). As such they are not applicable for K_o conditions other than 1, complicated geological media, different shapes of openings and multistage excavation. To analyse underground openings, therefore, the above methods are not adequate and numerical methods such as finite element method (Desai and Abel 1972, Zienkiewicz 1977) are best suited for these cases.

A review of the available literature shows that in case of underground openings the studies carried out are restricted to the elastic (EL) analysis and in the few elastoplastic (EP) analysis, Mohr-Coulomb or Drucker-Prager yield criterion has generally been used. These yield criteria assume the strength behaviour as linear and neglect *the non-linearity of the* failure envelope inspite of the fact that the nonlinearity is quite significant at low stress level. Recently, a non-linear yield criterion has been proposed by Hoek and Brown (1980). This yield criterion has been derived using available experimental and field data in published literature and has several advantages over the conventional yield criteria used in finite element analysis so far. It's use in elasto-plastic finite element analysis has been reported and advocated by Srivastava et al. (1986).

In the present study, a deep circular tunnel excavated in basalt in single and in two stages has been analysed. Hoek and Brown yield criterion has been used for the elastoplastic analysis. A comparison of elastic and elasto-plastic analysis for single stage excavation has been carried out to show clearly the differences and desirability of elasto-plastic analysis. Then a comparison of elasto-plastic analysis for single and two stage excavations has been presented to show the effect of excavation in multiple stages. Three insitu stress ratios (K_o = 0.5,1.0 and 1.5) have been considered in the analysis. The deformed shapes, displacement paths, principal stress contours and variation of principal stresses, along some typical radial directions, have been plotted.

Elasto-Viscoplasticity

In the present study, the elasto-viscoplastic theory (Zienkiewicz and Cormeau, 1974) has been adopted and used as an artifice to obtain elasto-plastic solution. The theory is briefly presented in this section. The state of stress at a point is represented by the stress vector $\{\sigma\}$ ($\{\sigma\}^T = [\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \sigma_{zx}]$) and total strain by the vector $\{\epsilon\}$ ($\{\epsilon\}^T = [\epsilon_x\ \epsilon_y\ \epsilon_z\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}]$).

The total strain at a point is the sum of elastic and viscoplastic strains, i.e.,

$$\{\epsilon\} = \{\epsilon^e\} + \{\epsilon^{vp}\} \qquad \ldots(1)$$

where $\{\epsilon^e\}$ and $\{\epsilon^{vp}\}$ are the elastic and viscoplastic strain vectors, respectively.

The stresses are related to the total and viscoplastic strains through the relation

$$\{\sigma\} = [D]\,(\{\epsilon\} - \{\epsilon^{vp}\}) \qquad \ldots(2)$$

where $[D]$ is the elasticity matrix. The elements of $[D]$ are function of Young's modulus and Poisson's ratio.

The yield function, in general, can be written in the form

$$F = F(\{\sigma\}, \{\epsilon^{vp}\}) = 0 \qquad \ldots(3)$$

Equation 3 represents in general the conditions of plasticity and hardening/softening at a point. The viscoplastic strain rates are given by the flow rule,

$$\{\dot{\epsilon}^{vp}\} = \mu < F > \frac{\partial F}{\partial\{\sigma\}} \qquad \ldots(4)$$

in which $\{\dot{\epsilon}^{vp}\}$ is the viscoplastic strain rate vector, μ is the fluidity parameter (taken as 1 for elasto-plastic analysis), $< >$ is used to indicate that if $F \leqslant 0$, $< F > = 0$ and if $F > 0$, $< F > = F$.

Yield or Failure Criterion

The empirical failure criterion as proposed by Hoek and Brown (1980) can be written as

$$\sigma_1 = \sigma_3 + (m\sigma_c\sigma_3 + s\sigma_c^2)^{1/2} \qquad \ldots(5)$$

where σ_1 is the major principal stress at failure,

σ_3 is the minor principal stress applied to the sample,

σ_c is the uniaxial compressive strength of the intact rock material in the specimen, and

m and s are the constants which depend upon the properties of the rock and upon the extent to which it has been broken before being subjected to stresses σ_1 and σ_3.

The uniaxial tensile strength σ_t of the specimen is given by substituting $\sigma_1 = 0$ in Eq. (5) and by solving the resulting quadratic equation for σ_3. Thus,

$$\sigma_t = \tfrac{1}{2}\,\sigma_c\,(m - (m^2 + 4s)^{1/2}) \qquad \ldots(6)$$

Thus, Eq. (5) includes both, failure criterion of rock and limited tension together. Thus, a separate *No Tension*, analysis is not required.

Equation (5) can be written in the form

$$F = \sigma_1 - \sigma_3 - (m\sigma_c\sigma_3 + s\sigma_c^2)^{1/2} = 0 \qquad \ldots(7)$$

Thus the original rock mass is linear elastic until $F \leqslant 0$ and goes to plastic state when $F > 0$. For the numerical computation it is convenient to write the yield function in terms of three stress invariants, σ_m (mean stress), $\bar{\sigma}$ (second invariant of deviatoric stresses) and θ_o (Lode angle) as proposed by Nayak and Zienkiewicz (1972). In terms of three stress invariants, the yield criterion (Eq. 7) can be written as

$$F = \frac{4\,\bar{\sigma}^2 \cos^2\theta_o}{\sigma_c} + m\,(\cos\theta_o + \frac{\sin\theta_o}{\sqrt{3}})\,\bar{\sigma} - m\sigma_m - s\sigma_c = 0 \quad ...(8)$$

Computer Program

A computer program has been developed on ICL 2960 for elasto-viscoplastic finite element analysis for plane strain condition. Eight noded isoparametric elements and 2×2 Gauss point integration have been used. Hoek-Brown yield criterion has been incorporated in the program.

The procedure adopted to simulate excavation is that proposed by Chandrasekaran and King (1974). The elements which cover the area to be excavated are made AIR elements i.e. their contribution to the global stiffness matrix is reduced to almost zero. This is done by reducing the Young's modulus of the elements to be excavated to about 10^{-6}th of the original value. For the present study, the tunnel is simulated to be excavated in single stage (i.e. full face) and in two stages (the upper half in the first stage and the lower half in the second stage).

Cases Analysed

For the present study, a circular tunnel of 8 m diameter, at a depth of 250 m from the ground surface has been analysed. Three insitu stress ratios (K_o = 0.5, 1.0 and 1.5) have been considered. For each insitu stress ratio, the tunnel is simulated to be excavated in single stage (1ST) and then in two stages (2ST). The rock properties used in the analysis are from a river valley project in India and are as follows:

Rock type	= Good quality basalt
Young's modulus	= 3.5×10^6 t/m^2
Poisson's ratio	= 0.21
Unconfined compressive strength	= 12236.4 t/m^2
Tensile strength	= 24.47 t/m^2
Insitu stress, σ_v	= 675.0 t/m^2
m = 1.7, s = 0.004.	

In Fig. 1, the various cases analysed are shown and explained schematically.

Analysis

Taking into consideration the symmetry of excavation, for the single stage (full face) excavation, only quarter tunnel and for two stage evcavation, half the tunnel has been analysed. In the finite element discretization mesh the total number of elements is 26 and that of nodes is 101 for the quarter tunnel (figure not shown). The discretization mesh for half tunnel is exactly double the size of that for quarter tunnel and is shown in Fig. 2. The total number of elements is 52 and nodes is 181 in this case. The results of single and two stage excavations of elastic and elasto-plastic analyses are discussed in the following section.

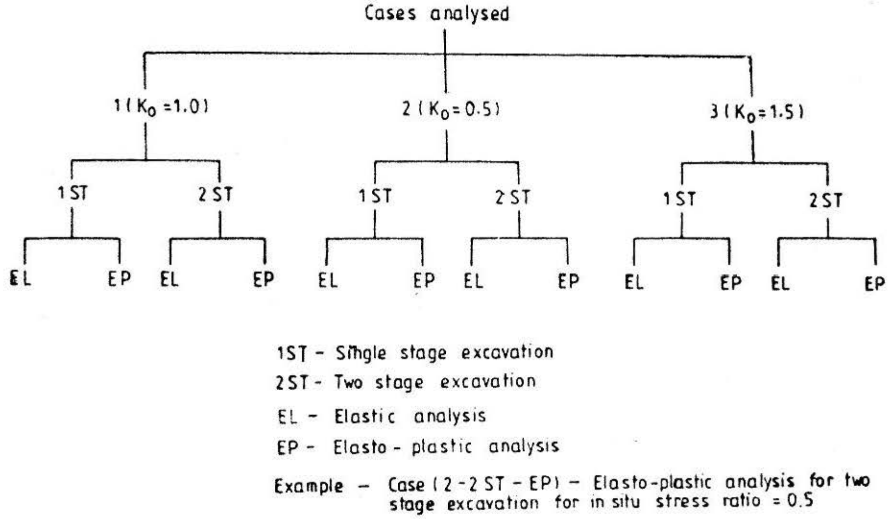


FIGURE 1 Schematic Diagram Indicating Cases Analysed

Results and Discussion

Case 1 (K_o = 1.0)

The elastic and elasto-plastic deformed shapes of the tunnel are shown in Fig. 3 for case 1ST (on the left hand side) and for case 2ST (on the right hand side). For case 1ST, the deformed shape of the tunnel is uniform and concentric to the tunnel boundary for both elastic and elasto-plastic analyses. For the case 2ST, after the first stage of excavation for elastic analysis, displacement is maximum at the crown and minimum at the springing of the tunnel, but after the final stage of excavation, it is uniform and concentric to the tunnel boundary and is same as obtained for case 1ST. This is expected also, because in case of elastic analysis, there should be no effect of sequence of excavation as reported by Chandrasekaran and King (1974). But for the elasto-plastic analysis for case 2ST, the final deformed shape obtained is non-uniform.

Considering case 1ST, the difference found in displacements at the tunnel boundary between elastic and elasto-plastic analyses is 16.4 percent.

A comparison of the displacements obtained from elasto-plastic analysis of the two cases, shows that at crown, springing and invert of the tunnel, the displacements are more in case 1ST (3.0, 15.3 and 1.2 percent, respectively) but between the crown and springing, the displacements are greater at some points for case 2ST (maximum about 48 percent).

The displacement paths (for elasto-plastic analysis), for four typical points on the tunnel boundary have been shown in Fig. 4. The displacement paths show the movement of the points at the tunnel boundary from initial to final displaced position in case 1ST and from initial to first stage displacements to final stage displacements in case 2ST. The displacement

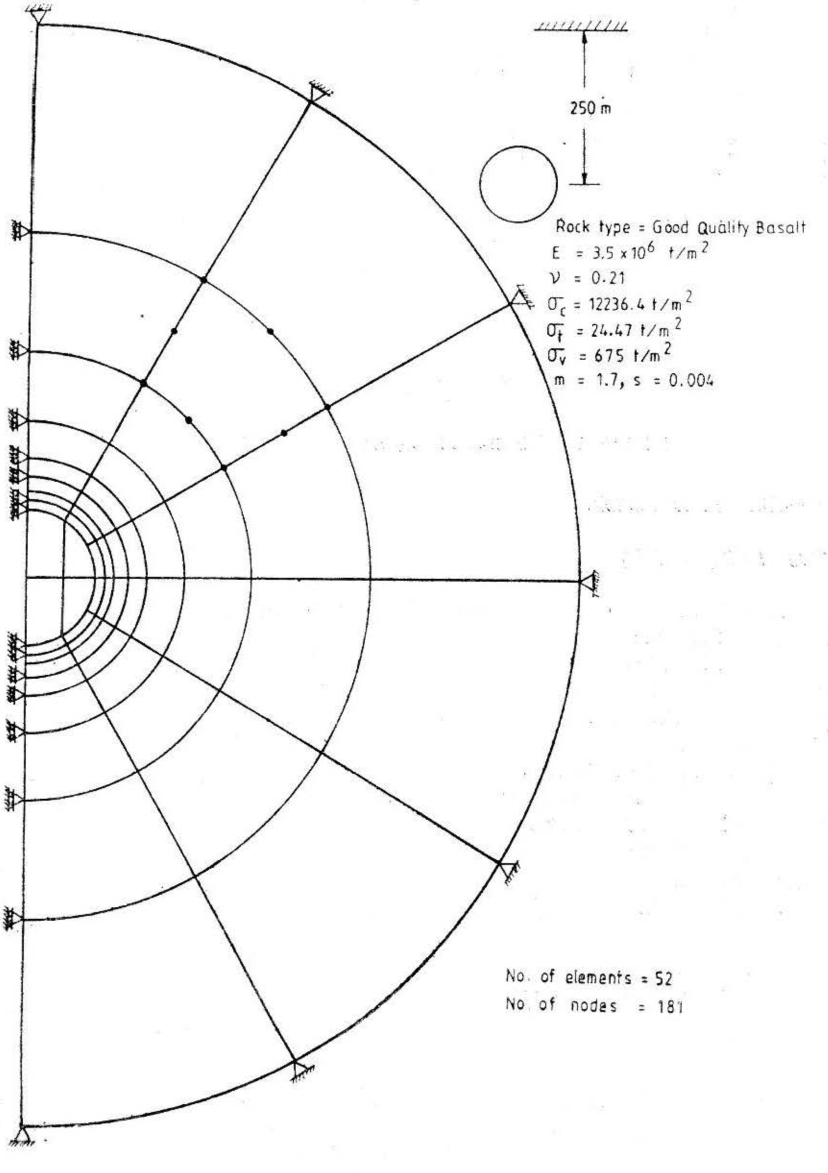


FIGURE 2 Finite Element Discretization Mesh (Half Circular Tunnel)

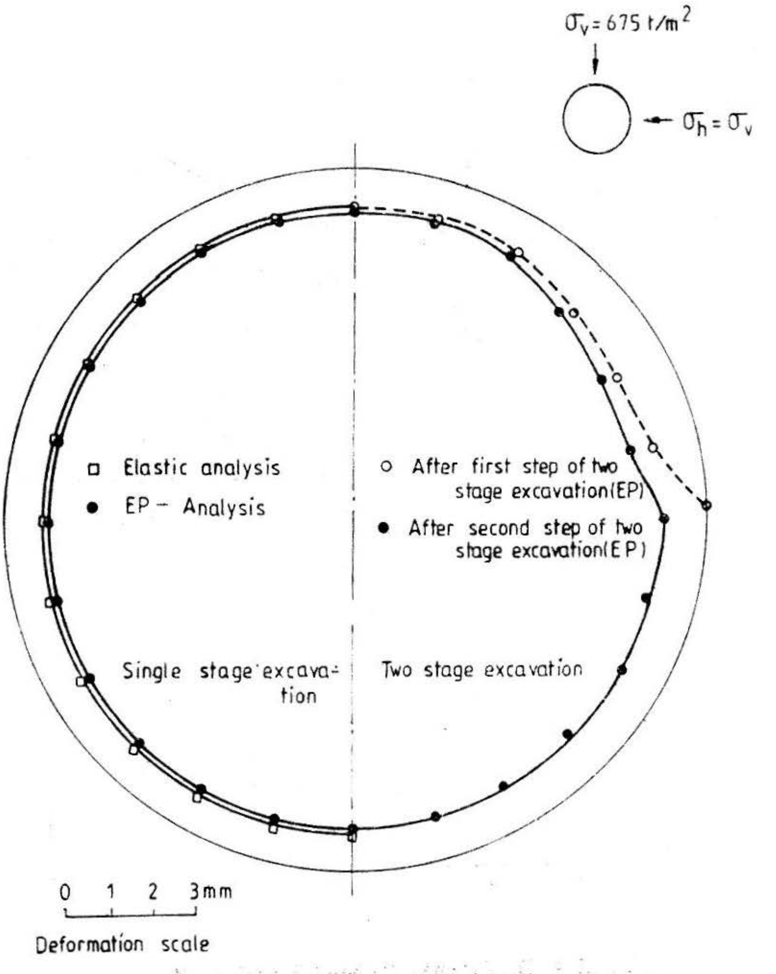


FIGURE 3 Deformed Shape of Tunnel (Ko = 1.0)

paths for the two types of excavations indicate that the points near the springing of the tunnel are markedly affected by two stage excavation scheme.

The principal stress contours (all figures not shown) for elastic analysis for both single and two stage excavations are same, For case 1ST, the stress contours for elastic as well as elasto-plastic analyses are concentric to the tunnel shape (i.e. uniform circular shape). The concentric nature of the contours is lost in elasto-plastic analysis for case 2ST (Fig. 5).

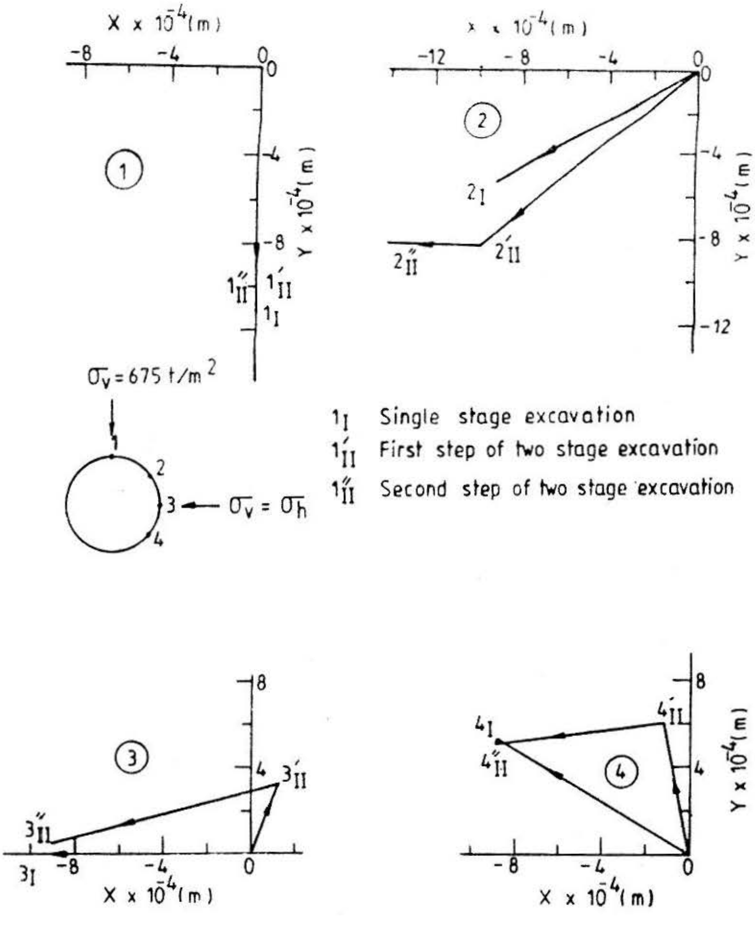


FIGURE 4 Displacement Paths—EP Analysis (K_o = 1.0)

The principal stresses have been plotted along a radial direction also for case 1ST (for elastic and elasto-plastic analyses) as shown in Fig. 6. The nature of the curves obtained are similar to that reported in literature (Obert and Duvall 1967, Goodman 1980).

A comparison of elastic and elasto-plastic analyses results shows that the maximum difference in major principal stresses is 11.4 percent and in minor principal stresses is 13.6 percent, very close to the tunnel boundary.

In case 2ST, the principal stresses have been plotted along two typical radial directions as shown in Fig. 7 (for elasto-plastic analysis only).

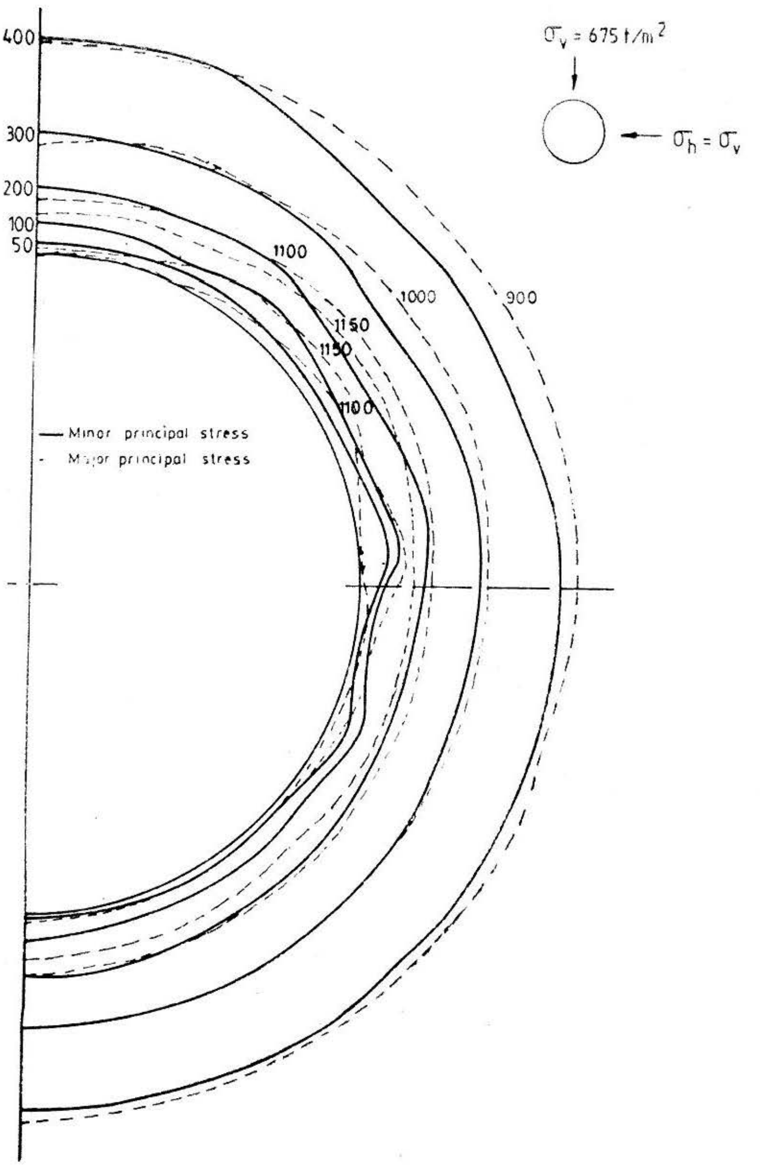


***FIGURE 5* Principal Stress Contours—Two Stage Excavation—EP Analysis (K_o = 1.0)**

A comparison of elasto-plastic stresses in the two cases shows that near the tunnel boundary the maximum difference is 37.2 percent in major principal stress and 103.6 percent in minor principal stress (this large difference in minor principal stress is due to small magnitude of stress values).

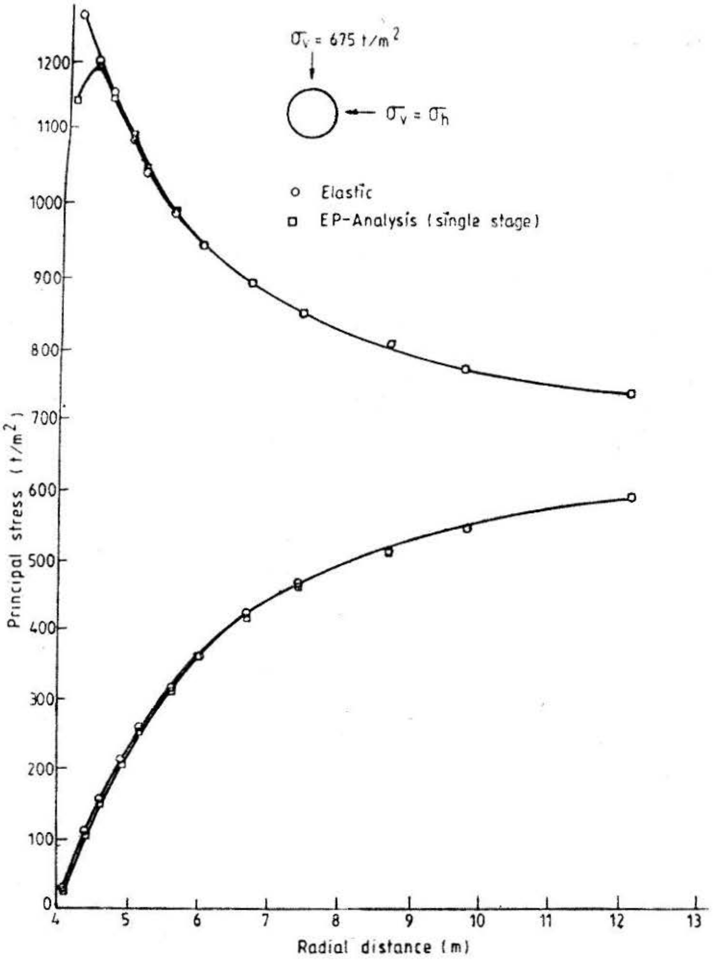


FIGURE 6 Principal Stresses along a Radial Direction (K_o = 1.0)

For single stage excavation, development of any tension around the tunnel is not indicated in the analysis, whereas for the two stage excavation tension develops near the springing level after the first stage of excavation, for the persent excavation scheme, and this is taken care of by Hoek-Brown yield criterion. A separate *No Tension* analysis is not required as mentioned earlier. The maximum value of tensile stress is-209.4 t/m² and this is reduced to—1.6 t/m² after the elasto-plastic solution of first stage is obtained and to 16.3 t/m² (compressive) after the final stage solution is obtained. The tensile strength of the rock as mentioned earlier is—24.47 t/m².

CASE 2 ($K_o = 0.5$)

The elastic and elasto-plastic deformed shape of the tunnel are shown in Fig. 8 for case 1ST (on the left hand side) and for case 2ST (on the right hand side). The maximum deformation takes place at the crown of the tunnel and minimum at the springing of the tunnel in case of elastic analysis. In case of elasto-plastic analysis, the deformations near the springing increase markedly as compared to those obtained from elastic analysis. In case of single stage excavation, it can be seen that near the tunnel boundary, the deformation tend to be uniform all around the tunnel in case of elasto-plastic analysis. The deformed shape obtained from elasto-plastic analysis for case 2ST is non-uniform in this case also.

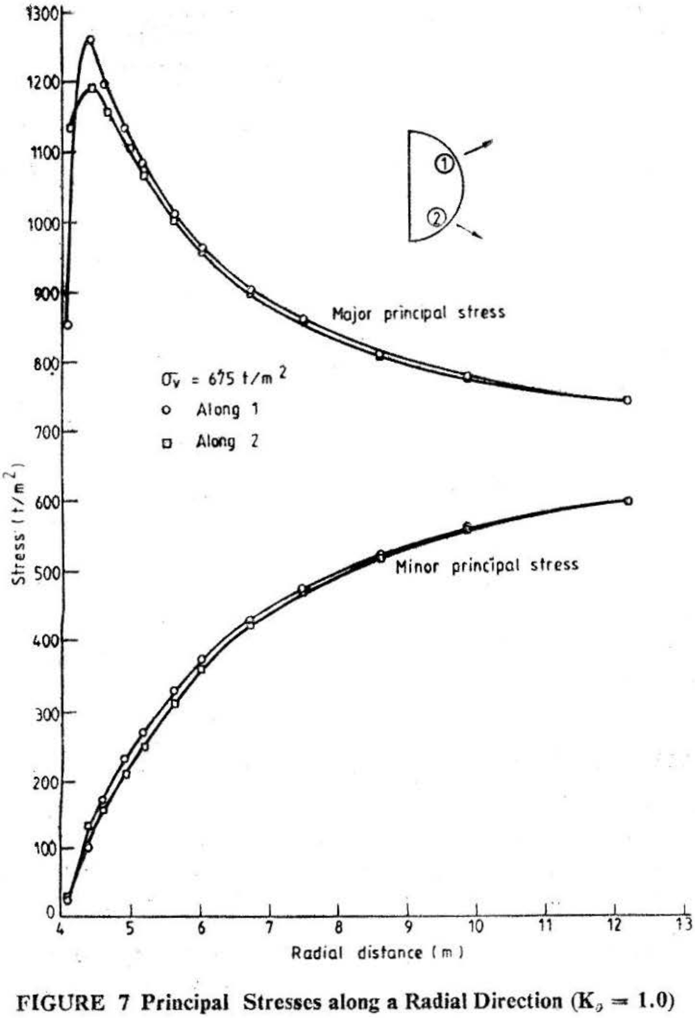


FIGURE 7 Principal Stresses along a Radial Direction ($K_o = 1.0$)

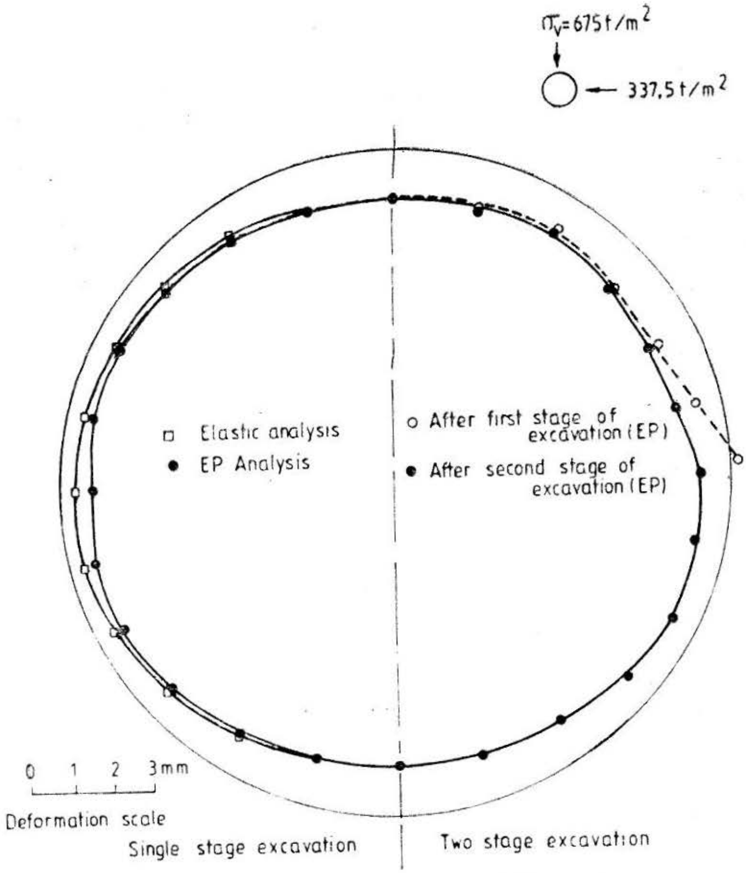


FIGURE 8 Deformed Shape of Tunnel (K_o = 0.5)

Considering case 1ST, the difference found in displacements at the tunnel boundary between elasto-plastic and elastic analyses is 0.9 percent at the crown and 233.68 percent at the springing level of the tunnel.

A comparison of the displacements obtained from elasto-plastic analysis for case 2ST and case 1ST shows that at crown and invert of the tunnel, the displacements are more in case 2ST (0.26 and 5.5 percent, respectively) and at springing the displacements are lesser in case 2ST (17.06 percent). But between springing and crown, the displacements in case 2ST are markedly higher (maximum about 80 percent).

The displacement paths (for elasto-plastic analysis) for four typical points on the tunnel boundary have been shown in Fig. 9. This case also shows that the nature and magnitude of the movement is markedly influenced by single and multiple stage excavation scheme.

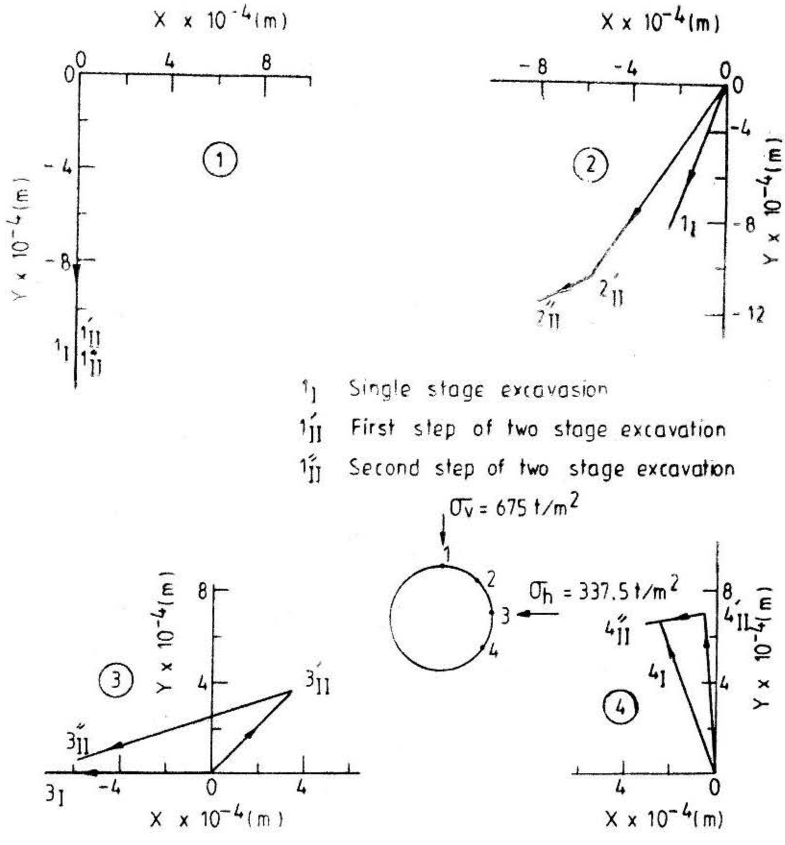


FIGURE 9 Displacement Paths—EP Analysis ($K_o = 0.5$)

The principal stress contours for case 1ST and case 2ST are shown in Figs. 10 to 12. For case 1ST stress contours for both elastic and elasto-plastic analyses are shown and for case 2ST, stress contours for elasto-plastic analysis only are shown. The contours show that both the principal stresses are affected markedly by excavation in multistages.

The principal stresses have been plotted along two typical radial directions as shown in Fig. 13 (for elasto-plastic analysis only) for the two cases. A comparison of elasto-plastic stresses for the two cases shows that near the tunnel boundary the maximum difference in minor principal stresses is about 47.48 percent and in major principal stresses is about 30.5 percent. These differences in the stresses occur in the upper half of the tunnel which has been excavated in the first stage.

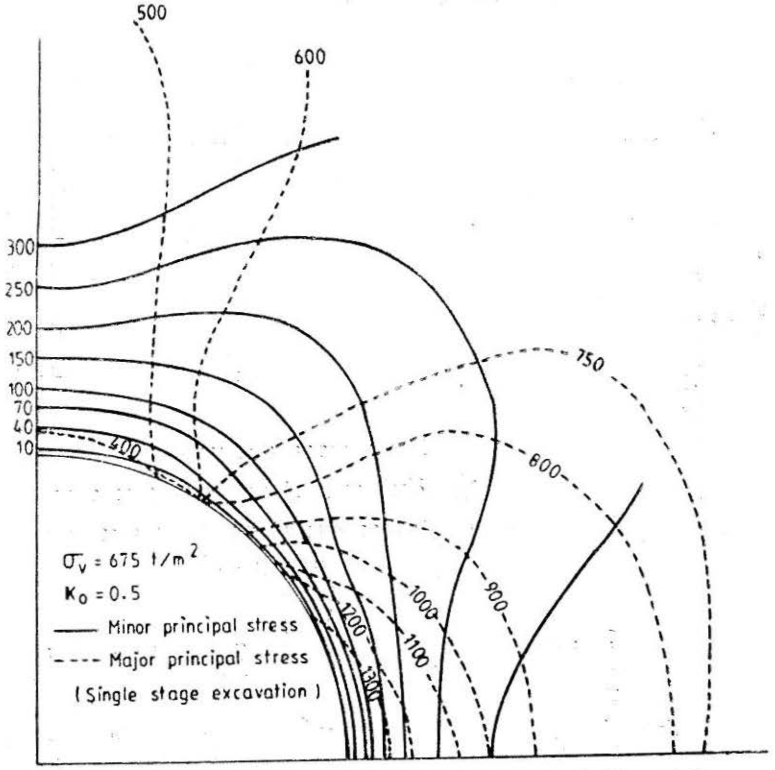
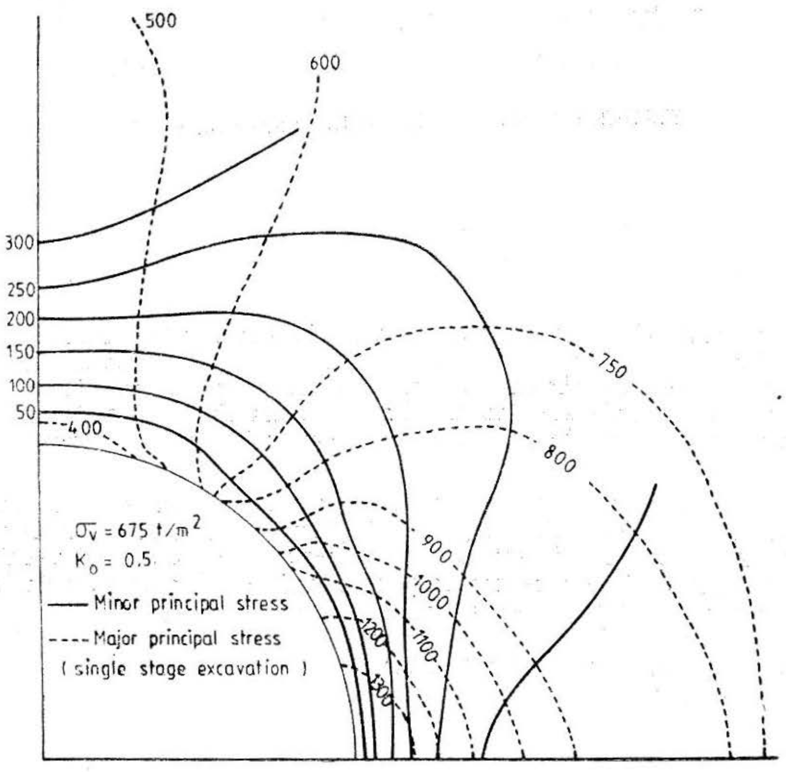


FIGURE 10 Principal Stress Contours—EL Analysis (K_o = 0.5)



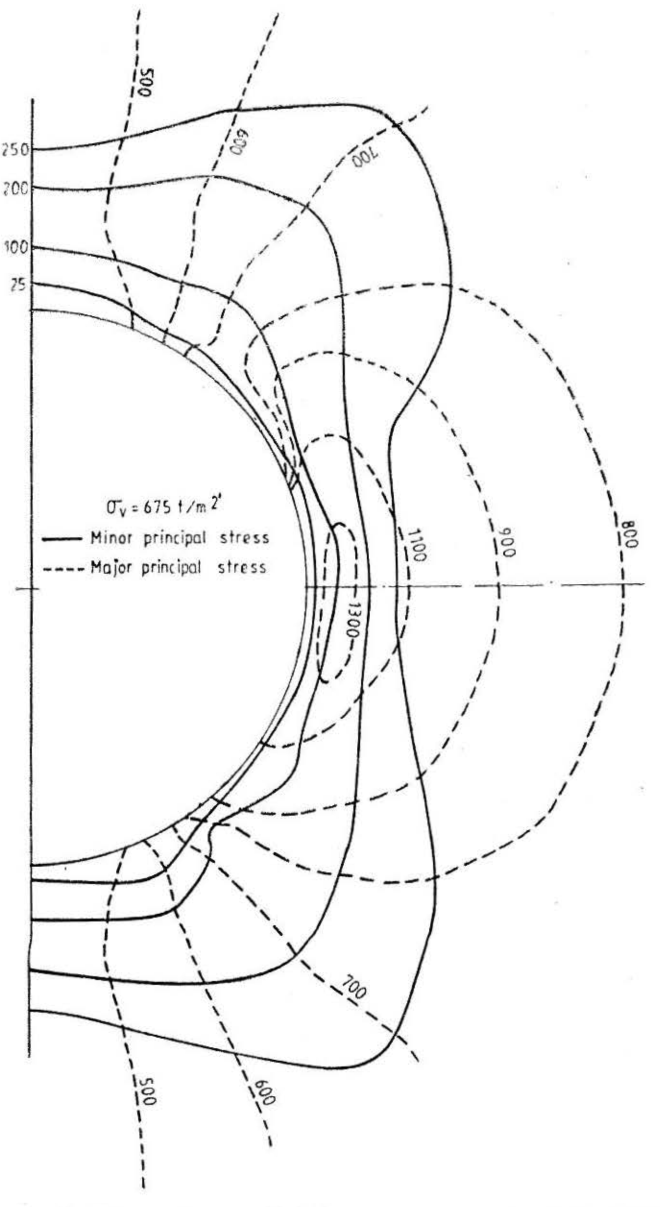


FIGURE 12 Principal Stress Contours Two Stage Excavation—EP Analysis (K_o = 0.5)

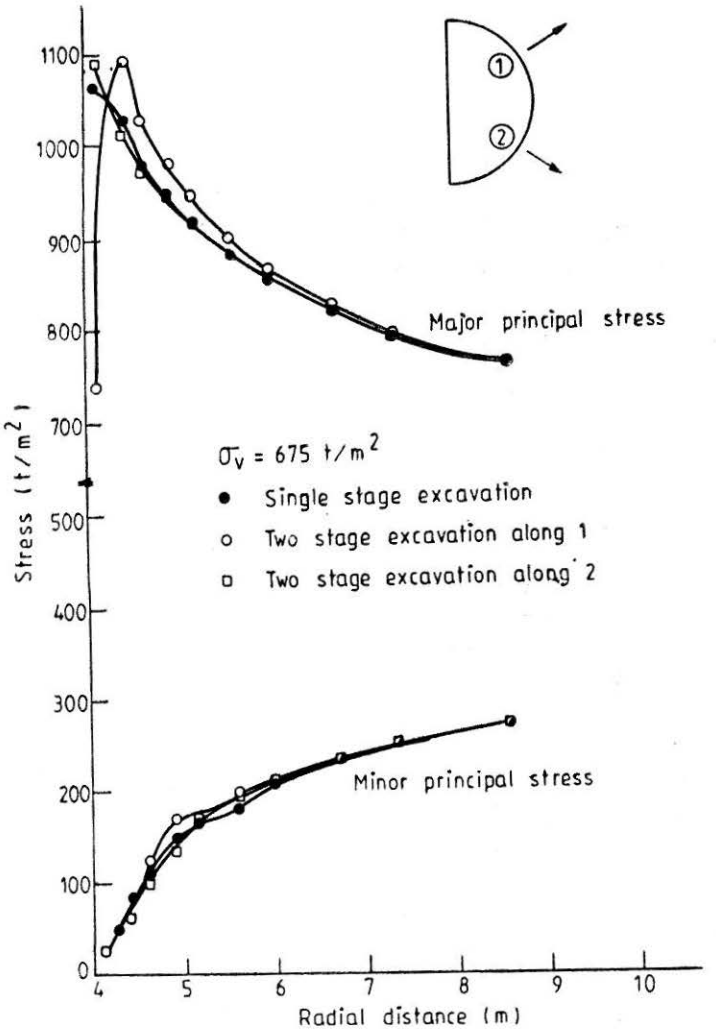


FIGURE 13 Principal Stresses along a Radial Direction (K_o = 0.5)

The maximum value of tensile stress indicated after first stage of excavation is near springing level and it is—119.6 t/m². This is reduced to 6.4 t/m² (compressive) in the elasto-plastic solution obtained after the completion of first stage excavation and becomes 6.5 t/m² (compressive) after ths elasto-plastic solution of the final stage excavation is obtained.

CASE 3 ($K_o = 1.5$)

The elastic and elasto-plastic deformed shapes of the tunnel are shown in Fig. 14 for case 1ST (on the left hand side) and case 2ST (on the right

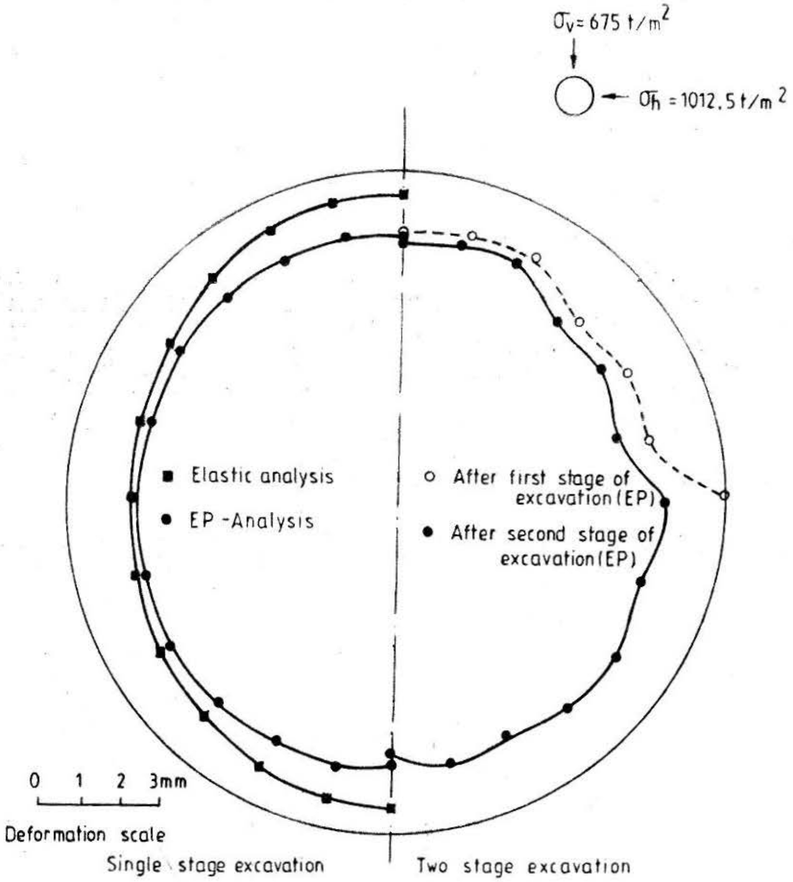


FIGURE 14 Deformed Shape of Tunnel (K_o = 1.5)

hand side). In case of elastic analysis the maximum displacement takes place at the springing of the tunnel and minimum at the crown of the tunnel. In case of elasto-plastic analysis, the deformations near the crown of the tunnel increase markedly as compared to those obtained from elastic analysis. The deformed shape obtained from elasto-plastic analysis for case 2ST is non-uniform in this case also.

Considering case 1ST, the difference found in displacement at the tunnel boundary between elastic and elasto-plastic analysis is 165.62 percent at the crown and 2.1 percent at the springing of the tunnel. The difference shown between diplacements at crown and springing obtained from elasto-plastic analysis is only 8.4 percent.

A comparison of the displacements obtained from elasto-plastic analysis for case 2ST and case 1ST shows that in general, the displacements in case 2ST are higher and in the upper half of the tunnel the difference in the displacements is more as compared to the lower half of the tunnel. Between the crown and the springing maximum difference is 8.4 percent and between springing and invert it is about 14 percent.

The displacement paths (for elasto-plastic analysis only) for four typical points on the tunnel boundary have been shown in Fig. 15. In this case also they indicate that the points around the springing of the tunnel are significantly affected in two stage excavation scheme.

The major and minor principal stress contours for both the cases are shown in Figs. 16 to 18. For case 2ST, stress contours for elasto-plastic analysis only are shown. The contours show that in this case, the major principal stresses are significantly affected by the yielding of the rock as compared to minor principal stresses.

The principal stresses have been plotted along two typical radial directions for the two cases as shown in Fig. 19 (for elasto-plastic analysis only). A comparison of principal stresses for the two cases shows that near the tunnel boundary, the stresses are affected more in the upper half of the tunnel. The difference shown in minor principal stresses is large, 80.03 percent (possibly due to small values of stresses involved) and in major principal stress, it is 23.38 percent. In the upper half of the tunnel, the principal stresses are small in case 2ST and in the lower half of the tunnel, they are slightly higher compared to case 1ST.

The maximum value of tensile stress indicated after first stage excavation is-314.04 t/m² and this is reduced to 5.6 t/m² (compressive) after the

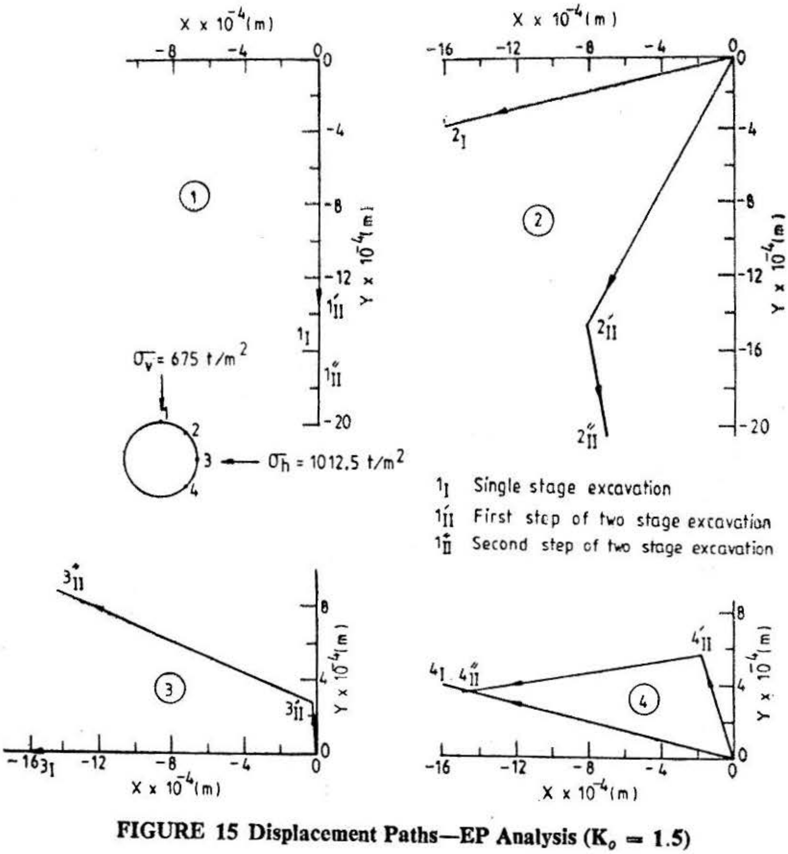


FIGURE 15 Displacement Paths—EP Analysis (K_o = 1.5)

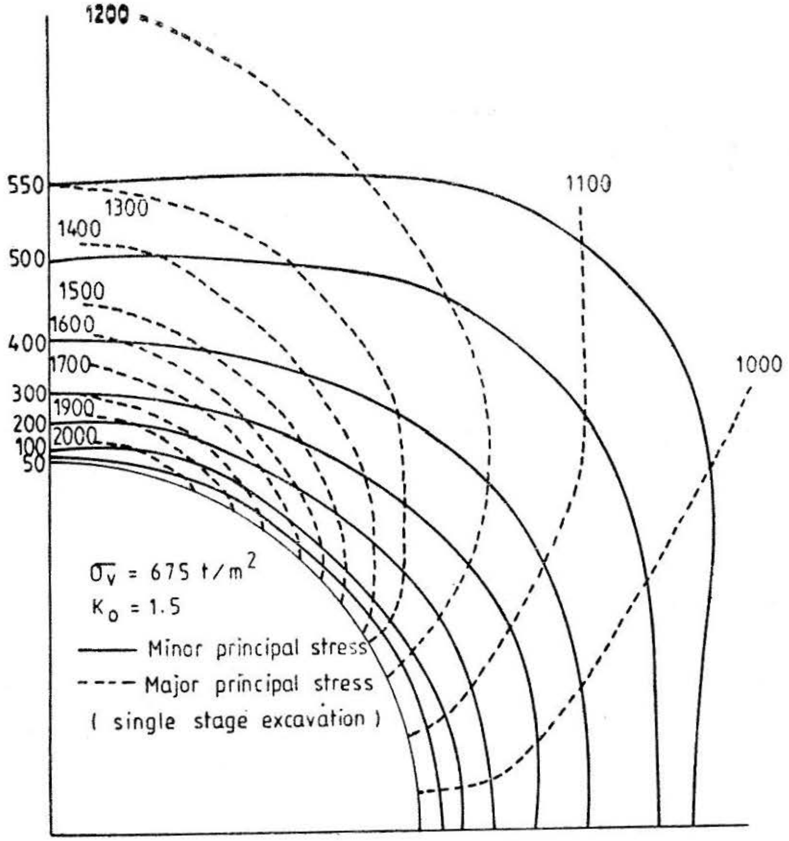
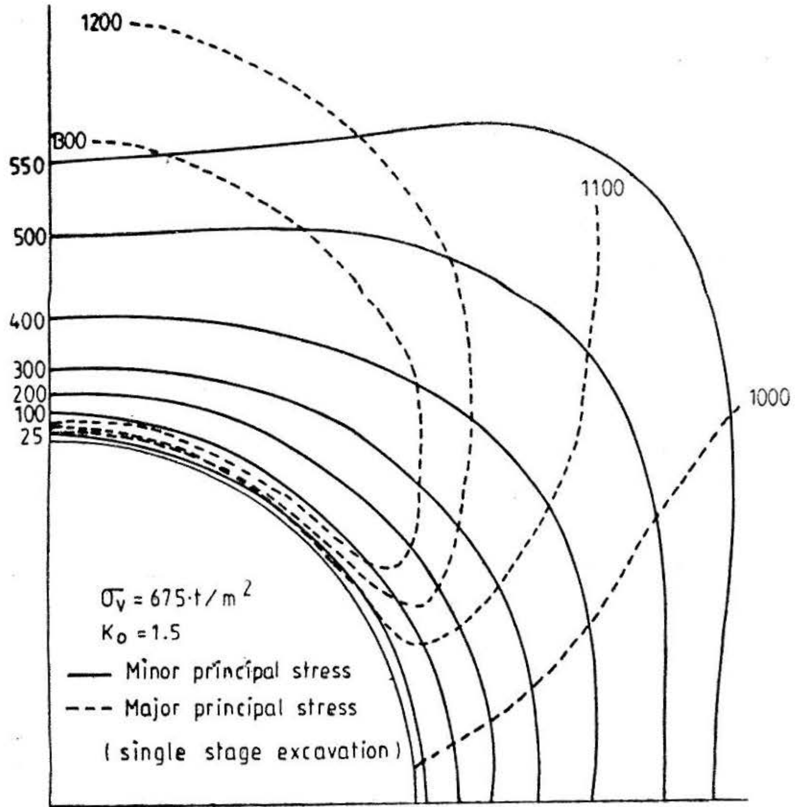


FIGURE 16 Principal Stress Contours—EL Analysis (K_o = 1.5)



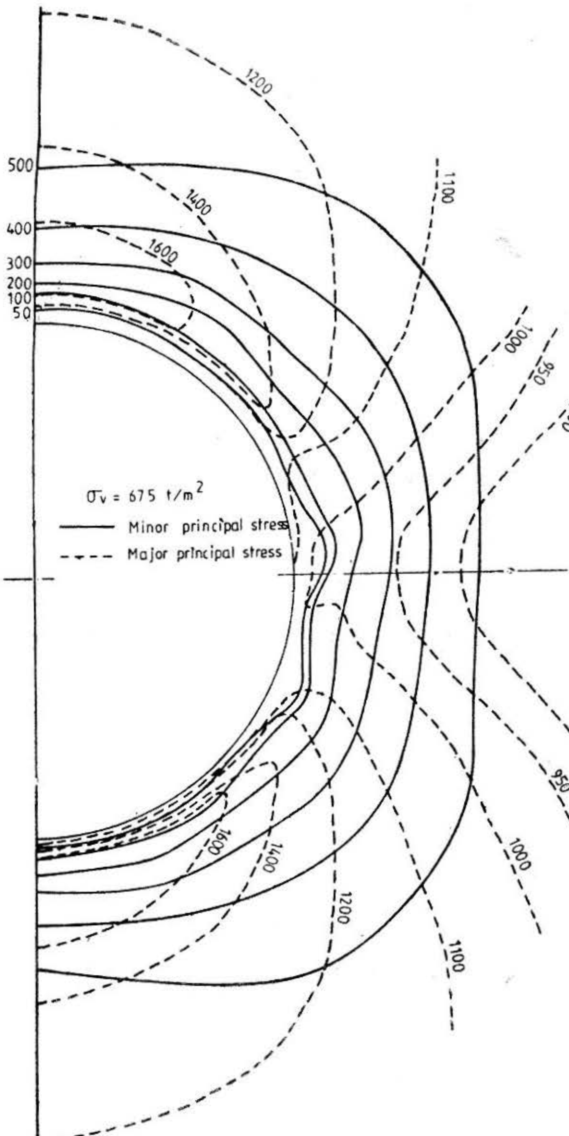


FIGURE 18 Principal Stress Contours Two Stage Excavation—EP Analysis ($K_o = 1.5$)

elasto-plastic solution of first stage excavation is obtained and becomes 37.0 t/m² (compressive) after the final stage elasto-plastic solution is obtained.

Conclusions

From the study carried out for elastic and elasto-plastic analysis of circular tunnels excavated in single and two stages for three insitu stress

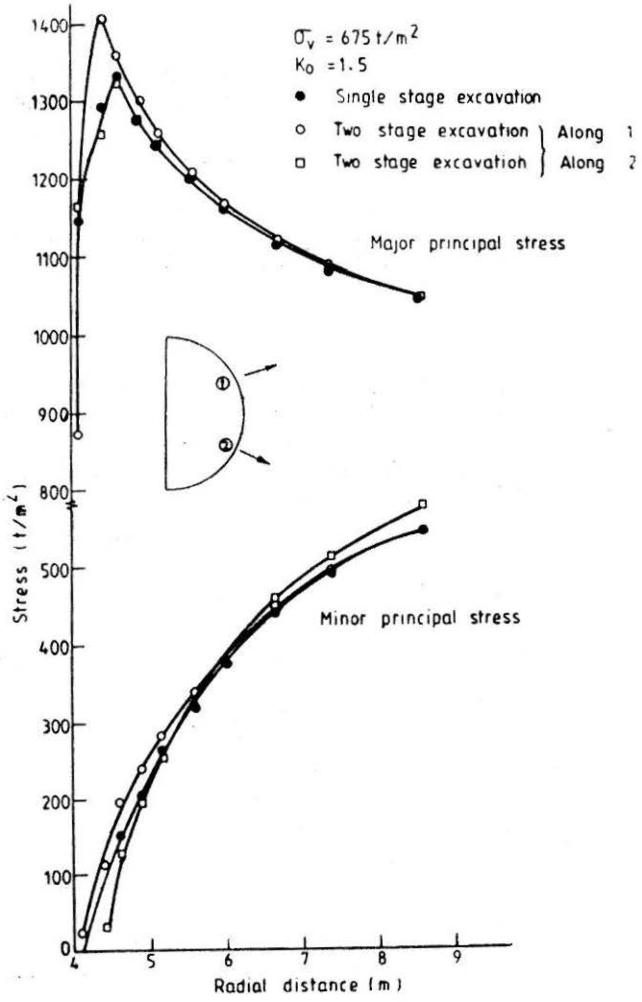


FIGURE 19 Principal Stresses along a Radial Direction (K_o = 1.5)

ratios, it may be concluded that in case of single stage excavation, maximum difference in displacements between elastic and elasto-plastic analyses is for $K_o = 0.5$ and it is at the springing of the tunnel.

In case of elasto-plastic analysis, for the properties used, near the tunnel boundary the displacements tend to be uniform all around the tunnel boundary even though insitu stress values are non-hydrostatic.

A comparison of displacements for elasto-plastic analysis for single and two stage excavation shows that in general, the displacements obtained from the two stage excavation are higher and the difference reflected is more in case of $K_o = 0.5$ in the upper half of the tunnel.

Comparing displacement paths, it is found that the magnitude and direction of movement is affected maximum (at all places except crown) for the higher insitu stress ratio, i,e., $K_o = 1.5$ for the present sequence of excavation.

From a comparison of principal stress contours for the two types of excavation schemes, it has been found that in general in the upper half of the tunnel, there is significant variation in contours for all K_o values and the maximum variation is observed for $K_o = 1$ case.

It has been found that in case of two stage excavation, the rock is subjected to tensile stresses near the springing of the tunnel. In general, the multiple stage excavation scheme introduces non-uniformity in stress and displacement variation.

References

CHANDRASEKARAN, V.S. AND KING, G.I.W. (1974): "Simulation of Excavation Using Finite Elements" *Journal of Geotech. Engg. Div.*, ASCE, 100:1087-1089.

DESAI, C.S. AND ABEL, J.F. (1972): "*Introduction to the Finite Element Method*". Van Nostrand Reinhold, New York.

GOODMAN, R.E. (1980): "*Introduction to Rock Mechanics*". John Wiley & Sons, New York.

HOEK, E. AND BROWN, E.T. (1980): "*Underground Excavations in Rock*". Institution of Mining and Metallurgy, London.

NAYAK, G.C. AND ZIENKIEWICZ, O.C. (1972) : "Convinient form of Stress Invariants for Plasticity". *Journal Struct. Div.*, ASCE, 98:949-954.

OBERT, L. AND DUVALL, W.I. (1967): "*Rock Mechanics and the Design of Structures in Rock*". John Wiley & Sons, New York.

SRIVASTAVA, R.K., SHARMA, K.G. AND VARADARAJAN, A. (1986) : "Finite Element Analysis of Tunnels using different Yield Criteria", *Proc. Second International Conference on Numerical Models in Geomechanics*, Ghent, Belgium, pp. 381-389.

ZIENKIEWICZ, O.C. (1977) : "*The Finite Element Method*". McGraw-Hill, London.

ZIENKIEWICZ, O.C. AND CORMEAU, I.C. (1974) : "Viscoplasticity, Plasticity and Creep in Elastic Solids: A Unified Numerical Solution Approach". *Int. J. Num. Meth. Engg.*, 8: 821-845.