# A New Lumped Parameter Model for the Coupled Rocking and Sliding Vibrations of Embedded Footings

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## Introduction

A study of the dynamic behaviour of footings subjected to coupled rocking and sliding mode of vibrations is very important in the design of foundations for various types of structures such as radar towers, chimneys, off-shore platforms etc. and also in the design of machine foundations. Many procedures pertaining to the study of vibrations of footings, idealise the footing to be circular, as thus shape simplifies the analysis considerably. Rectangular shapes are more commonly used as machine foundations. These footings produce sliding and rocking components during horizontal excitations.

During coupled rocking and sliding vibrations of embedded rectangular footings, the spring and damping resistance for rocking and sliding on account of embedment are developed on the faces perpendicular to the plane of rocking. There is a likelihood of slip at the vertical interfaces of the rectangular block parallel to the plane of rocking and at the horizontal interface of the base and the soil below giving rise to the mobilisation of frictional forces. The magnitude of these forces depend on the angle of intergranular friction, overburden pressure, moisture content, nature of contact surface between embedded block and soil, etc.

In this paper a two-degree-of-freedom lumped parameter analogue and its closed form solution have been described to predict the behaviour of rigid embedded block foundations under coupled rocking and sliding mode with following assumptions: (1) The base shape of the embedded block foundation is rectangular. However, footings with circular base shape can be approximated as an equivalent equare of same base area (2) Though the development of the lumped parameter analogue model implies the embedment of the foundation blocks in linearly elastic, isotropic and homogeneous half-space, it is possible to accommodate variable properties of the soil above the base level of the footing by suitable modifications in the proposed lumped parameters, (3) The proposed theoretical model assumes superposition of the effect of embedment with the response of surface footings, (4) The damping and spring constants for surface footings under coupled rocking and sliding mode of vibration can be chosen either from Hall's (1967) analogue or from the theoretical model of Beredugo and Novak (1972), (5) Stiffness and damping constants due to

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*(The revised paper was received in February, 1986 and is open for discussion till the end of September, 1986)*

embedment of the block for horizontal translation can be obtained by a suitable adoption of the lumped paramaters of Lysmer's (1965) analogue for surface footings, (6) Stiffness and damping constants for rocking due to embedment can be obtained by a suitable adoption of the lumped parameters of Hall's (1967) analogue for surface footings, (7) There is a likelihood of slip at the vertical interfaces of the embedded rectangular block parallel to the plane of rocking, (i.e. perpendicular to the axis of rocking) with the surrounding soil and at the horizontal interface of the base with the soil below, giving rise to the mobilisation of frictional forces. These forces can be lumbed as a single force $F$ acting at the base of the rectangular footing (Fig. 1).

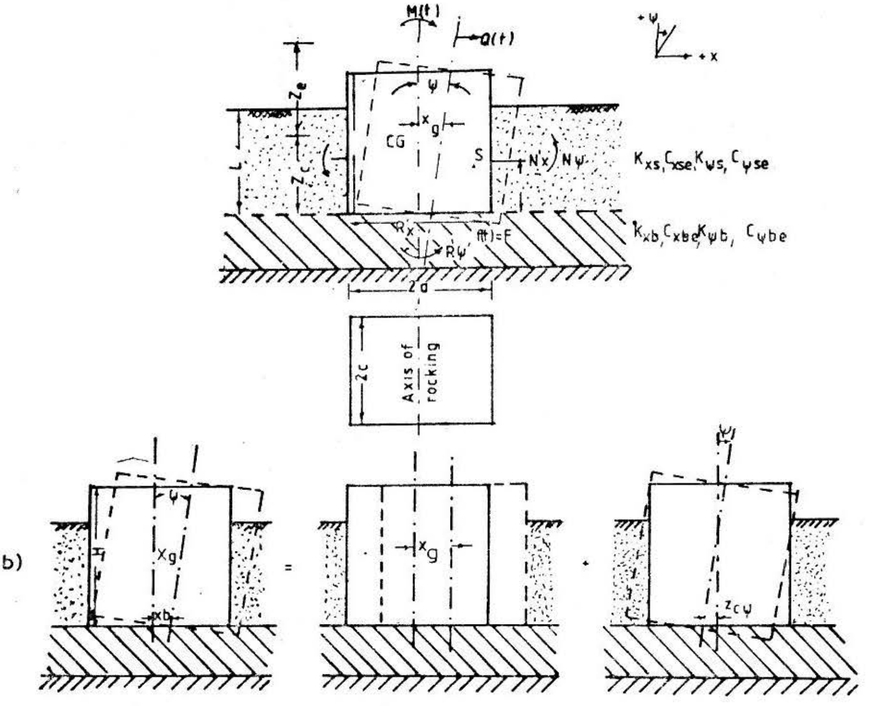


**FIGURE 1 Proposed Theoretical Model of an Embedded Footing Under Coupled Vibrations**

## Analysis

Figure 1 represents the proposed theoretical model of a rectangular embedded footing of plan dimensions $2c \times 2d$ under coupled rocking and sliding vibrations and embedded to a depth $L$ in a homogeneous, isotropic elastic half-space. Let it be subjected to an exciting force

$$Q(t) = Q_o \cos(\omega t + \phi) \quad \text{...(1)}$$

and moment

$$M(t) = M_o \cos(\omega t + \phi) \quad \text{...(2)}$$

about the center of gravity. With the sign conventions and notations indicated, summation of forces about the center of gravity yield the following equations of dynamic equilibrium:

$$m\ddot{x}_g + R'_x + N'_x = Q_o \cos(\omega t+\phi) \pm F \qquad \ldots(3)$$

$$I\ddot{\varphi} - R'_x z_c + R'_\varphi - N'_x\, z_c - L/2) + N'_\varphi = M_o \cos(\omega t+\phi) \mp F z_c \qquad \ldots(4)$$

where

$$\left.\begin{aligned}
x_b &= x_g - z_c\varphi \\
x_s &= x_g + (L/2 - z_c)\varphi \\
R'_x &= k_{xb}\, x_b + c_{xbe}\, \dot{x}_b \\
R'_\varphi &= k_{\varphi b}\, \varphi + c_{\varphi be}\, \dot{\varphi} \\
N'_x &= k_{xs}\, x_s + c_{xse}\, \dot{x}_s \\
N'_\varphi &= k_{\varphi s}\, \varphi + c_{\varphi se}\, \dot{\varphi} \\
c_{xbe} &= c_{xb} + b_{xb}/\omega \\
c_{xse} &= c_{xs} + b_{xs}/\omega \\
c_{\varphi be} &= c_{\varphi b} + b_{\varphi b}/\omega \\
c_{\varphi se} &= c_{\varphi s} + b_{\varphi s}/\omega
\end{aligned}\right\} \qquad \ldots(5)$$

**Notations Used in the Above Equations**

$m$-mass of the vibrating body, $Q_o$ and $M_o$—Real amplitudes of exciting forces and moments, $\omega$-Frequency of excitation, $\phi$-Phase angle, $I$-mass moment of inertia about center of gravity, $x_g$ and $\varphi$ = translation and rotation of center of grarity at any instant of time, $x_o$ and $\varphi_o$—maximum amplitude of translation and rotation of center of gravity. $x_b$—horizontal component of displacement of the footing base at any instant of time, $x_s$—horizontal component of displacement of the point of applications of the soil reaction $N'_x$ due to embedment at any instant of time, $c_{xbe}$ and $c_{\varphi be}$ — equivalent damping constants for horizontal translation and rocking respectively of the soil layer below the footing base, $c_{xse}$ and $c_{\varphi se}$ — equivalent damping constants for horizontal translation and rocking respectively due to embedment $k_{xb}$ and $k_{\varphi b}$ —stiffness constants for horizontal translation and rocking respectively due to the soil layer below footing base, $c_{xb}$ and $c_{\varphi b}$—Damping constants for horizontal translation and rocking respectively of the soil layer below the footing base, $k_{xs}$ and $k_{\varphi s}$

stiffness constants for horizontal translation and rocking respectively due embedment, $c_{xs}$ and $c_{\varphi s}$ — damping constants for horizontal translation and rocking respectively due to embedment, $z_c = r_f$—height of center of gravity of the vibrating mass above base, $z_e$ — eccentricity of the applied force with respect to the CG of the system, $R'_x$ and $R'_\varphi$ — soil reaction for translation and rocking respectively at the footing base, $N'_x$ and $N'_\varphi$ — soil reactions for translation and rocking respectively due to embedment, $b_{xb}$ and $b_{xs}$— internal damping coefficients for translation, $b_{\varphi s}$ and $b_{\varphi b}$—internal damping coefficients for rocking and dot represents differentiation with respect to time.

The sign of $F$ in Eqs. (3) and (4) respectively depend on the velocity vector $\dot{x}_b$. If the frictional force is small compared to the driving force amplitudes, the resulting motion will be continuous while for larger values of F, a single cycle of motion map consist of regions of stand still. The procedure indicated herein assumes smaller values of $F$ and correspondingly continuous motion without any standstill. However, the present analysis can also be extended to motions with standstills through suitable modifications in the boundary conditions.

**Introducing the Notations**

$$\left.\begin{aligned}
q_1 &= c_{xbe} + c_{xse} \\
q_2 &= k_{xb} + k_{xs} \\
q_3 &= -c_{xbe}\, z_c + c_{xse}(L/2\text{-}z_c) \\
q_4 &= -k_{xb}\, z_c + k_{xs}(L/2\text{-}z_c) \\
s_1 &= c_{xbe}\; z_c^2 + c_{\varphi be} \qquad + c_{xse}\,(L/2\text{-}z_c)^2 + c_{\varphi se} \\
s_2 &= k_{xb}\; z_c^2 + k_{\varphi b} \qquad + k_{xs}\,(L/2\text{-}z_c)^2 + k_{\varphi s} \\
s_3 &= -c_{xbe}\, z_c + c_{zse}\,(L/2\text{-}z_c) \\
s_4 &= -k_{xb}\, z_c + k_{xs}\,(L/2\text{-}z_c)
\end{aligned}\right\} \qquad \ldots(6)$$

and the boundary conditions

$$\begin{aligned}
&\text{at } t = 0,\ x_b = x_{bmax} \quad \dot{x}_b = 0 \quad \varphi = \dot{\varphi}_o \quad \varphi = \dot{\varphi}_o \\
&\text{at } t = \pi/\omega,\ x_b = -x_{bmax} \quad \dot{x}_b = 0 \quad \varphi = -\varphi_o \quad \dot{\varphi} = -\dot{\varphi}_o \qquad \ldots(7) \\
&\quad 0 \leqslant t \leqslant \pi/\omega \qquad\qquad \dot{x}_b \leqslant 0.
\end{aligned}$$

the dynamic equilibrium equations become

$$m\,\ddot{x}_g + q_1\,\dot{x}_g + q_2 x_g + q_3\,\dot{\varphi} + q_4\,\varphi = Q(t) + F \qquad \ldots(8)$$

$$I\,\ddot{\varphi} + s_1\,\dot{\varphi} + s_2\,\varphi + s_3\,\dot{x}_g + s_4\,x_g = M(t) - F\,z_c \qquad \ldots(9)$$

These equations can be solved by the superposition of complementary and particular solutions of the differential equations. Let

$$x_g = A \exp(\eta t) \text{ and } \varphi = B \exp(\eta t), \text{ then} \qquad \ldots(10)$$

for non-trivial values of $A$ and $B$, the quartic equation

$$mI\eta^4+(ms_1+q_1I)\,\eta^3+(ms_2+q_1\,s_1+q_2\,I-q_3\,s_3)\,\eta^2+(q_1\,s_2+q_2\,s_1 -q_3\,s_4-q_4\,s_3)\,\eta+(q_2\,s_2-q_4\,s_4)=0 \quad \ldots(10A)$$

with characteristic roots $\eta_1=-\alpha+i\beta$, $\eta_2=-\alpha-i\beta$, $\eta_3=-\gamma+i\delta$, $\eta_4=-\gamma-i\delta$ and $\alpha$, $\beta$, $\gamma$ and $\delta$ as positive, real constants can be solved by the theory of equations. The ratio of $B/A$ for each root can be obtained as :

For $\eta_1$,

$$[B_1/A_1]=[(-m\alpha^2+m\beta^2+q_1\alpha-q_2)+i(2m\alpha\beta-q_1\beta)]/[(-q_3\alpha+q_4)+iq_3\beta] \quad \ldots(11)$$
$$= a+ib$$

For $\eta_2$, $[B_2/A_2]=a-i\,b$

For $\eta_3$,

$$[B_3/A_3]=[(-m\gamma^2+m\delta^2+q_1\gamma-q_2)+i\,(2m\gamma\,\delta-q_1\delta)]/[(-q_3\gamma+q_4)+iq_3\delta)] \quad \ldots(12)$$

$$= c+id$$

For $\eta_4$, $[B_4/A_4]=c-i\,d$

Then the complementary solutions are

$$x_g=\exp(-\alpha t)[C_1\cos\beta t+C_2\sin\beta t]+\exp(-\gamma t)[C_3\cos\delta t+C_4\sin\delta \mathrm{t}] \quad \ldots(13)$$

$$\varphi=\exp(-\alpha t)(D_1\cos\beta t+D_2\sin\beta t]+\exp(-\gamma t)[D_3\cos\delta t+D_4\sin\delta t] \quad \ldots(14)$$

and

$$\begin{array}{l|l|l|l}
C_1=A_1+A_2 & C_2=(A_1-A_2)i & C_3=A_3+A_4 & C_4=(A_3-A_4)i \\
D_1=B_1+B_2 & D_2=(B_1-B_2)i & D_3=B_3+B_4 & D_4=(B_3-B_4)i \\
D_1=aC_1+C_2b & D_2=aC_2-bC_1 & D_3=cC_3+C_4d & D_4=cC_4-C_3d
\end{array} \quad \ldots(15)$$

can be obtained.

The particular solution of the above Eqs. (8) and (9)

$$x_g=U_1\cos(\omega t+\phi)+U_2\sin(\omega t+\phi)+V_1F \quad \ldots(16)$$

$$\varphi=U_3\cos(\omega t+\phi)+U_4\sin(\omega t+\phi)+V_2F \quad \ldots(17)$$

can be evaluated in a similar manner as reported by Yeh (1966)

$$V_1 = (s_2 + r_f q_4)/(-s_4 q_4 + s_2 q_2) \quad \text{and} \quad V_2 = -(q_2 r_f + s_4)/(-s_2 q_2 + s_4 q_4) \qquad \ldots(18)$$

$$|a| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} (q_2/m-\omega^2) & (q_1\omega/m) & (q_4/m) & (q_3\omega/m) \\ (-q_1\omega/m) & (q_2/m-\omega^2) & (-q_3\omega/m) & (q_4/m) \\ (s_4/I) & (s_3\omega/I) & (s_2/I-\omega^2) & (s_1\omega/I) \\ (-s_3\omega/I) & (s_4/I) & (-s_1\omega/I) & (s_2/I-\omega^2) \end{vmatrix} \qquad \ldots(19)$$

$$|a| \; U_j = [Q_o/m]\, A_{1j} + [M_o/I]\, A_{3j}, \; j = 1, 2, 3, 4 \qquad \ldots(20)$$

and

$A_{11} \ldots A_{44}$ are the cofactors of $a_{11} \ldots a_{44}$

The general solutions for the system of differential Eqs. (8) and (9) are written as

$$x_g = \exp(-\alpha t)\,[C_1 \cos \beta t + C_2 \sin \beta t] + \exp(-\gamma t)\,[C_3 \cos \delta t + C_4 \sin \delta t] + U_1 \cos(\omega t + \phi) + U_2 \sin(\omega t + \phi) + V_1 F \qquad \ldots(21)$$

$$\varphi = \exp(-\alpha t)\,[D_1 \cos \beta t + D_2 \sin \beta t] + \exp(-\gamma t)[D_3 \cos \delta t + D_4 \sin \delta t] + U_3 \cos(\omega t + \phi) + U_4 \sin(\omega t + \phi) + V_2 F \qquad \ldots(22)$$

**Introducing the Notations**

$p_1 = [-\alpha + r_f(\alpha a + b\beta)]$, $p_2 = [\beta + r_f(b\alpha - a\beta)]$, $p_3 = [-\gamma + r_f(\gamma c + d\delta)]$, $p_4 = [\delta + r_f(\gamma d - c)]$.

$$|b| = \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix} = \begin{vmatrix} (1-ar_f) & -br_f & (1-cr_f) & -dr_f \\ p_1 & p_2 & p_3 & p_4 \\ a & b & c & d \\ (-a\alpha-b\beta) & (-\alpha b+a\beta) & (-c\gamma-d\delta) & (-\gamma d+\delta c) \end{vmatrix} \qquad \ldots(23)$$

$$\left.\begin{array}{ll} p = \exp(-\alpha\pi/\omega)\cos(\beta\pi/\omega), & q = \exp(-\alpha\pi/\omega)\sin(\beta\pi/\omega) \\ r = \exp(-\gamma\pi/\omega)\cos(\delta\pi/\omega), & s = \exp(-\gamma\pi/\omega)\sin(\delta\pi/\omega) \end{array}\right] \qquad \ldots(24)$$

$a_1 = [p + r_f(-ap + bq)]$, $a_2 = [q + r_f(-bp - aq)]$, $a_3 = [r + r_f(-rc + ds)]$,
$a_4 = [s - r_f(rd + cs)]$, $a_5 = [ap - bq]$, $a_6 = [bp + aq]$, $a_7 = [rc - ds]$,
$a_8 = [rd + cs]$, $a_9 = [-a_1\alpha - a_2\beta]$, $a_{10} = [-a_2\alpha + a_1\beta]$,

$$a_{11}=[-a_3\gamma = a_4\delta],\ a_{12}=[-a_4\gamma + a_3\delta],\ a_{13}=[-a_5\alpha - a_6\beta],$$

$$a_{14}=[-a_6\alpha + a_5\beta],\ a_{15}=[-a_7\gamma - a_8\delta],\ a_{16}=[-a_8\gamma + a_7\delta], \quad \ldots(25)$$

$$[D] = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \quad \ldots(26)$$

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11}+|b|(-U_1+U_3 r_f) & m_{12}+|b|(-U_2+r_f U_4) & m_{13} & m_{14} \\ m_{21}+|b|\omega(-U_2+U_4 r_f) & m_{22}+|b|\omega(U_1-r_f U_3) & m_{23} & m_{24} \\ m_{31}-|b|\ U_3 & m_{32}-|b|\ U_4 & m_{33}+|b| & m_{34} \\ m_{41}-|b|\omega\ U_4 & m_{42}+|b|\omega\ U_3 & m_{43} & m_{44}+|b| \end{bmatrix} \quad \ldots(27)$$

$$\begin{bmatrix} c_{15} & c_{16} \\ c_{25} & c_{26} \\ c_{35} & c_{36} \\ c_{45} & c_{46} \end{bmatrix} = \begin{bmatrix} m_{15}-|b| & m_{16}-|b|(V_1-V_2 r_f) \\ m_{25} & m_{26} \\ m_{35} & m_{36}-|b|V_2 \\ m_{45} & m_{46} \end{bmatrix} \quad \ldots(28)$$

$$\begin{Bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{Bmatrix} = [D] \begin{Bmatrix} -(U_1-r_f U_3)B_{11}-\omega(U_2-U_4 r_f)B_{21}-U_3 B_{31}-\omega U_4 B_{41} \\ -(U_1-r_f U_3)B_{12}-\omega(U_2-U_4 r_f)B_{22}-U_3 B_{32}-\omega U_4 B_{42} \\ -(U_1-r_f U_3)B_{13}-\omega(U_2-U_4 r_f)B_{23}-U_3 B_{33}-\omega U_4 B_{43} \\ -(U_1-r_f U_3)B_{14}-\omega(U_2-U_4 r_f)B_{24}-U_3 B_{34}-\omega U_4 B_{44} \end{Bmatrix} \quad \ldots(29)$$

$$\begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \\ m_{42} \end{bmatrix} = [D] \begin{bmatrix} -(U_2-r_f)U_4)B_{11}+\omega(U_1-U_3r_f)B_{21}-U_4B_{31}+\omega U_3B_{41} \\ -(U_2-r_f)U_4)B_{12}+\omega(U_1-U_3r_f)B_{22}-U_4B_{32}+\omega U_3B_{42} \\ -(U_2-r_f)U_4)B_{13}+\omega(U_1-U_3r_f)B_{23}-U_4B_{33}+\omega U_3B_{43} \\ -(U_2-r_f)U_4)B_{14}+\omega(U_1-U_3r_f)B_{24}-U_4B_{34}+\omega U_3B_{44} \end{bmatrix} \quad ...(30)$$

$$\begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \\ m_{34} \end{bmatrix} = [D] \begin{bmatrix} B_{31} \\ B_{32} \\ B_{33} \\ B_{34} \end{bmatrix} \text{ and } \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \\ m_{44} \end{bmatrix} = [D] \begin{bmatrix} B_{41} \\ B_{42} \\ B_{43} \\ B_{44} \end{bmatrix} \quad ...(31)$$

$$\begin{bmatrix} m_{15} \\ m_{25} \\ m_{35} \\ m_{45} \end{bmatrix} = -[D] \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \end{bmatrix} \text{ and } \begin{bmatrix} m_{16} \\ m_{26} \\ m_{36} \\ m_{46} \end{bmatrix} = -[D]$$

$$\begin{bmatrix} -V_1B_{11}+V_2(r_fB_{11}-B_{31}) \\ -V_1B_{12}+V_2(r_fB_{12}-B_{32}) \\ -V_1B_{13}+V_2(r_fB_{13}-B_{33}) \\ -V_1B_{14}+V_2(r_fB_{14}-B_{34}) \end{bmatrix} \quad ...(32)$$

and $B_{11} \, .. \, B_{44}$ to represent the cofactors of $b_{11}$ to $b_{44}$, $C_{11}$ to $C_{44}$ to represent the cofactors of $c_{11}$ to $c_{44}$ and applying Eq. (7) the following could be obtained :

$$|C| \cos \phi = \sum_{j=1}^{4} C_{j1} \, (c_{j5} x_{bmax} + c_{j6} F) = |C| \, [u_1 x_{bmax} + u_2 F] \quad ...(33)$$

$$|C| \cos \phi = \sum_{j=1}^{4} C_{j2} \, (c_{j5} x_{bmax} + c_{j6} F) = |C| \, [v_1 x_{bmax} + v_2 F] \quad ...(34)$$

$$|C| \, \varphi_o = \sum_{j=1}^{4} C_{j3} \, (c_{j5} x_{bmax} + c_{j6} F) \quad ...(35)$$

$$|C| \, \dot{\varphi}_o = \sum_{j=1}^{4} C_{j4} \, (c_{j5} x_{bmax} + c_{j6} F) \quad ...(36)$$

Eliminating $\phi$ between Eqs. (33) and (34) an equation

$$(u_1^2 + v_1^2) x_{bmax}^2 + 2F x_{bmax} (u_1 u_2 + v_1 v_2) + (u_2^2 + v_2^2) F^2 - 1 = 0 \quad ...(37)$$

is obtained and the positive value of $x_{bmax}$ should be considered. Substituting in Eqs. (21) and (22) $x_o$ and $\varphi_o$ can be computed. The problem is then completely solved for a given set of parameters. As $\dot{x}_b \leqslant 0$, substitution of the above equations together with Eqs. (21), (22) and (5) provide an expression for the region of validity (Raghavendra Rao, 1981).

**Evaluation of Stiffness and Damping Constants**

**Evaluation of $k_{xb}$, $c_{xb}$, $k_{\varphi b}$ and $c_{\varphi b}$**

These parameters are evaluated either using Hall's (1967) analogue model or Beredugo and Novak's (1972) model for rocking and sliding of a cylindrical block of radius $r_o$ resting on the surface of an elastic half-space. Thus according to Hall (1967)

$$k_{xb} = [32(1-\mu)\, Gr_o\,]/[\,7-8\mu] \quad \ldots(38)$$

$$k_{\varphi b} = [8\, G r_0^3\,]/[3(1-\mu)] \quad \ldots(39)$$

$$c_{xb} = [18.4\ (1-\mu)\, r_0^2\, (\rho G)^{1/2}]/[7-8\mu] \quad \ldots(40)$$

$$c_{\varphi b} = [0.8\ r_0^4\, (\rho G)^{1/2}\,]/[1-\mu)\ (1+B'_\varphi\,)] \quad \ldots(41)$$

$$B'_\varphi = [3\,(1-\mu)\ I_b\,]/[8\rho\, r_0^5\,] \quad \ldots(42)$$

and according to Beredugo and Novak (1972)

$$k_{xb} = G\, r_o\, C_{ul} \quad \ldots(43)$$

$$k_{\varphi b} = G\, r_0^3\, C_{\varphi 1} \quad \ldots(44)$$

$$c_{xb} = (Gr_o/\omega)\ C_{uc} \quad \ldots(45)$$

$$c_{\varphi b} = (Gr_0^3\,/\omega)\, C_{\varphi 2} \quad \ldots(46)$$

where $C_{u1}$ and $C_{u2}$—elastic half-space stiffness and damping parameters for translation, $C_{\varphi 1}$ and $C_{\varphi 2}$ — elastic half-space stiffness and damping parameters for rocking, $G$—shear modulus of the elastic medium, $\rho$—mass density of the medium, $\mu$—Poisson's ratio, $I_b$—mass moment of inertia about an axis through center of base. Richart et al. (1970) has observed that with the value of $r_o$ chosen for appropriate mode of vibration, the above expressions can be used for rectangular contact areas with length to breadth ratio upto 2.

The stiffness of rectangular contact areas $2c \times 2d$ can be computed alternatively with the help of expressions given by Richart et al. (1970). Thus the sliding and rocking stiffnesses respectively, are given by

$$k_{xb} = [4(1+\mu)\ G\, \beta_x\, (cd)^{1/2}] \quad \ldots(47)$$

$$k_{\varphi b} = [8 \quad G\,\beta_\varphi\,(cd^2)\;]/[1-\mu] \qquad \ldots(48)$$

$\beta_x$ and $\beta_\varphi$ depend on $d/c$ ratio.

**Evaluation of $k_{x_s}$ and $c_{x_s}$**

Lysmer (1965) developed a single-degree-of-freedom lumped parameter analogue model to predict the dynamic response of a circular footing under vertical vibrations with spring and damping constants as

$$k_z = 4\,G\,r_o/[1-\mu] \qquad \ldots(49)$$

$$c_z = 3.4 r_0^2\,(\rho G)^{1/2}/[1-\mu] \qquad \ldots(50)$$

The above expressions depend only on the area of contact and the elastic properties of the soil medium. Also, during vertical vibrations, the soil below the footing is in a state of elastic uniform compression (Barkan, 1962).

Coupled rocking and sliding vibrations of an embedded rectangular footing produce normal pressure on soil mass adjacent to the two vertical faces, parallel to and on either sides of the axis of rocking. This soil mass is also likely to be in a state of elastic uniform compression, if the horizontal component of coupled motion alone is considered. As half of the semi-infinite soil medium is effective with respect to each vertical face of the footing, it is reasonable to assume that the damping $c_{x_s}$ and lateral stiffness, $k_{x_s}$ are approximately equal to the values of $c_z$ and $k_z$ given by Eqs. (50) and (49) respectively, taking care to use the appropriate values of $r_o$ and other elastic properties.

Therefore, for an embedded rectangular footing of dimensions $2c \times 2d$ with embedment $L$, approximate values of $c_{x_s}$ and $k_{x_s}$ can be computed using the following expressions:

$$c_{x_s} = 1.7\, r_{eq}^2\,(\rho G)^{1/2}/[1-\mu] \qquad \ldots(51)$$

$$k_{x_s} = 2\, r_{eq}\, G/[1-\mu] \text{ or} \qquad \ldots(52)$$

$$k_{x_s} = G\,\beta_z\,(cL)^{1/2}/[1-\mu] \qquad \ldots(53)$$

$$r_{eq} = [4c\,L/\pi]^{1/2} \qquad \ldots(54)$$

and $\beta_z$ depends on $L/c$ ratio.

**Evaluation $k_{\varphi s}$ and $c_{\varphi s}$**

Hall (1967) developed stiffness and damping parameters for rocking vibrations of a circular footing on the surface of an elastic half space and the expressions to evaluate them are given in Eqs. (38) to (42).

The soil, below a footing undergoing rocking mode of vibration is in a state elastic non-uniform compression (Barken, 1962). Coupled rocking and sliding vibrations of an embedded rectangular footing would also bring the soil adjacent to the two vertical faces parallel to and on either side of the axis of rocking into a state of elastic non-uniform compression, if the rocking component alone is considered.

Therefore, for an embedded rectangular footing, of dimensions $2c \times 2d$, with embedment $L$, the lateral stiffness $k_{\varphi s}$ can be assumed as the difference between the stiffnesses offered by two rectangular surface footings of dimensions $2c \times (2d+2L)$ and $2c \times 2d$. The value of damping parameter $c_{\varphi s}$ can be evaluated in a similar manner. Thus both $c_{\varphi s}$ and $k_{\varphi s}$ can be evaluated using Hall's (1967) expressions given in Eqs. (38) to (42) taking to care to use oppropriate values of $r_o$ and other properties as given below :

$$k_{\varphi s} = [8G\,(r_2^3 - r_1^3\,\beta)]/[3\,(1-\mu)] \qquad \ldots(55)$$

Alternatively,

$$k_{\varphi s} = G\,(2c)\,(2d)^2\,[\beta_{\varphi e}\,(1+L/d)^2 - \beta_{\varphi s}\,]/[1-\mu] \qquad \ldots(56)$$

where $\beta_{\varphi e}$ is the value $\beta_\varphi$ (for ratio $[L+d]/c$. $\beta_{\varphi s}$ is the value of $\beta_\varphi$ corresponding to ratio $[d/c]$.

$$c_{\varphi s} = [0.8\,(G\rho)^{1/2}/(1-\mu)][r_2^4\,/(1+B_{\varphi 2}\,) - r_1^4\,/\,(1+B_{\varphi 1}\,] \ldots(57]$$

$$B_{\varphi 1} = [3\,(1-\mu)\,I_b]/[8\,\rho\,r_1^5\,] \qquad \ldots(58)$$

$$B_{\varphi 2} = [3\,(1-\mu)\,I_b\,]/[8\,\rho\,r_2^5\,] \qquad \ldots(59)$$

$$r_1 = [16\,c\,d^3\,/3\pi]^{1/4} \qquad \ldots(60)$$

$$r_2 = [(2c)\,(2d+2L)^3/3\,\pi]^{1/4} \qquad \ldots(61)$$

For the case of frequency dependant excitation the above analysis can be applied by replacing $Q_o$ by the frequency dependant forcing function $m_o\,e_o\,\omega^2$, where $m_o$ is the mass at an eccentricity, $e_o$, rotating with an angular velocity $\omega$.

**Presentation of Results**

The closed form solution as developed above is then obtained using electronic digital computer (IBM 370) for various values of dimensionless frequency

$$a_o = \omega\,r_o\,(\rho/G)^{1/2}. \qquad \ldots(62)$$

Using the notation $\epsilon = F/[m_o\,e_o\,\omega_c^2\,]$ ...(63)

called the friction factor and $\omega_c = [k_{xb}/m]^{1/2}$ ...(64)

typical response curves have been plotted for a square footing with sides sides $2_c \times 2_c$ (equivalent radius $r_o$) and embedment $L$ in Fig. 2 in terms of dimensionless horizontal amplitude $(mx_o/m_o\,e_o)$, dimensionless rocking amplitude $(I\varphi_o/m_o e_o z_e)$ for quadratic excitation with various friction factors

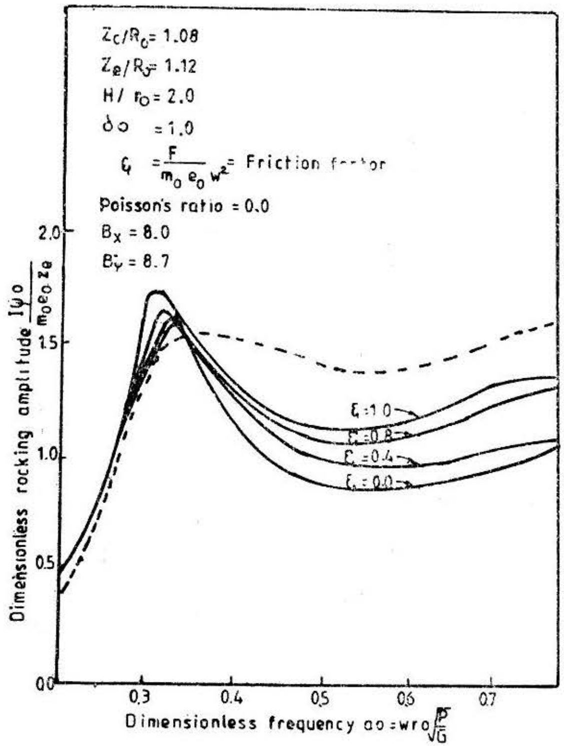


FIGURE 2*a*. Theoretical Response Curves for Rotation, Embedment Factor 1=1.0

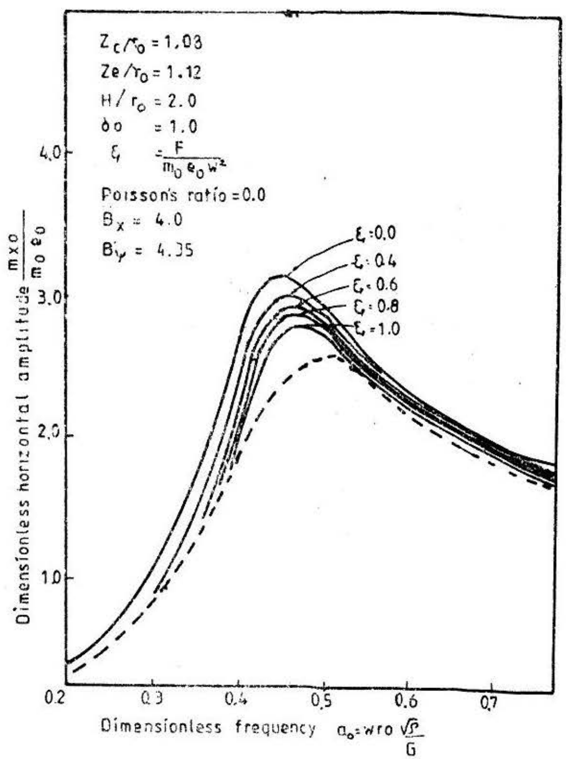


FIGURE 2*b*. Theoretical Response Curves for Translation, Embedment Factor=1.0

(Embedment factor $\delta_o = L/r_o = 1$). The dotted line in the figure represents the boundary from where continuous motion occurs and the present analysis is valid. The introduction of friction damper reduces resonant amplitudes and shifts resonant frequency considerably.

**Evaluation of Constant Frictional Force**

The frictional force mobilised depends mainly on the physical characteristics of the interface between the soil and the foundation walls, depth of embedment, lateral earth pressure acting normal to the interfaces etc. As the base and two sides parallel to the plane of rocking are effective in offering friction, for a $c$—$\phi$ soil,

$$F = \mu_f W + 2\, k_o\, \gamma_s\, L^2\, (a\mu_f) + 4dc_a\, [c + L] \qquad \ldots(65)$$

where $k_o = \mu/(1-\mu)$, the coefficient of lateral earth pressure at rest, $\gamma_s$ bulk density of soil, $\mu_f$—coefficient of kinematic wall friction $C_a$—wall adhesion between soil and surface of foundation under dynamic conditions, $W$—weight of the vibrating body.

Since the coefficient of kinematic friction is always less than coefficient of static friction, for a footing under vertical vibrations, Anandakrishan and Krishnaswamy (1973) and Krishnaswamy (1972) have recommended a value of $\mu_f$ as

$$\mu_f = \tan\, (\phi_1/3) \qquad \ldots(66)$$

where $\phi_1$—the angle of internal friction of soil and $C_a$ to be of the order of 1—2 percent of the actual cohesive strength of soil. For torsional vibrations, Bhaskaran Nair (1980) has recommended $\mu_f$ as

$$\mu_f = \tan,\, \phi_1/6) \qquad \text{(Precast footing)}$$

$$\mu_f = \tan\, (\phi_1/6) \qquad \text{(Cast-in-situ footing)} \quad \ldots(67)$$

From the present study of coupled rocking and sliding vibrations of footings the value of $\mu_f$ has been recommended as

$$\mu_f = \tan\, (\phi_1/9) \qquad \text{(Precast footing)}$$

$$\mu_f = \tan\, (\phi_1/6) \qquad \text{(Cast-in-situ footing)} \quad \ldots(68)$$

**Field Tests**

Field vibratory tests on several precast and cast-in-situ reinforced cement concrete footings were conducted to study the applicability of the theory for the prediction of coupled rocking and sliding response. The site by the northern side of the Soil Dynamics laboratory of the Indian Institute of Technology, Madras was selected for the investigations. The average unit weight and water content of soil were found to be 19.3 KN/$m^3$ and 11 per cent respectively. The angle of internal friction and cohesion of soil were determined as 31° and 23.54 KN/$m^2$ respectively. Figure show the soil profile at test site and variation of shear modulus with depth. Foundation bolts were provided in the footings to enable the vibrator and other attachments to be installed on the footings during testing. Wooden plugs were used for fixing the vibrator of dimensions 310×250×175 mm and mass 43.1kg was used. Six eccentric masses $m_o$ (6 kg in all) at an

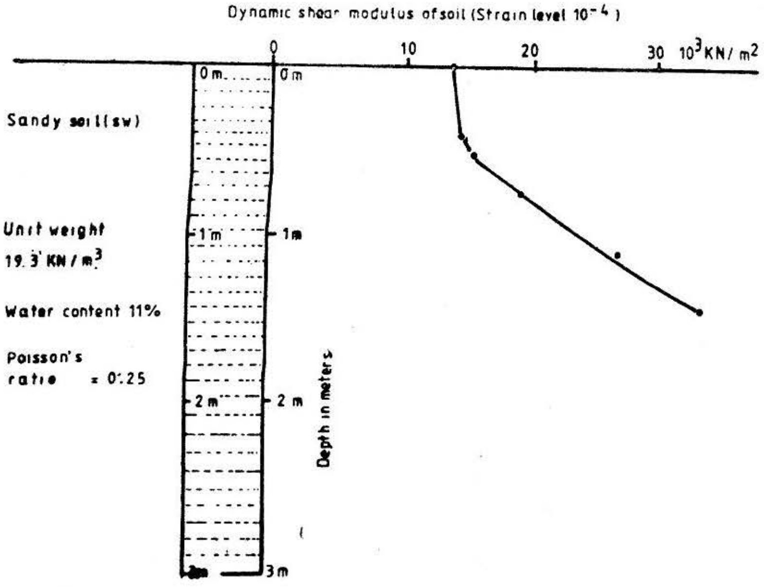


FIGURE 3 Soil Profile With Variation of Shear Modulus (After Bhaskaran Nair, 1980)

eccentricity of $e_o = 38.3$ mm from the center of the revolving shafts produced coupled rocking and sliding excitation. The vibrator was provided with an arrangement to change the relative position of the rotating eccentric masses over a wide range. A 5 HP speed controlled motor was used to run the vibrator through a flexible shaft. Electrodynamic vibration pickups were used in conjunction with the amplitude measuring apparatus measure peak vibration amplitudes. A Digital speed indicating teacheometer was used to read the spot speed of the revolving shafts. A mild steel base plate $600 \times 600 \times 16$ mm and mass 40 kg was used above the footings for all tests. The vibrator was mounted firmly as shown in Fig. 4 to produce coupled rocking and siliding vibrations. The overall weight of the set up was increased by adding over the vibrator circular cast iron plates. For various speeds of rotation the amplitudes of motion were measured. The properties of the footings used are indicated in Tables 1 and 2.

The experimental data, thus obtained, have been analysed with the help of the proposed theoretical model as well as by the theoretical model of Beredugo and Novak (1972). These results are being published elsewhere as the Tables of test data are too lengthy to be included in the present paper. The agreement between the results predicted by the proposed theoretical model and the experimental data is found to be satisfactory.

## Summary and Conclusions

A two-degree-of-freedom lumped parameter analogue and its closed form solution has been developed to investigate the study state coupled rocking and sliding response of rigid embedded block foundations. Field vibratory test in coupled rocking and sliding mode were conducted on several pre-cast-in-situ reinforced cement concrete block foundations to

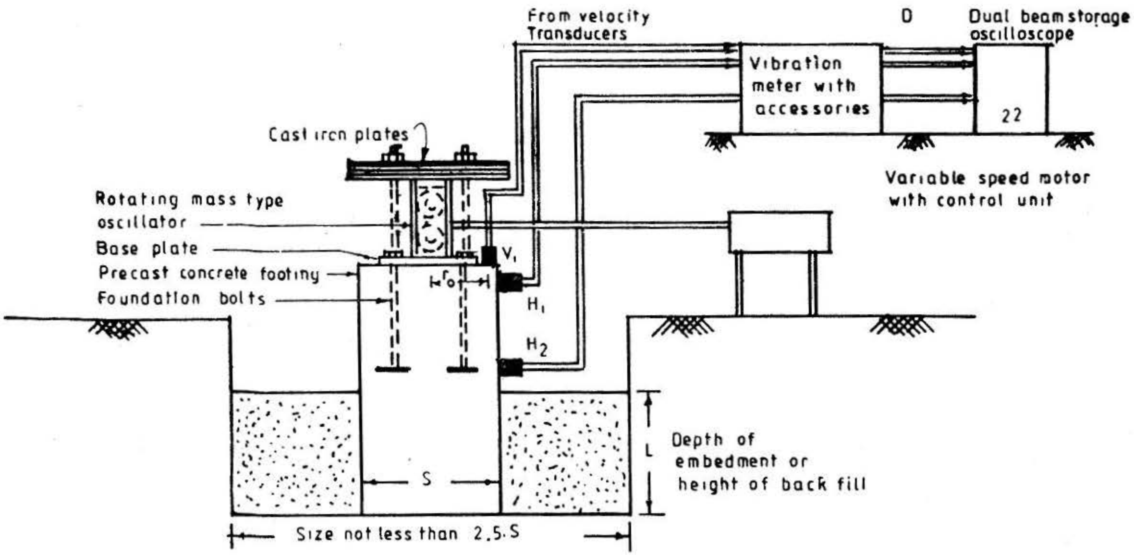


**FIGURE 4 Experimental Set-Up**

**TABLE 1**

**Dimensions of Test Footings**

| Base No. | Type of Base | Dimensions (m) | | |
|---|---|---|---|---|
| | | Length | Breadth | Height |
| 1 | Precast Rigid Block | 0.900 | 0.900 | 0.450 |
| 2 | Precast Rigid Block | 0.700 | 0.700 | 0.500 |
| 3 | Precast Rigid Block | 0.600 | 0.600 | 0.450 |
| 4 | Precast Rigid Block | 0.500 | 0.500 | 0.500 |
| 5 | Precast Rigid Block | 0.500 | 0.700 | 1.200 |
| 6 | Precast Rigid Block | 0.677 | Diameter | 1.200 |
| 7 | Precast Rigid Block | 0.700 | 0.500 | 1.200 |
| 8 | Cast-in-situ Block | 0.500 | 0.500 | 0.450 |
| 9 | Cast-in-situ Block | 0.500 | 0.500 | 0.500 |
| 10 | Cast-in-situ Block | 0.400 | 0.400 | 0.500 |
| 11 | Cast-in-situ Block | 0.350 | 0.350 | 0.500 |
| 12 | Cast-in-situ Block | 0.275 | 0.275 | 0.500 |

**TABLE 2**

**Properties of Test Footings**

| Base No. | Test No. | Equivalent Radius, $r_o$ m | Additional Mass Kg | $\frac{z_e}{r_o}$ | $\frac{z_c}{r_o}$ | Mass Moment of Inertia about Base $I_b$ Kg/m² | $B_x$ | $B'_\varphi$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.2 | 0.5106 | 210 | 0.54 | 0.69 | 286 | 0.93 | 1.19 |
| | 2.3 | 0.5106 | 315 | 0.46 | 0.77 | 363 | 1.01 | 1.50 |
| | 1.4 | 0.5106 | 420 | 0.39 | 0.83 | 443 | 1.08 | 1.84 |
| | 1.5 | 0.5106 | 525 | 0.33 | 0.90 | 528 | 1.18 | 2.18 |
| 2 | 2.4 | 0.3972 | 420 | 0.40 | 1.31 | 440 | 1.82 | 6.35 |
| 3 | 3.4 | 0.3404 | 420 | 0.27 | 1.58 | 365 | 2.37 | 11.45 |
| 4 | 4.2 | 0.2850 | 210 | 0.58 | 1.79 | 222 | 2.70 | 16.80 |
| 5 | 5.2 | 0.3672 | 210 | 1.59 | 2.16 | 1239 | 2.83 | 25.52 |
| 6 | 6.2 | 0.3385 | 210 | 1.72 | 2.35 | 1179 | 3.62 | 37.80 |
| | 6.4 | 0.3385 | 420 | 1.49 | 2.68 | 1726 | 4.19 | 55.50 |
| 7 | 7.2 | 0.3105 | 210 | 1.88 | 2.56 | 1212 | 4.69 | 60.00 |
| 8 | 8.2 | 0.2850 | 210 | 0.40 | 1.72 | 194 | 2.56 | 14.70 |
| | 8.4 | 0.2850 | 420 | 0.15 | 2.06 | 351 | 3.52 | 26.60 |
| 9 | 9.2 | 0.2850 | 210 | 0.58 | 1.79 | 222 | 2.70 | 16.80 |
| | 9.4 | 0.2850 | 420 | 0.23 | 2.14 | 400 | 3.68 | 30.30 |
| 10 | 10.2 | 0.2280 | 210 | 0.47 | 2.46 | 217 | 4.62 | 50.20 |
| | 10.4 | 0.2280 | 420 | 0.05 | 2.88 | 396 | 6.48 | 91.00 |
| 11 | 11.2 | 0.2000 | 210 | 0.44 | 2.89 | 208 | 6.40 | 93.00 |
| | 11.4 | 0.200 | 420 | 0.05 | 2.98 | 395 | 9.17 | 176.00 |
| 12 | 12.2 | 0.156 | 210 | 0.34 | 3.94 | 206 | 12.37 | 319.00 |

study the applicability of the proposed theory. The agreement between the results predicted by the proposed theoretical model and the experimental data is satisfactory. Thus the proposed theoretical model provides an interesting alternative to the existing theoretical model of Beredugo and Novak (1972).

## References


ANANDAKRISHNAN, M. AND KRISHNASWAMY, N.R. (1973): "Response of Embedded Footings to Vertical Vibrations: *J. Soil Mech. Found. Div., Proc.* ASCE, 99 : SM 10 : 863-883.

BARKAN, D.D. (1962) : "*Dynamics of Bases and Foundations*", Mc-Graw Hill Book Co. Inc., New York, 434 pp.

BEREDUGO, Y.O. and NOVAK, M. (1972): "Couplied Horizontal and Rocking Vibrations of Embedded Footings", *Can Geotech. J.*, 9 : 4 : 477-497.

BHASKARAN NAIR, P.G. (1980) : "*Analytical and Experimental Studies on the Torsional Vibrations of Embedded Footings*", Ph. D. Dissertation, Indian Institute of Technology, Madras.

HALL, J.R. Jr. (1967) : "Coupled Rocking and Sliding Oscillations of Rigid Circular Footings", *Symp. Wave Prop. Dyn. Prop. Earth. Mat.*, Albuqerque, New Mexico, pp. 139-148.

KRISHNASWAMY, N.R. (1972) : "*A Study of the Effect of Embedment and and Related Factors on the Dynamic Behaviour of Footings*", Ph. D. Dissertation I.I.T., Kanpur, 293 pp.

LYSMER, J. (1965) : "*Vertical Motion of Rigid Footings*", Dept. of Civil Eng., Univ. Eng., Univ. of Michigan Report to WES Contract Report No. 3—115, Under Contract No. DA-22-079-Eng-340 : also a Ph. D. Dissertation, University of Michigan.

RAVI, N. (1985), : "*Dynamic Studies on The Vibrations of Embedded and Pile Supported Footings*", Ph. D. Dissertation, submitted to the Indian Institute of Technology, Madras, 378 pp.

PRADEEP KUMAR, A.V. (1982) : "*Coupled Rocking and Sliding Vibrations of Embedded Footings*", M. Tech. Thesis, IIT, Madras.

RAGHAVENDRA RAO, M.V. (1981) : "*Coupled Rocking and Sliding Vibrations of Footings*", unpublished M. Tech. Thesis, IIT, Madras.

RICHART, F.E., Jr., HALL, J.R. Jr., and WOODS, R.D. (1970) : *Vibrations of Soils and Foundations*", Prentice-Hall Inc., Englewood Cliffs, New Jersey, 414 pp.

YEH, G.K. (1966) ; "Forced Vibration of a Two-Degree-of Freedom-System with Combined Coulomb and Viscous Damping', *J. Acoust. Soc. Am.*, 39 ; 14-24.