Short Communication

Stresses and Displacements in a Transversely Isotropic Medium-Having a Lined Cylindrical Hole

by

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Introduction

Stresses and displacements in a semi-infinite medium weakened by a cylindrical hole (shaft) are of importance in mining and geotechnical engineering. Blenkaran and Wilhoit (1962) were the first to present a solution for an isotropic semi-infinite medium having a cylindrical hole subjected to radial load on the boundary of the hole. Later Vasill'ev (1977) presented a solution for a semi-infinite isotropic medium having a cylindrical hole, with frictionless rigid lining and without lining subjected to axisymmetric normal load on the surface. However it has been shown (Lapkin, 1975, Pickering, 1970) that soil and rock can be considered as transversely isotropic instead of isotropic. Following this assumption several investigators (eg. Bardon, 1963, Gerrakd and Harrison, 1970, Milovic, 1972) have solved foundation problems of Boussinesq type. Recently the authors (Chandrashekhara and Gopalakrishnan, 1982) have presented a solution for the stresses and displacements in a transversely isotropic medium having a cylindrical hole with or without elastic lining. The solution of this problem is fairly complex and involves finally solution of a system of integral equations. However, a simple solution can be obtained for the case of a cylindrical hole, with a smooth rigid lining, in a semi-infinite medium. It is the object of this note to present such a solution and later on compare the vertical displacement at the surface obtained from this solution with those of no lining with friction (Chandrashekhara and Gopalakrishnan, 1982) and treating the medium as isotropic (Vasill'ev, 1977).

Basic Equations and Boundary Conditions

For an axisymmetric problem of a transversely isotropic body, the solution can be obtained in terms of a stress function φ (Lekhnitskii, 1963) satisfying the differential equation,

$$\nabla_1^2 \nabla_2^2 \varphi = 0 \qquad \dots (1)$$

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where

$$\nabla_i^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{S_i^2}\frac{\partial^2}{\partial z^2} \right), \ (i = 1, 2)$$

and

$$S_{1,2} = \sqrt{\frac{A+C \pm \sqrt{(A+C)^2 - 4D}}{2D}}$$

$$A = \frac{-\nu_{rz}(1+\nu_{rz})}{(\eta - \nu_{rz}^2)},$$

$$C = \frac{-\nu_{rz}(1+\nu_{r\theta}) + (E_r/G_{rz})}{(\eta - \nu_{rz}^2)}$$

$$D = \frac{1 - \nu_{r\theta}^2}{(\eta - \nu_{rz}^2)}$$

The stresses and displacements are given by

$$\sigma_r = -\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial r^2} + \frac{B}{r}\frac{\partial}{\partial r} + A\frac{\partial^2}{\partial z^2} \right)\varphi$$

$$\sigma\theta = -\frac{\partial}{\partial z}\left(B\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + A\frac{\partial^2}{\partial z^2} \right)\varphi$$

$$\sigma_z = \frac{\partial}{\partial z}\left(C\frac{\partial^2}{\partial r^2} + \frac{C}{r}\frac{\partial}{\partial r} + D\frac{\partial^2}{\partial z^2} \right)\varphi \qquad \ldots (2)$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + A\frac{\partial^2}{\partial z^2} \right)\varphi$$

and

$$u = A^* \frac{\partial^2 \varphi}{\partial r\, \partial z} \qquad \ldots (3)$$

$$w = B^* \left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} \right) + C^* \frac{\partial^2 \varphi}{\partial z^2}$$

where

$$B = \frac{\nu_{rz}\left(-\nu_{rz} + \frac{E_r}{G_{rz}} \right) + \nu_{r\theta}\eta}{(\eta - \nu_{rz}^2)}$$

$$A^* = \frac{1}{E_r}(1+\nu_{r\theta})(B-1)$$

$$B^* = \frac{1}{G_{rz}}$$

$$C^* = \frac{1}{E_r}(\eta D+2A\,\nu_{rz})$$

The following boundary conditions may be written for the problem (Fig. 1).

$$\text{at } z=o,\ \sigma_z = p(r) = \begin{cases} q, & -a \leqslant r \leqslant b \\ o, & r > b \end{cases} \qquad \ldots (4)$$

$$\tau_{rz} = o$$

$$\text{at } r=a,\ \tau_{rz}=o \qquad \ldots (5)$$

$$u = o$$

Solution

A stress function φ, satisfying Eq. 1, can be taken as

$$\varphi = \int_0^\infty \frac{1}{\beta^3}\left[C(\beta)\,e^{-S_1\beta z} + D(\beta)\,e^{-S_2\beta z}\right] V_o(\beta r)\,d\beta \qquad \ldots (6)$$

where

$C(\beta)$ and $D(\beta)$ are functions of β and

$V_0(\beta r) = J_0(\beta r)\ Y_1(\beta a) - Y_0(\beta r)\ J_1(\beta a)$

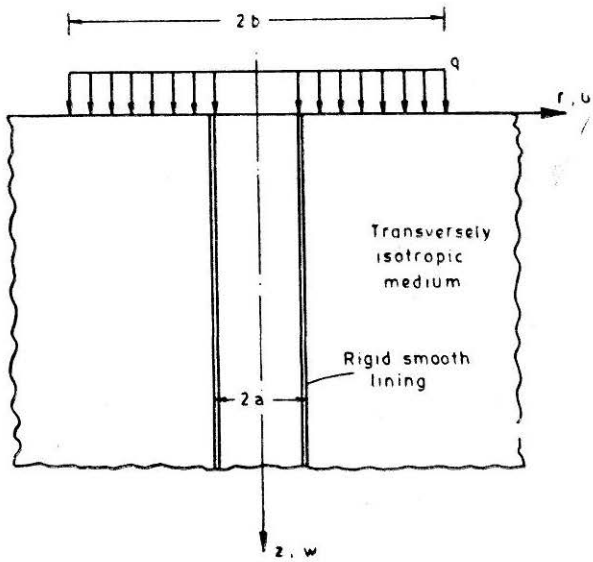

FIGURE 1 A Semi-Infinite Medium Having a Cylindrical Hole with Right Smooth Lining

It may be verified that Eq. 6 satisfies the boundary conditions given in Eq. 5.

The axisymmetric normal loading on the surface can be expressed as

$$p(r) = \int_0^\infty P(\beta)\, \beta\, V_0(\beta r)\, d\beta \qquad a < r < \infty \qquad \ldots(7)$$

where $P(\beta)$ is the Hankel transform of $p(r)$ given by

$$P(\beta) = \int_0^\infty \frac{p(r)\, r\, V_0(\beta r)\, dr}{J_1^2(\beta a) + Y_1^2(\beta a)} \qquad \ldots(8)$$

For the loading considered here $P(\beta)$ will be

$$P(\beta) = \frac{q\, b\, V_1(\beta b)}{\beta\, [J_1^2(\beta a) + Y_1^2(\beta a)]} \qquad \ldots(9)$$

Using the boundary conditions given in Eq. 4, together with Eqs. 7 and 9, the unknown functions $C(\beta)$ and $D(\beta)$ can be determined. These can then be substituted in Eq. 6 from which, the stresses and displacements can be determined from Eqs. 2 and 3 as

$$\sigma_r = -\frac{qb}{f(S)}\Bigg\{ (1-AS_1^2) \int_0^\infty (S_1 e^{-S_1\beta z} - S_2\, e^{-S_2\beta z}) \frac{V_1(\beta b)\, V_0(\beta r)}{F(\beta a)}\, d\beta$$

$$(+B-1) \int_0^\infty [S_1\, e^{-S_1\beta z} - \frac{(1-AS_1^2)}{(1-AS_2^2} S_2\, e^{-S_2\beta z}] \frac{V_1(\beta b)\, V_1(\beta r)}{F(\beta a)\, \beta r}\, d\beta \Bigg\}$$

$$\ldots(10)$$

$$\sigma_0 = -\frac{qd}{f(S)}\Bigg\{ \int_0^\infty [S_1 (B-AS_1^2)\, e^{-S_1\beta z} - \left(\frac{1-AS_1^2}{1-AS_2^2}\right) S_2 (B-AS_2^2\, e^{-S_2\beta z}].$$

$$\frac{V_1(\beta b)\, V_0(\beta r)}{F(\beta a)}\, d\beta - (B-1 \int_0^\infty [S_1\, e^{-S\beta z} - \left(\frac{1-AS_1^2}{1-AS_2^2}\right) S_2\, e^{-S\beta z}].$$

$$\frac{V_1(\beta b)\, V_1(\beta r)}{F(\beta a)\, \beta r}\, d\beta \Bigg\} \qquad \ldots(11)$$

$$\sigma_z = \frac{qb}{f(S)} \int_0^\infty [S_1 (C-DS_1^2\, e^{-S_1\beta z} - \left(\frac{1-AS_1^2}{1-AS_2^2}\right) (C-D\, S_2^2)\, S_2\, e^{-S\beta z}].$$

$$\frac{V_1(\beta b)\, V_0(\beta r)}{F(\beta a)}\, d\beta \qquad \ldots(12)$$

$$\tau_{rz} = \frac{qb}{f(S)} \; (1 - AS_1^2) \int_0^\infty [e^{-S_1\beta z} - e^{-S_2\beta z}] \frac{V_1(\beta b)\, V_1(\beta r)}{F(\beta a)} \, d\beta \qquad ...(13)$$

$$u = \frac{qd\, A^*}{f(S)} \int_0^\infty [S_1\, e^{-S_1\beta z} - \left(\frac{1 - AS_1^2}{1 - AS_2^2} \right) S_2\, e^{-S_2\beta z}] \frac{V_1(\beta b)\, V_1(\beta r)}{F(\beta a)\, \beta} \, d\beta \qquad ...(14)$$

$$w = \frac{qb}{f(S)} \int_0^\infty [\, (B^* - S_1^2\, C^*)\, e^{-S_1\beta z} - \left(\frac{1 - AS_1^2}{1 - AS_2^2} \right) (B^* - S_2^2\, C^*) e^{-S_2\beta z}] . \frac{V_1(\beta b)\, V_0(\beta r)}{F(\beta a)\, \beta} \, d\beta \qquad ...(15)$$

where

$$f(S) = (C - DS_1^2)\, S_1 - \left(\frac{1 - AS_1^2}{1 - AS_2^2} \right) (C - DS_2^2)\, S_2$$

$$F(\beta a) = J_1^2(\beta a) + Y_1^2(\beta a)$$

$$V_1(\beta b) = J_1(\beta b)\, Y_1(\beta a) - Y_1(\beta b)\, J_1(\beta a)$$

Numerical Results and Discussions

Numerical results for stresses and displacements have been obtained for the following two different sets of elastic constants valid for soil. (Koning, 1957)

(a) $\eta = 2$, $\nu_{r\theta} = 0.125$, $\nu_{rz} = 0.75$, $G_{rz} = E_r/4.5$

(b) $\eta = 4$, $\nu_{r\theta} = 0.125$, $\nu_{rz} = 0.75$, $G_{rz} = E_r/6.25$

It may be seen that for determining the stresses and displacements infinite integrals have to be evaluated. The infinite integrals have been numerically evaluated using Weddle's formula (Salvadori and Baron, 1966). The upper limit of integration has been fixed at a limiting value such that the contribution of the integral beyond the limiting value is less that 10^{-6}. The vertical stress (σ_z), tangential stress (σ_θ), radial stress (σ_r), shear stress (τ_{rz}) as well as vertical (w) and radial (u) displacement variations for the above two cases are shown in Figures. 2 to 7.

The following observations can be made from Figures 2 to 7. The vertical stress and vertical displacement tend to decrease with increase in anisotropy while radial, tengential and shear stresses as well as radial displacement tend to increase with increase in anisotropy. In Figure 8 a comparison has been made for the vertical displacements at the surface both for anisotropic and isotropic medium having a cylindrical hole (*i*) with no lining (*ii*) with smooth rigid lining (*iii*) with rigid lining having perfect bond

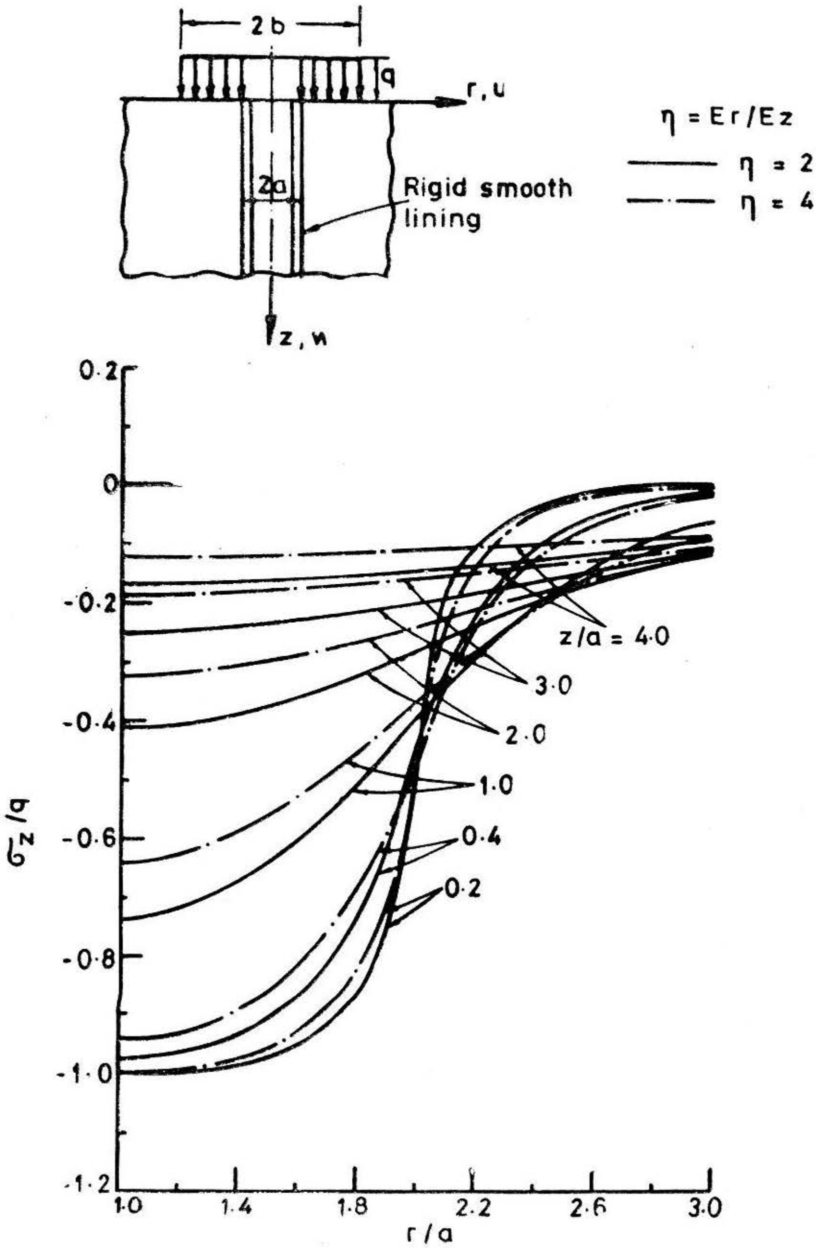

FIGURE 2 Distribution of Vertical Stress (σ_z) along Different Horizontal Sections ($b/a = 2$) with Rigid Lining

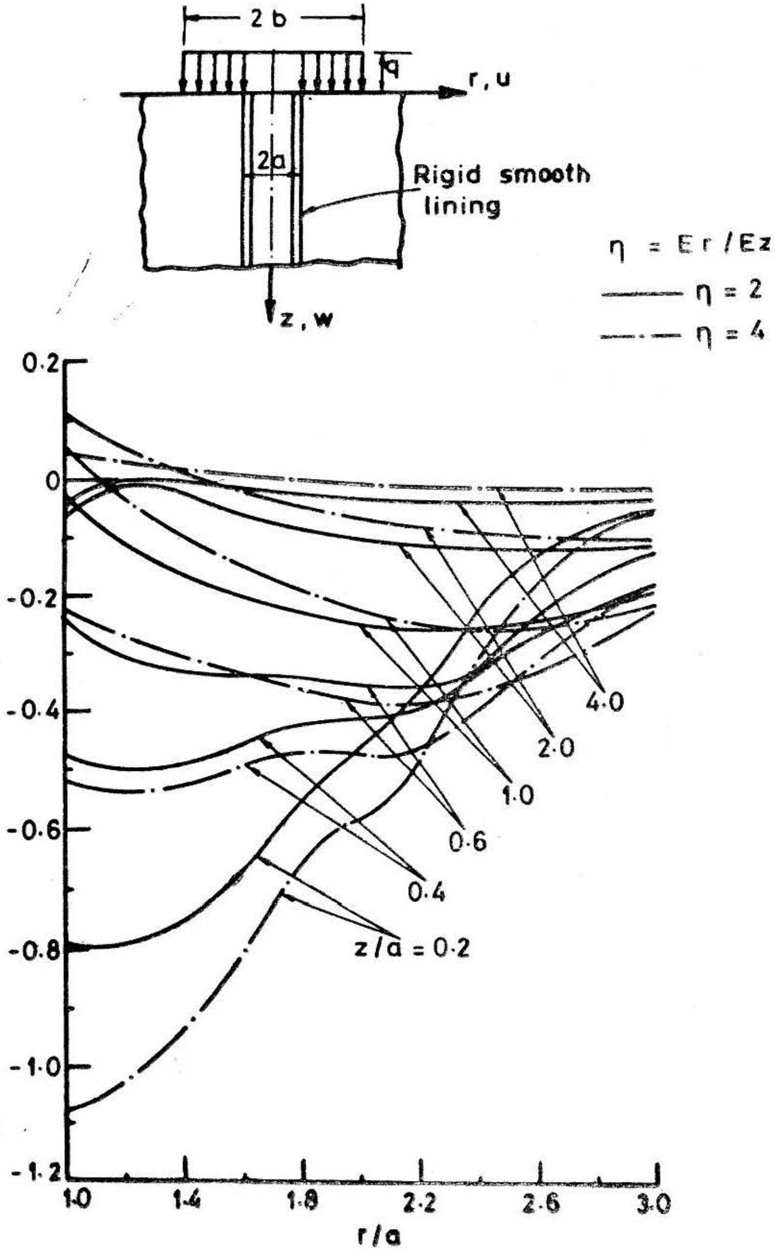

FIGURE 3 Distribution of Radial Stress (σ_r) along Different Horizontal Sections ($b/a = 2$) with Rigid Lining

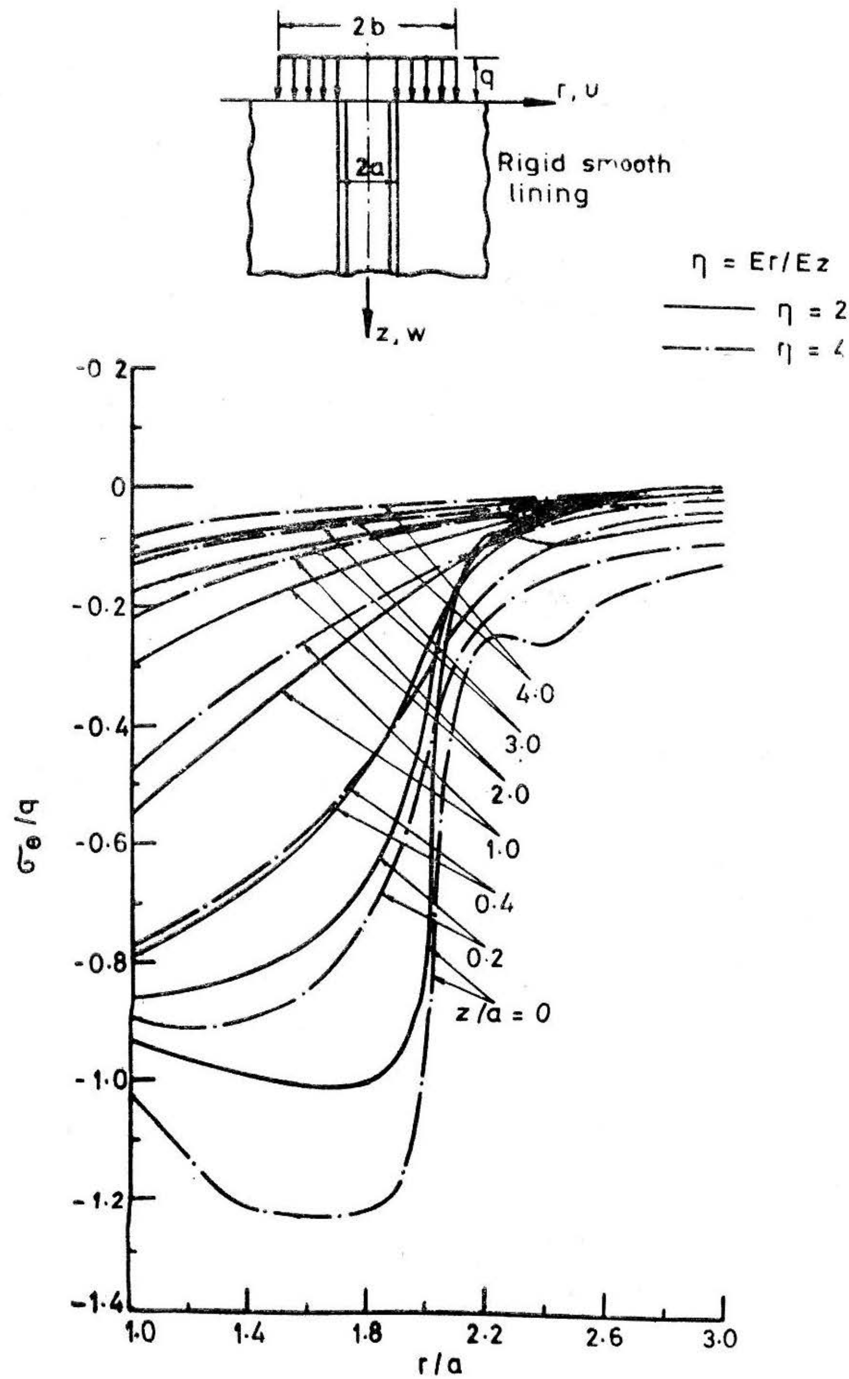

FIGURE 4 Distribution of Circumferential Stress (σ_θ) along Different Horizontal Sections ($b/a = 2$) with Rigid Lining

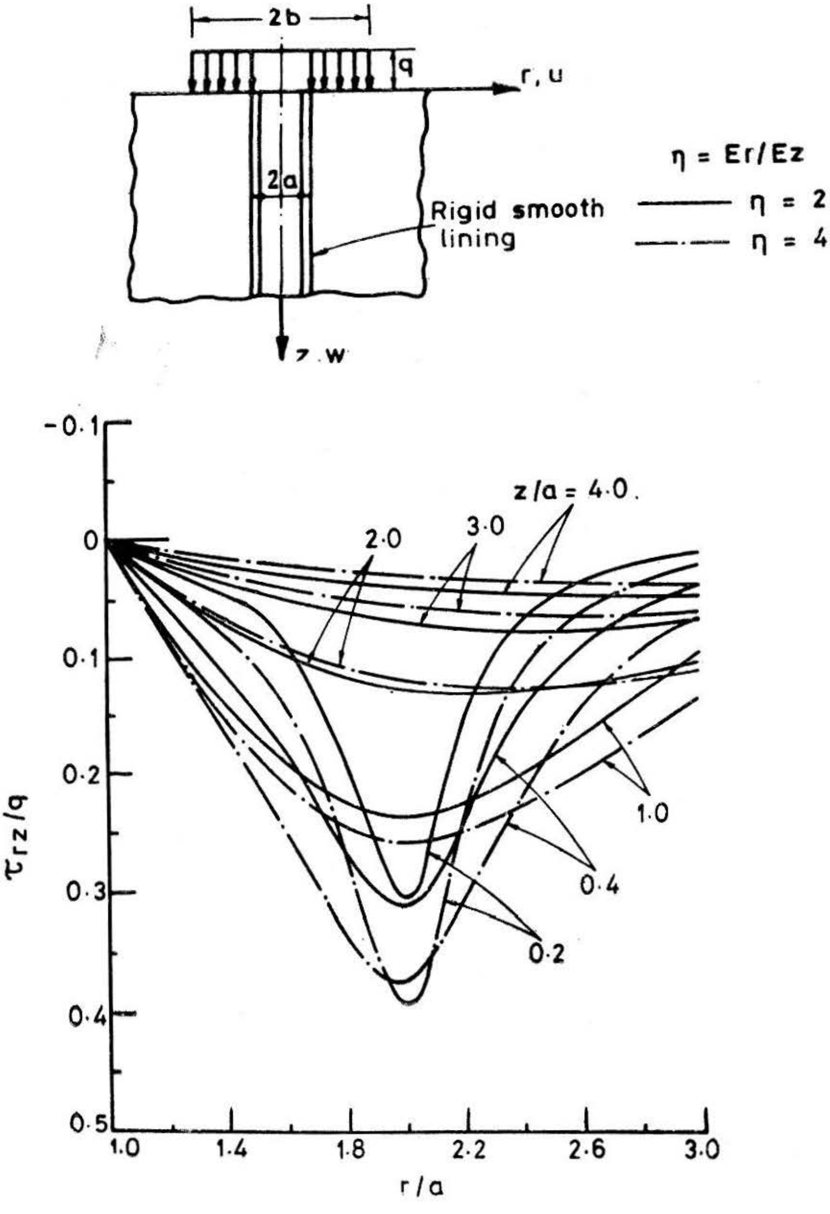

FIGURE 5 Distribution of Shear Stress (τ_{rz}) along Different Horizontal Sections ($b/a = 2$) with Rigid Lining

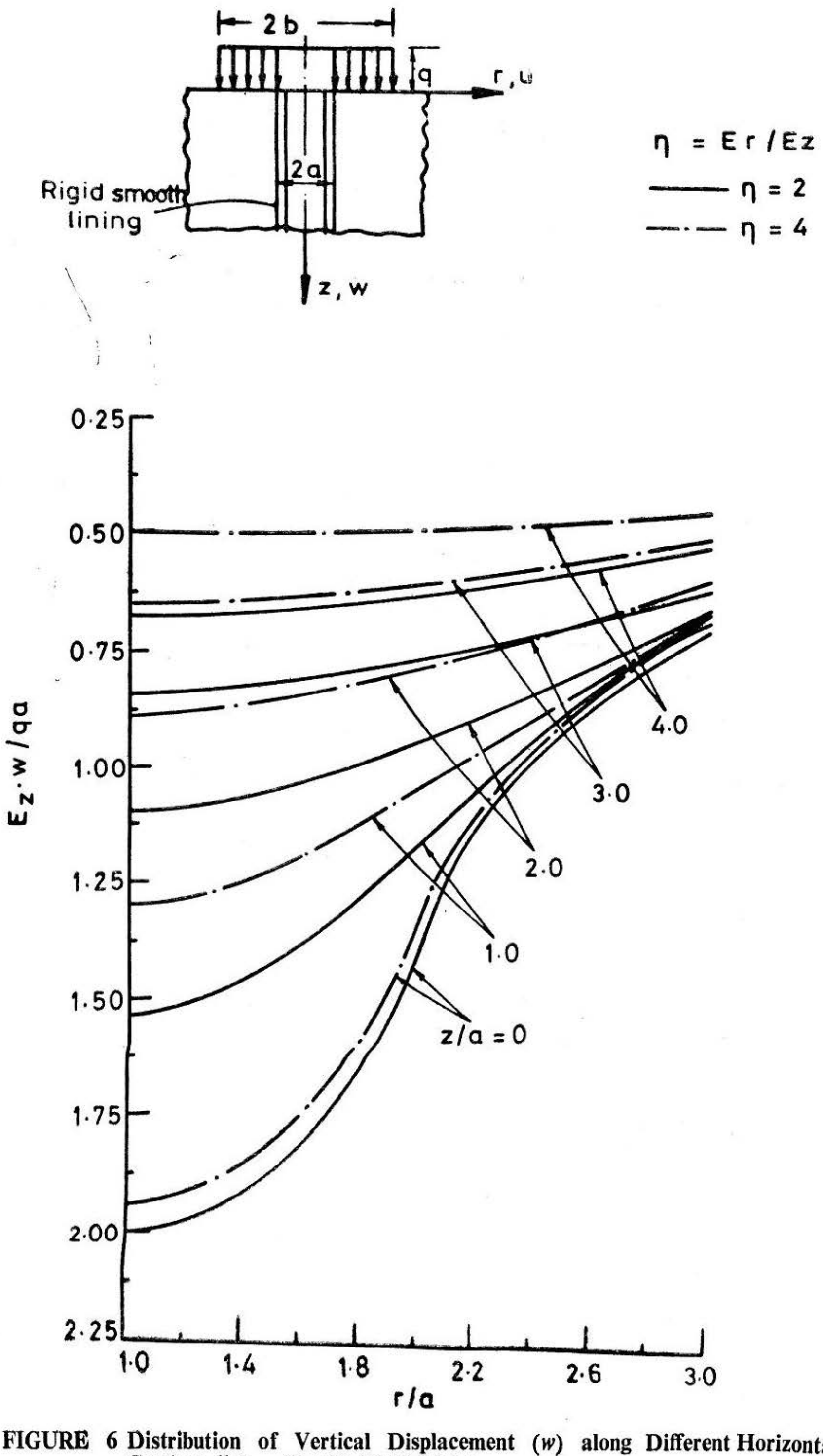

FIGURE 6 Distribution of Vertical Displacement (w) along Different Horizontal Sections ($b/a = 2$) with Rigid Lining

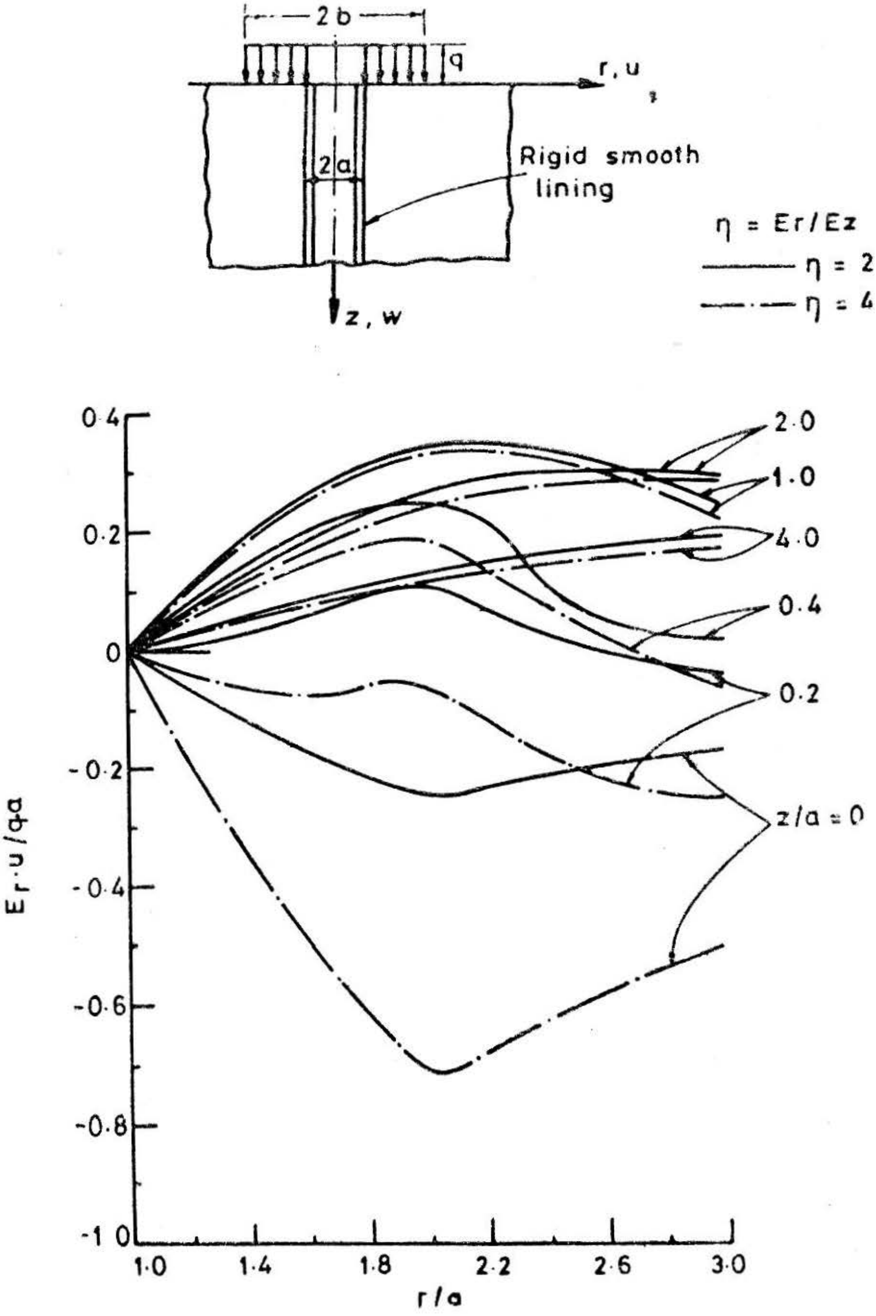

FIGURE 7 Distribution of Radial Displacement (u) along Different Horizontal Sections (b/a) = 2) with Rigid Lining

(friction). In general the maximum vertical displacement is higher for an isotropic medium compared to that of an anisotropic medium.

Further the maximum vertical displacement tends to decrease with increase in anisotropy. Also the maximum vertical displacement reduces copsiderably, both for isotropic and anisotropic medium, with the provision of a rigid lining (with friction). For example (Figure 8) the vertical displacement can be reduced by as much as 60 per cent by providing a rigid lining (with friction). However, there would be only a marginal reduction in the displacement when a smooth rigid lining is provided. Hence it may be concluded that the vertical displacements are smaller for an anisortopic medium compared to isotropic medium, and the vertical displacement can be drastically reduced by providing a rigid lining (with friction).

Notation

a = radius of the cylindrical hole

E_r, E_z = Young's modulus of elasticity in r and z directions

G_{rz} = Shear modulus in rz plane

J_0, J_1 = Bessel functions of first kind, zero and first order

$p(r)$ = axisymmetric normal load on the plane boundary

Y_0, Y_1 = Bessel functions of second kind, zero and first order

u = radial displacement

w = vertical displacement

σ_r, σ_θ, σ_z = normal stresses

$\nu_{r\theta}$, ν_{rz} = Poisson's ratio

τ_{rz} = Shear stress

η = ratio (E_r/E_z)

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