Dynamic Response of a Beam on a Foundation of Finite Depth

by

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Introduction

The theory of a beam on an elastic foundation has found extensive application (Hetenyi, 1946) in many branches of civil engineering. An area wherein this is frequently used is the study of rigid pavements (Siva Reddy and Pranesh, 1971). Pavements are subjected to a variety of dynamic forces during their life time. Among these, the forces that come during an earthquake are probably the most severe. The structural model one would like to analyse in this connection is the response of a beam with both ends free (free-free beam) resting on a finite elastic medium, which is subjected to a vertical dynamic displacement at its lower edge. However, this problem even in the two-dimensional case is quite complicated. The major difficulty is in solving the wave equation for the foundation and at the same time satisfying the beam-foundation interface condition. Invariably, to avoid this difficulty, the foundation is assumed to be of the classical Winkler type. This allows one to use all the known results of the classical beam on an elastic foundation. The drawback of this approach is that it neglects the inertia of the foundation which may be more predominant than the spring effect, under dynamic conditions.

In the past there have been attempts to improve the foundation model under dynamic conditions. Notable among these are the works of Rades (1970). He has considered the vibration of a flexible beam resting on a Pasternak foundation under a harmonic force. An attempt has been made to take into consideration the effect of the foundation inertia also. This is done by introducing into the beam equation, second derivatives of the beam displacement with respect to space and time variables. Later Rades (1972) has extended this work to the response of a rigid beam resting on a three parameter foundation proposed by Kerr (1965).

The aim of the present paper is to suggest a simple approach for the free vibration and seismic response of a pavement modelled as a free-free beam, including the structure—soil interaction. Since the pavement is very long in one direction, the usual plane strain idealization of analysing a unit wide strip in the longitudial direction, automatically leads to the beam model for the pavement. To avoid excessive mathematical difficulties the following assumptions are made regarding the foundation :

(*i*) The foundation is of finite depth and is of width equal to the span of the beam.

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(*ii*) The foundation responds only in the vertical direction.

(*iii*) The foundation is made up of a series of columns, which do not interact with each other, but are connected at the top by the beam.

Fig. 1 shows a representative sketch of the beam-foundation system. The above foundation is a generalization of the Winkler model to suit dynamic conditions.

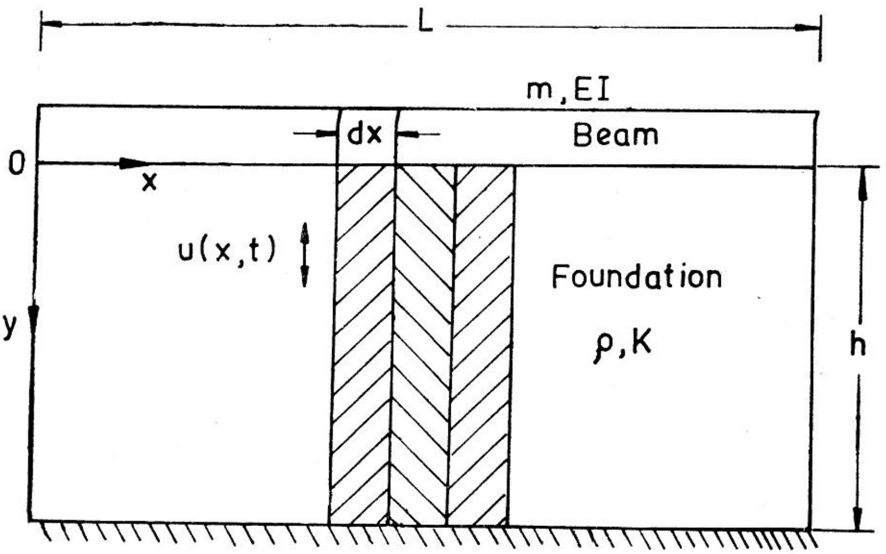

FIGURE 1 Beam-Foundation System

Equations of Motion

The equation of motion of the beam is

$$EI\frac{\partial^4 w}{\partial x^4} = -m\ddot{w} + q(x, t) \qquad \ldots(1)$$

Here

EI = Flexural rigidity of beam

w = deflection

m = mass per unit length

$\ddot{w}$ = acceleration

During free vibration, the force $q(x, t)$ will be due to the foundation reaction only.

The free-free boundary conditions for the beam are :

$$w''(0, t) = w'''(0, t) = 0$$
$$w''(L, t) = w'''(L, t) = 0 \qquad \ldots(2)$$

Here the primes denote differentiation with respect to x.

The vertical response $u(y, t, x)$ of the foundation at a point 'x' is governed by an equation similar to that of the longitudinal vibration of a bar. It follows that ;

$$k \frac{\partial^2 u}{\partial y^2} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(3)$$

Here k is the spring constant and ρ is the mass/unit area of the foundation.

Since it is assumed that the foundation is connected only at the top by the beam, x in the above equation is only a parameter and not a variable of differentiation. If the foundation is resting on a rigid layer, then the condition at the lower edge is

$$u(h, t)]_x = 0 \qquad \ldots(4)$$

The interface condition at $y = 0$ is :

$$u(0, t)]_x = w(x, t) \qquad \ldots(5)$$

The loading $q(x, t)$ on the beam is given by :

$$q(x, t) = k \frac{\partial u}{\partial y}]_{y=0} \qquad \ldots(6)$$

It may be noted here that a similar foundation model has been used by Saito and Murakami (1969), in studying the propagation of waves in an infinite Timoshenko beam resting on an elastic medium.

Free Vibration

The harmonic solution of the differential Eqns (1) and (3) can be taken in the usual separable form ;

$$w(x, t) = W(x) \sin \omega t \qquad \ldots(7)$$

$$u(y, t)]_x = U(y)]_x \sin \omega t \qquad \ldots(8)$$

This leads to the following Eqations :

$$\frac{d^4W}{dx^4} - \mu^4 W = 0 \qquad \ldots(9)$$

and

$$\frac{d^2U}{dy^2} + \lambda^2 U = 0 \qquad \ldots(10)$$

where

$$\mu^4 = \left(\frac{m\omega^2}{EI} - \frac{k\lambda}{EI} \cot \lambda h \right) \qquad \ldots(11)$$

$$\lambda^2 = \rho\omega^2/k \qquad \ldots(12)$$

ω = natural frequency in radians/second

W = shape function

The solutions of these equations are well known (Timoshenko, 1964). Eq. (9)

along with the conditions given by Eq. (2) leads to the following transcendental equation

$$\cosh \mu L \cos \mu L = 1 \qquad \ldots(13)$$

This has to be solved along with Eq. (11) to get the natural frequencies. Thus for every solution $\mu_i L$ of Eq. (13) the equation

$$\beta\,[\alpha\,(\lambda h)^2 - \lambda h \cot \lambda h] = \mu_i L^4 \qquad \ldots(14)$$

leads to infinite number of roots $\lambda_{ij} h$, α and β are non-dimensional parameters given by

$$\alpha = m/\rho\, h\ ;\ \beta = kL^4/EIh \qquad \ldots(15)$$

The first four roots of Eq. (13) are $\mu_1 L = 0$; $\mu_2 L = 4.730041$, $\mu_3 L = 7.853204$; $\mu_4 L = 10.99561$.

The fundamental natural frequency of the beam-foundation system is associated with the root $\mu_1 L = 0$. For this case Eq. (14) reduces to

$$\lambda h \tan \lambda h = 1/\alpha \qquad \ldots(16)$$

To study the effect of including the foundation inertia it is more informative to take the frequency parameter

$$\phi_{ij} = \frac{m\omega^2_{ij} L^4}{EI} = \alpha\beta\,(\lambda_{ij} h)^2 \qquad \ldots(17)$$

Several limiting cases are possible. The most interesting of these is limit as ρ tends to zero. Taking the limit as $\rho \rightarrow 0$ in Eq. (14), we get

$$\phi_{ij} = (\mu_i L)^4 + \beta \qquad \ldots(18)$$

This is precisely the frequency parameter for a beam on a massless Winkler type elastic foundation (Timoshenko, 1964). The effective foundation modulus of such a beam would be

$$\bar{k} = k/h \qquad \ldots(19)$$

Natural Frequencies and Mode Shapes

Extensive numerical results have been obtained for the frequency parameter λh, by solving the transcendental Eq. (14) for various values of α and β.

Some of these are presented in Fig. 2 for two values of α. For each value of α the first four natural frequencies ($j = 1$; $i = 1, 2, 3, 4$) in the increasing order are shown in this figure. In the same figure the frequency variation of a beam without any foundation and of a beam with a massless foundation are also shown.

Eq. (13) is same as the frequency equation of an ordinary free-free beam (Timoshenko, 1964). Corresponding to the various values of $\mu_i L$ the mode shapes of the beam are obtained as the solutions of Eq. (9). Thus, for the first root $\mu_1 L = 0$, the beam executes rigid body modes

$$W_1 = 1\ ;\ W_2 = (1 - 2x/L) \qquad \ldots(20)$$

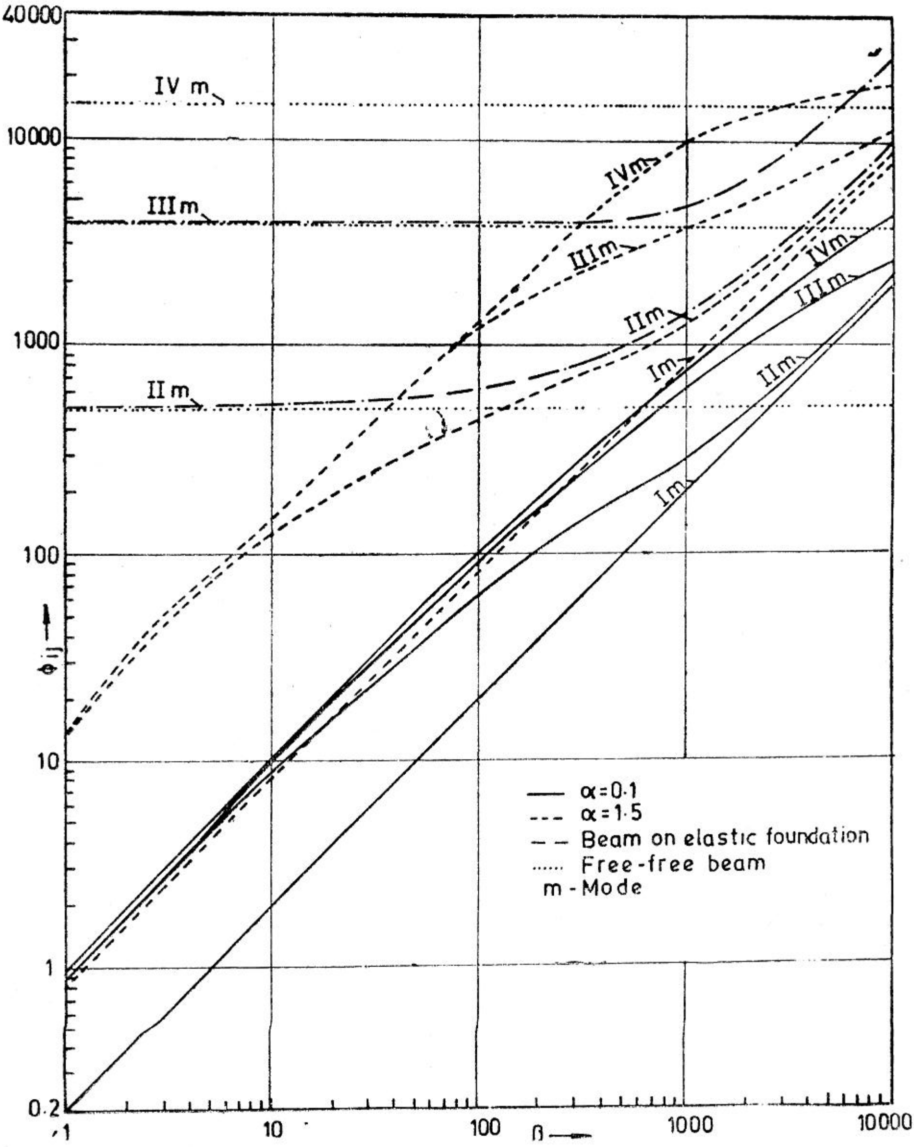

FIGURE 2 Variation of Natural Frequency

The higher modes are given by

$$W_i(x) = (\cos \mu_i x + \cosh \mu_i x) - \frac{\cosh \mu_i L - \cos \mu_i L}{\sinh \mu_i L - \sin \mu_i L} \cdot (\sin \mu_i x + \sinh \mu_i x) \quad ...(21)$$

Corresponding to every i, the foundation exhibits infinite number of natural frequencies and mode shapes. The foundation modes are given by

$$U_{ij}(y;x) = W_i(x)\,[\cos \lambda_{ij} y - \cot \lambda_{ij} h \sin \lambda_{ij} y]$$
$$= W_i(x)\, R_{ij}(y) \qquad \ldots(22)$$

The mode shapes corresponding to the lowest root of Eq. (14) for every i, may be denoted as the beam modes. The others exhibiting nodes in the foundation may be called the foundation modes. The roots of Eq. (14) approach approximately $j\pi$ for the case $\mu = O$. The effect of this on the frequency parameter ϕ_{ij} of Eq. (17) would be that it can be arranged in an increasing order as $\phi_{11}, \phi_{21} \ldots \phi_{i1}$, all corresponding to the beam modes. Even though this would mean that the foundation modes are not very important, the effect of the inertia of the foundation on ϕ_{ij} is brought out very clearly from Fig. 2. Also this figure shows that the frequencies of the ordinary beam are much above the beam-foundation system considered here. The addition of the classical Winkler type foundation increases the stiffness properties of such a beam and hence the frequencies further increase. However, the inclusion of the inertia of the foundation dramatically brings down the natural frequencies. The non-dimensional parameter α is representative of the inertial properties. The variation of the most important natural frequency, namely, the fundamental frequency with respect to $1/\alpha$ is shown in Fig. 3. The value of ϕ_{11}/β for the classical beam on Winkler's elastic foundation is unity. The effect of considering the depth and hence the inertia of the foundation is to substantially decrease this value.

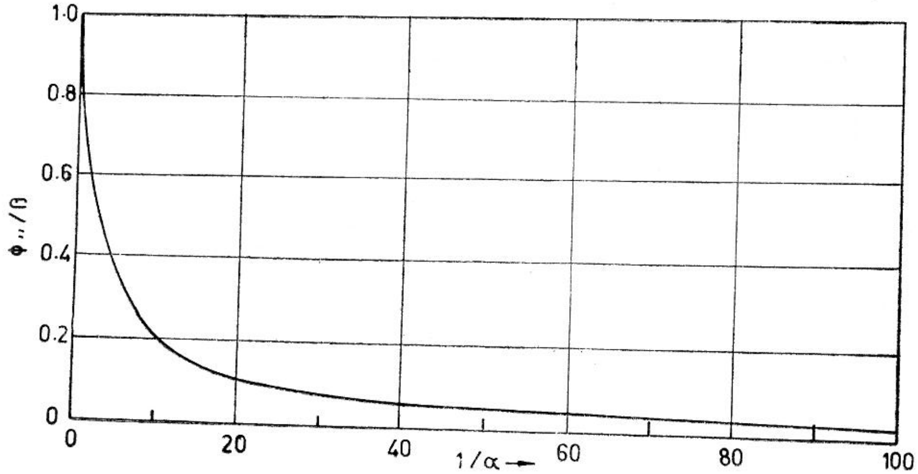

FIGURE 3 Influence of Depth on the Natural Frequency

Response Under Ground Motion

During an earthquake, the beam—foundation system is subjected to three components of displacements at the ground level $y = H$. The most important component will be the vertical component inducing bending stresses in the beam. The equations of motion are

$$EI \frac{\partial^4 w}{\partial x^4} + m\ddot{w} + c_1 \dot{w} = K \frac{\partial u}{\partial y}\Big]_{x=O} \qquad \ldots(23)$$

$$k \frac{\partial^2 u}{\partial y^2} - \rho \ddot{u} - c_2 (\dot{u} - \dot{v}_g) = 0 \qquad ...(24)$$

The boundary conditions for the beam remain same as Eq. (2). For the foundation, the time dependent boundary condition :

$$u(h, t)]_x = v_g(t) \qquad ...(25)$$

has to be satisfied. In the above Eqns. $v_g(t)$ is the ground motion. Now, introducing the relative displacement :

$$z = u - v_g \qquad ...(26)$$

the foundation equation becomes:

$$\rho \ddot{z} - k \frac{\partial^2 z}{\partial y^2} + c_2 \dot{z} = \rho \ddot{v}_g \qquad ...(27)$$

The solution for z is taken in terms of the eigen functions in the form :

$$z(y, t, x) = \sum_i \sum_j a_{ij}(t)\, W_i(x)\, R_{ij}(y) + (y/h - 1)\, v_g(t) \qquad ...(28)$$

This satisfies the time dependent boundary condition of Eq. (25). Also the eigen functions satisfy the special orthogonality conditions :

$$\int_0^L W_i(x)\, W_j(x)\, dx = 0 \; (i \neq j) \qquad ...(29)$$

$$\int_0^h R_{ik}(y)\, R_{in}(y)\, dy + (m/\rho) = 0 \; (k \neq n) \qquad ...(30)$$

Now, if $v_g(t)$ is independent of x, substitution of Eq. (28) in Eq. (24) leads to

$$\sum_k [\ddot{a}_{1k} + (c_2/\rho)\, \dot{a}_{1k} + \omega_{1k}^2\, a_{1k}]\, R_{1k} = -(y/\rho h)\, v_g - (c_2/\rho)\,(y/h - 1)\, \dot{v}_g \quad ...(3\ 1)$$

It may be observed here that only modes corresponding to $i = 1$ get excited, since the ground displacement is uniformly distributed in x. From the interface condition

$$w(x, t) = \sum \sum a_{in}(t)\, W_i(x) \qquad ...(32)$$

Thus from the beam Eq, (23)

$$\sum_n [\ddot{a}_{1n} + (c_1/m)\, \dot{a}_{1n} + \omega_{1n}^2\, a_{1n}] = (k/mh)\, v_g(t) \qquad ...(33)$$

These equations can be uncoupled further if the damping constants are taken as $c_1/m = c_2/\rho$. After multiplying Eq. (31) by R_{1n}, integrating in (o, h) and adding to Eq. (33), the orthogonality condition of Eq. (30) gives

$$\ddot{a}_{1n} + 2\eta\omega_{1n}\, \dot{a}_{1n} + \omega_{1n}^2\, a_{1n} = P_{1n}\, V_g - P_{2n} \dot{v}_g - P_{3n}\, \ddot{v}_c \qquad ...(34)$$

Here η is the usual damping coefficient and

$$P_{1n} = (k/\rho h^2)/P_4$$

$$P_{2n} = 2\,\eta\omega_{1n}\,(a-1/\lambda_{1n}^2\;h^2)/P_4$$

$$P_{3n} = [\lambda_{1n}\,h\,(1+a^2\,\lambda_{1n}^2\;h^2)^{1/2}-1]/P_4$$

$$P_{4n} = a+0.5\,(1-a+a^2\,\lambda_m^2\,h^2) \qquad \ldots(35)$$

Numerical Example

The above equation for a_{1n} is in the standard form but needs not only the ground acceleration but also the velocity and displacement. However the solution can be carried out easily on a computer. As an example a concrete pavement with the following parameters is considered :

D : Depth = 30.5 cm

h : Depth of foundation = 9.15 m, 12.15 m

L : Length = 3 m

I : Moment of Inertia = 0.000721 m^4

G : Shear modulus of soil = 175 kg cm^{-2}

E : Young's Modulus = 2.1×10^5 kg cm^{-2}

γ : Poisson's Ratio = 0.25

γ_c : Unit weight of concrete = 2350 kg m^{-3}

γ_s : Unit weight of soil = 1920 kg m^{-3}

η : Damping coefficient = 0.2

The excitation to the system is taken as the vertical component of the Taft 1952 earthquake shown in Fig. 4. The integration has been carried out for a duration of 11 s. It is noted that the maximum response is attained towards the end of the interval. The reason for this is due to the fact that the input reaches its maximum only very near 11 s. To economize computer time further integration has not been carried out. The displacement response of the pavement for two different depths of foundation are shown in Fig. 5. The contact pressure developed at the interface has also been computed. The maximum value of the contact

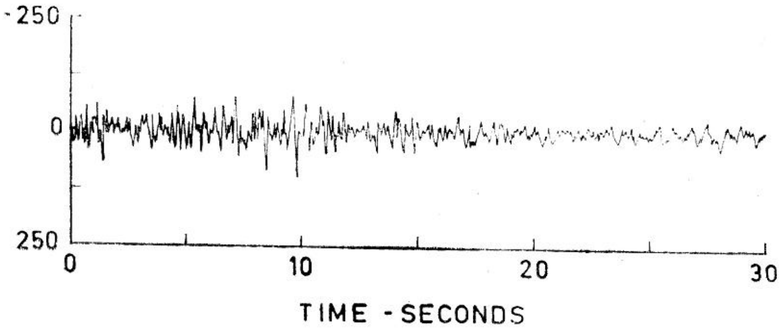

FIGURE 4 Taft July 21, 1952 Accelerogram, Vertical component

pressure for $h = 9.15$ m is 19.6 kg cm^{-2}. As the depth is increased to $h = 12.15$ m the maximum contact pressure during the earthquake reduces to 14.75 kg cm^{-2}.

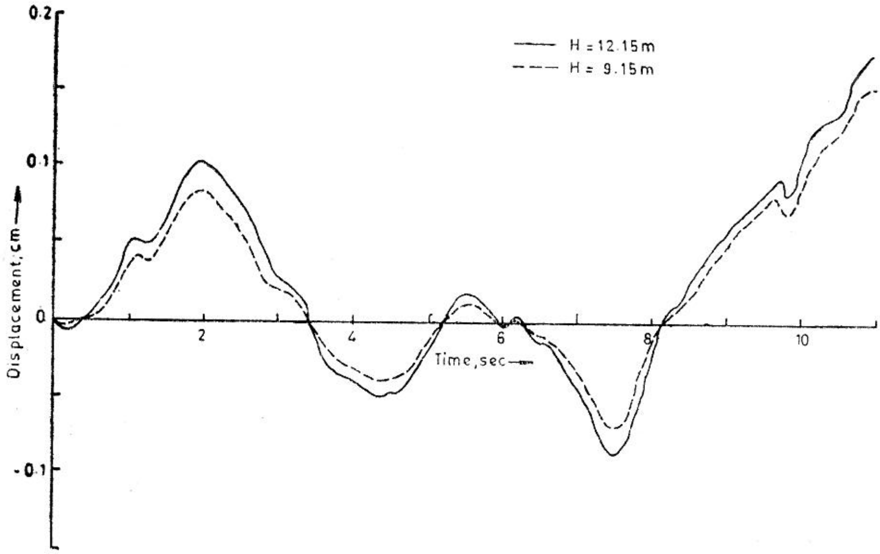

FIGURE 5 Response of Pavement to Vertical Component of Taft (1952) Earthquake

Discussion and Conclusion

The beam—foundation model developed in the present study is a generalization of the Winkler's model to include foundation inertia. The basic question with many of the dynamic soil– structure interaction models is how well the soil behaviour is simulated. The assumption that foundation is an elastic material is perhaps the most severe limitation, with the suggested model. However, this can be improved by considering the foundation to be a series of inelastic shear beams. The inelastic behaviour may be easily incorporated by adding an auxiliary equation as shown by Iyengar (1979) in analyzing vibration of inelastic beams. The other limitation is the assumed one dimensional nature of the foundation elements, The present foundation model is not strictly unidimensional since the foundation mode shapes are functions of both x and y. However, it is not a strictly correct two dimensional model. Finite element approaches can be used to model the foundation in a better manner but the software requirements of such an approach is time consuming and expensive. But, to evaluate the simple model proposed in the present paper. It is recognized that comparison with other approaches is needed. Such comparisons and further extension to include inelastic foundation behaviour are currently under progress.

In conclusion it may be mentioned that for vertical vibration of pavement—foundation systems the generalization of the classical Winkler's model along the lines of this study is natural and simple. This is particularly attractive in analyzing seismic response. The method can be used in studying vertical response under other types of forces like moving loads also.

Acknowledgement

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References

HETENYI, M. (1946) : *"Beams on Elastic Foundation"*, University of Michigan Press, Ann Arbor, USA.

KERR, A.D. (1965) : "A study of a new foundation model", *Acta Mechanics*, 1 : 135-147.

RADES, M. (1970) : "Steady-state Response of a Finite Beam on Pasternak Foundation", *Int. Jnl. of Solids and Structures*, 6 : 739-756.

RADES, M. (1972) : *"Dynamical Analysis of an Inertial Foundation Model"*, *Int. Jnl. of Solids and Structures*, 8 : 1353-1372.

SAITO, H. and MURAKAMI, T. (1969) : "Vibrations of an Infinite Beam on an Elastis Foundation with Consideration of Mass of a Foundation", *Bulletin of Japan Society of Mechanical Engineers*, 12 : 202-205.

SIVA REDDY, A. and PRANESH, M.R. (1971) : "Warping Stresses in Slabs on Reissner Foundation", *Transportation Engineering Journal, ASCE*, 97 : TE 3 : 511-526.

TIMOSHENKO, S. (1964) : *"Vibration Problems in Engineering"*, East-West Student Edition.

IYENGAR, R.N. (1979) : 'Inelastic Response of Beams under Sinusoidal and Random Loads', *Jnl. of Sound and Vibration*, 64 : 2 : 161-172.

Notation

a_{in} = Coefficient

C_1, C_2 = Constants

D = Depth (cm)

d = non-dimensional parameter

E = Young's modulus (kg/cm^2)

G = Shear modulus of soil (kg/cm^2)

h = Depth of foundation

I = Moment of inertia

i, j = mode indices

k = Spring constant (kg/cm)

$\overline{k}$ = Effective foundation modulus

L = Length of beam

m = mass (kg)

P_{in} = Coefficient

$q(x, t)$ = force due to foundation reaction

R_{ij} = foundation eigen function

t = time

u = foundation displacement

U_{ij} = Foundation modes

$v_g(t)$	=	ground motion
w	=	beam displacement
z	=	relative displacement
α	=	parameter
β	=	parameter
γ	=	Poisson's ratio
γ_c	=	Unit weight of concrete
γ_s	=	Unit weight of soil
η	=	damping coefficient
μ^4	=	Parameter
ρ	=	mass/unit area
ϕ_{ij}	=	frequency parameter
ω	=	frequency
(.)	=	derivative w. r. t. time
(')	=	derivative w. r. t. space