An Hyperelastic Model For Granular Materials Under Monotonic Loading

by

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Introduction

In recent years, a tremendous amount of research work has been undertaken for developing sophisticated constitutive models for soil. As a result of these efforts advanced constitutive models based on elasticity, plasticity, visco-elasticity, viscoplasticity micromechanics, damage, etc. have evolved for characterizing behaviour of soils under arbitrary loading conditions (Desai and Gallagher (1983), Desai and Siriwardane (1984), Saleeb and Chen (1980). Zienkiewiez et. al. (1975)). While a number of currently available models are mathematically elegant, their implementation into existing computer programs and evaluation of associated material constants from laboratory tests appear to be an extremely difficult task. Mainly because of these reasons a majority of these models are primarily used by researchers rather than practitioners. Many researchers actively involved in constitutive modeling feel that a model which is simple but capable of representing basic soil characteristics (Fig. 1) is more desirable. In this paper an attempt is made to develop an hyperelastic model, based on Green's formulation (Eringen, 1962), for granular materials subjected to monotonic increasing loads.

Brief Review

The first application of hyperelastic model to soils was demonstrated by Chang et. al. (1967), by using a second order model with five material parameters. The application of this model to Ottawa sand was found satisfactory in predicting hydrostatic behaviour, but the simple shear and triaxial shear conditions were not adequately represented. It may be noted that a second order model predicts a parabolic stress-strain relationship, and the quadratic term present in such a model will yield a symmetric behaviour even when the signs of strain components are changed. Ko et. al., (1976) used a third order model having nine material constants to predict the behaviour of Ottawa sand. The strain hardening feature was adequately represented. However, the prediction of dilatant behaviour was relatively poor. A similar model was also developed by Ayala et al., (1976). More recently, Saleeb and Chen (1980) have discussed various aspects of nonlinear hyperelastic model (s) in a workshop organized by the McGill University, Canada.

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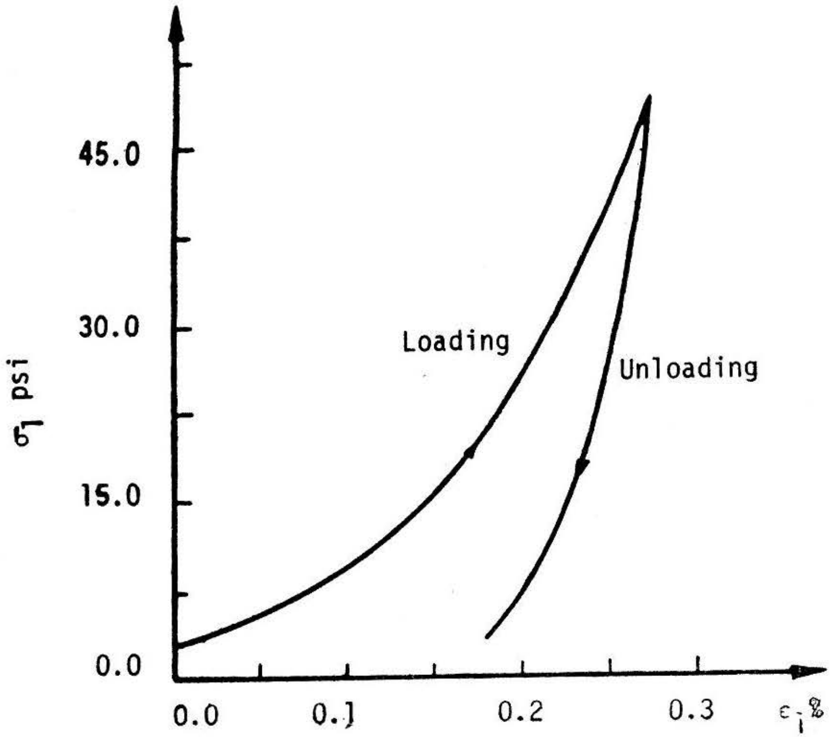


FIGURE 1 (a). Typical One Dimensional Compression Test (After Eringen, 1962) (1 kg/cm² = 14.2 psi)

Scope and Objective

The main objective of this paper is to develop an improved and simplified third order hyperelastic constitutive model for frictional soil subjected to monotonic loading. The nine material constants present in the general third order model developed by Ko and Masson (1976) are reduced to five by imposing some constraints based on the observed behaviour of granular materials in the laboratory. The model accounts for the effects of initial state on material behavior and predicts dilatancy with reasonable accuracy. The model is implemented in a finite element computer program and a boundary value problem is solved to demonstrate its application.

Hyperelastic Model

Green's formulation assumes an existence of strain energy density function U defined by

$$U = U(\varepsilon_{ij})$$

where ε_{ij} is the strain tensor. For an isotropic and homogeneous material U can be represented as

$$U = U(I_1, I_2, I_3)$$

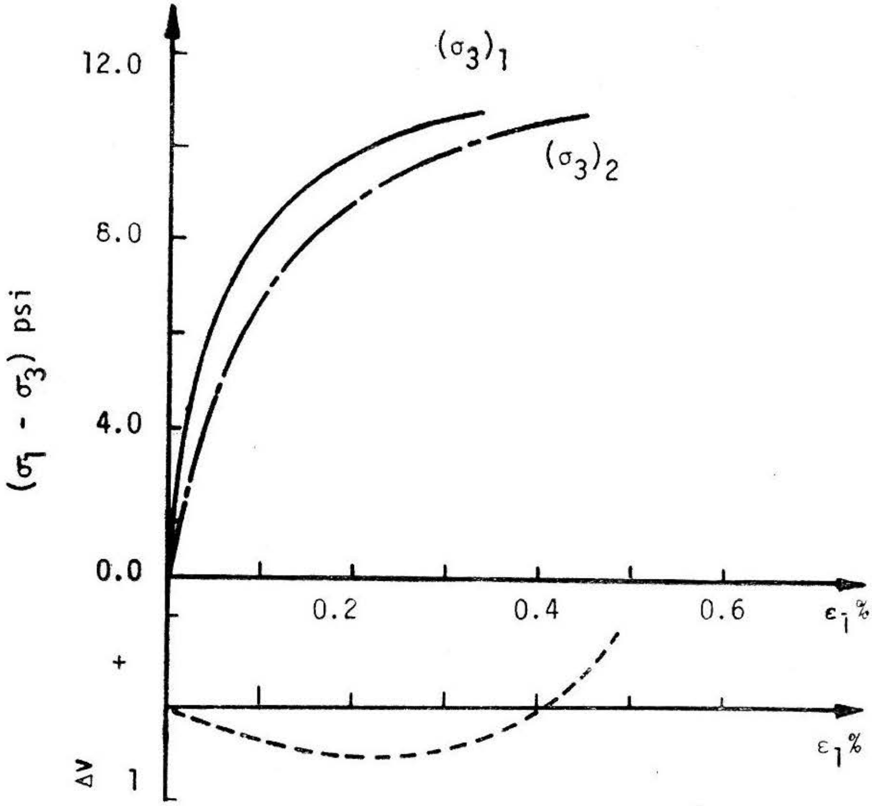


FIGURE 1 (b). Typical Dilatant Behavior of Granular Material (1 kg/cm² = 14.2 psi)

where I_1, I_2, and I_3 are the invariants of strain tensor ε_{ij}, and defined by

$$I_1 = \varepsilon_{ii}$$

$$I_2 = \tfrac{1}{2}\,\varepsilon_{ij}\,\varepsilon_{ji} \qquad \ldots(1)$$

and

$$I_3 = \tfrac{1}{3}\,\varepsilon_{ij}\,\varepsilon_{jk}\,\varepsilon_{ki}$$

It may be noted that the summation convention for indices ($i, j, k,$) applies to Eq. (1). Also, indicial notation is used throughout this paper. The stress tensor σ_{ij} is related to ε_{ij} by

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \qquad \ldots(2)$$

Using the chain rule of differentiation we can write

$$\sigma_{ij} = \frac{\partial U}{\partial I_1}\,\frac{\partial I_1}{\partial \varepsilon_{ij}} + \frac{\partial U}{\partial I_2}\,\frac{\partial I_2}{\partial \varepsilon_{ij}} + \frac{\partial U}{\partial I_3}\,\frac{\partial I_3}{\partial \varepsilon_{ij}} \qquad \ldots(3)$$

In view of Eq. (1), Eq. (3) can be written as

$$\sigma_{ij} = \frac{\partial U}{\partial I_1} \delta_{ij} + \frac{\partial U}{\partial I_2} \varepsilon_{ij} + \frac{\partial U}{\partial I_3} \varepsilon_{ik} \varepsilon_{kj} \qquad \ldots(4)$$

where δ_{ij} is the Kronecker delta. Eq. (4) is the general stress-strain relationship. By assuming different functions for U, various types of hyperelastic models can be derived.

Proposed Hyperelastic Model

In the present formulation, the strain energy density function U is assumed as

$$U = \alpha_1 I_1^2 + \alpha_2 I_2 + \sigma_3 I_1^4 + \alpha_4 I_1^2 I_2 + \alpha_5 I_1 I_3 + \alpha_6 I_2^2 + \alpha_7 I_1 I_2$$
$$+ \alpha_8 I_3 \qquad \ldots(5)$$

In the above equation I_1 contributes toward volumetric behaviour and I_2 and I_3 contribute to both volumetric and shear behaviours. It may be noted that the assumed polynomial function is not complete. The last two terms containing α_7 and α_8 account for the effects of initial state of the material.

Substitution of Eq. (5) into Eq. (4) and required differentiation will lead to

$$\sigma_{ij} = (2\alpha_1 I_1 + 4\alpha_3 I_1^3 + 2\alpha_4 I_1 I_2 + \alpha_5 I_3 + \alpha_7 I_2) \delta_{ij}$$
$$+ (\alpha_2 + \alpha_4 I_1^2 + 2\alpha_6 I_2 + \alpha_7 I_1) \varepsilon_{ij} + (\alpha_5 I_1 + \alpha_8) \varepsilon_{ik} \varepsilon_{kj} \qquad \ldots(6)$$

Using expressions for I_1, I_2 and I_3 (Eq. 1), Eq. (6) can be written as

$$\sigma_{ij} = (2\alpha_1 \varepsilon_{kk} + 4\alpha_3 \varepsilon_{kk}^3 + \alpha_4 \varepsilon_{kk}\varepsilon_{mn}\varepsilon_{mn} + \tfrac{1}{3} \alpha_5 \varepsilon_{lk} \varepsilon_{km} \varepsilon_{ml}$$
$$+ \tfrac{1}{2} \alpha_7 \varepsilon_{mn} \varepsilon_{mn}) \delta_{ij} + (\alpha_2 + \alpha_4 \varepsilon_{kk}^2 + \alpha_6 \varepsilon_{kl} \varepsilon_{kl} + \alpha_7 \varepsilon_{kk}) \varepsilon_{ij}$$
$$+ (\alpha_5 \varepsilon_{ll} + \alpha_8) \varepsilon_{ik} \varepsilon_{kj} \qquad \ldots(7)$$

It is interesting to note that if higher order terms are neglected from Eq. (7), it essentially specializes to an equivalent form of well known Hooke's Law of linear elasticity (Desai and Siriwardane (1984) and Eringen (1962))

$$\sigma_{ij} = 2\alpha_1 \varepsilon_{kk} \delta_{ij} + \varepsilon_{ij} a_2 \qquad \ldots(8)$$

Equation (7) is the general form of stress-strain relation for the assumed strain energy density function (Eq. 3) in which α_i ($i = 1, 2, \ldots, 8$) are material constants which are to be determined from appropriate laboratory test data.

In the following it will be shown that the eight material constants present in Eq. (7), can be reduced to five by imposing certain conditions observed in the laboratory. Imposition of such constraints will provide a non-linear model with fewer parameters, compared to existing similar models (Ko and Masson (1976) and Saleeb and Chen (1980)) and possibly increase computational efficiency in terms of solution of large boundary value problems. To accomplish this goal, Eq. (7) is written in an incremental form for various stress paths. The initial or reference state is assumed as hydrostatic compression with the following strains.

$$\overset{0}{\varepsilon}_1 = \overset{0}{\varepsilon}_2 = \overset{0}{\varepsilon}_3 = -b_0; \quad \varepsilon_{ij} = 0, i \neq j \qquad \ldots(9)$$

A state of deviatoric strain is then super-imposed on the previous initial state such that

$$\triangle\varepsilon_1 = -\varepsilon,\ \triangle\varepsilon_2 = \varepsilon,\ \triangle\varepsilon_3 = 0\ ;\ \triangle\varepsilon_{ij} = 0, i \neq j$$

in which $\triangle\varepsilon_1$, $\triangle\varepsilon_2$ and $\triangle\varepsilon_2$ are incremental strain components, and ε represents the amount of increment.

Thus, the total strain components become

$$\overset{1}{\varepsilon}_1 = -b_0 - \varepsilon,\ \overset{1}{\varepsilon}_2 = -b_0 + \varepsilon,\ \overset{1}{\varepsilon}_3 = -b_0\ ;\ \overset{1}{\varepsilon}_{ij} = 0, i \neq j \qquad \ldots(10)$$

where super-script o (zero) represents initial state and 1 represents final state ; also for convenience a repeated subscript is replaced by a single subscript. Substituting the total strain components of Eq. (10) into Eq. (7) a relation between the total stress and strain components can be written. A similar relation can be obtained for initial state of strain which can be subtracted from the total stress-strain relationship to get relation between the super-imposed quantities. Thus,

$$\triangle\sigma_1 = \overset{1}{\sigma}_1 - \overset{0}{\sigma}_1 = -(\alpha_2 + 9\alpha_4 b_0^2 + 6\alpha_5 b_0^2 + 3\alpha_6 b_0^2 - 3\alpha_7 b_0$$
$$- 2\alpha_8 b_0)\,\varepsilon - (6\alpha_4 b_0 + 5\alpha_5 b_0 + 2\alpha_6 b_0 - \alpha_7 - \alpha_8)\,\varepsilon^2 - 2\alpha_6\,\varepsilon^3 \qquad \ldots(11a)$$

$$\triangle\sigma_2 = \overset{1}{\sigma}_2 - \overset{0}{\sigma}_2 = (\alpha_2 + 9\alpha_4 b_0^2 + 6\alpha_5 b_0^2 + 3\alpha_6 b_0^2 - 3\alpha_7 b_0$$
$$- 2\alpha_8 b_0)\,\varepsilon - (6\alpha_4 b_0 + 5\alpha_5 b_2 + 2\alpha_6 b_0 - \alpha_7 - \alpha_8)\,\varepsilon^2 + 2\alpha_6\,\varepsilon^3 \qquad \ldots(11b)$$

$$\triangle\sigma_3 = \overset{1}{\sigma}_3 - \overset{0}{\sigma}_3 = -(6\alpha_4 b_0 + 2\alpha_5 b_0 + 2\alpha_6 b_0 - \alpha_7)\,\varepsilon^2 \qquad \ldots(11c)$$

$$\triangle\sigma_{ij} = 0, i \neq j \qquad \ldots(11d)$$

For simple shear state of stress it is known that

$$\triangle\sigma_1 = -\triangle\sigma_2 \qquad \ldots(12a)$$

$$\text{and } \triangle\sigma_3 = 0 \qquad \ldots(12b)$$

The condition of Eq. (12a), cannot be satisfied by Eqs. (11a) and (11b) unless the coefficients of ε^2 in both equations vanish such that

$$6\alpha_4 b_0 + 5\alpha_5 b_0 + 2\alpha_6 b_0 - \alpha_7 - \alpha_8 = 0 \qquad \ldots(13a)$$

Also, to satisfy the condition of Eq. (12*b*), the following relation must hold in Eq. (11*c*)

$$6\alpha_4\, b_0+2\alpha_5\, b_0+2\alpha_6 b_0-\alpha_7 = 0 \qquad \ldots(13b)$$

Triaxial Shear Test

Initial state : $\qquad \overset{0}{\varepsilon}_1 = \overset{0}{\varepsilon}_2 = \overset{0}{\varepsilon}_3 = -b_0\ ;\ \varepsilon_{ij}\ 0,\ i \neq j$

Superimposed state : $\triangle\varepsilon_1 = -\ \varepsilon,\ \triangle\varepsilon_2 = \triangle\varepsilon_3 = \frac{1}{2}\,\varepsilon,\ \triangle\varepsilon_{ij} = 0,\ i \neq j$

The resulting stress-strain relation is given by

$$\begin{aligned}\triangle\sigma_1 = \overset{1}{\sigma}_1 - \overset{0}{\sigma}_1 = &-(\alpha_2+9\alpha_4 b_0^2+6\alpha_5 b_0^2+3\alpha_6 b_0^2-3\alpha_7\, b_0\\ &-2\alpha_8\, b_0)\ \varepsilon-(\tfrac{9}{2}\alpha_4\, b_0+\tfrac{9}{2}\,\alpha_5\, b_0+\tfrac{3}{2}\,\alpha_6\, b_0-\tfrac{3}{4}\alpha_7)\ \varepsilon^2\\ &-(\tfrac{1}{4}\alpha_5+\tfrac{3}{2}\alpha_6)\ \varepsilon^3\end{aligned} \qquad \ldots(14a)$$

$$\begin{aligned}\triangle\sigma_2 = \overset{1}{\sigma}_2 - \overset{0}{\sigma}_2 = &(\tfrac{1}{2}\alpha_2+\tfrac{9}{2}\alpha_4 b_0^2+3\alpha_5 b_0^2+\tfrac{3}{2}\alpha_6 b_0^2-\tfrac{3}{2}\alpha_7\, b_0\\ &-\alpha_8\, b_0)\ \varepsilon-(\tfrac{9}{2}\alpha_4\, b_0+\tfrac{9}{4}\alpha_5\, b_0+\tfrac{3}{2}\,\alpha_6\, b_0-\tfrac{3}{4}\alpha_7-\tfrac{1}{4}\alpha_8)\ \varepsilon^2\\ &-\tfrac{1}{4}\alpha_5-\tfrac{3}{4}\alpha_6)\end{aligned} \qquad \ldots(14b)$$

$$\triangle\sigma_3 = \triangle\sigma_2\ ;\ \triangle\sigma_{ij} = 0,\ i \neq j \qquad \ldots(14c)$$

In this test, $\triangle\sigma_2 = \triangle\sigma_3 = -\frac{1}{2}\triangle\sigma_1$. This condition can be satisfied by Eqs. (14*a*) and (14*b*) if

$$\begin{aligned}-\tfrac{1}{2}\,(\tfrac{9}{2}\alpha_4\, b_0+\tfrac{9}{2}\alpha_5\, b_0+\tfrac{3}{2}\alpha_6\, b_0-\tfrac{3}{4}\alpha_7) = &\ \tfrac{9}{2}\alpha_4\, b_0+\tfrac{9}{4}\alpha_5\, b_0\\ &+\tfrac{3}{2}\alpha_6\, b_0-\tfrac{3}{4}\alpha_7-\tfrac{1}{8}\alpha_8\end{aligned} \qquad \ldots(15)$$

and

$$-\tfrac{1}{2}\,(\tfrac{1}{4}\alpha_5+\tfrac{3}{2}\alpha_6) = (\tfrac{1}{4}\alpha_5-\tfrac{3}{4}\alpha_6) \qquad \ldots(16)$$

From Eq. (16) it is seen that

$$\alpha_5 = 0 \qquad \ldots(17)$$

Subtracting Eq. (13*b*) from Eq. (13*a*) and substituting the value of α_5 (Eq. 17) into the resulting equation we get

$$\alpha_8 = 0 \qquad \ldots(18)$$

Knowing α_5 and α_8, α_7 can be expressed in terms of α_4, α_6 and b_0 using Eq. (15)

$$\alpha_7 = \alpha_7\, b_0 = (6\alpha_4+2\alpha_6)\, b_0 \qquad \ldots(19)$$

Thus, the number of parameters α_i is reduced from eight to five; the stress-strain relations developed with these five parameters satisfy the condition that $-\triangle\sigma_1 = \triangle\sigma_2$ and $\triangle\sigma_3 = 0$ for simple shear and $-1/2\triangle\sigma_1 = \triangle\sigma_2 = \triangle\sigma_3$ for triaxial shear tests. It is noted that α_5 and α_8 which become zero (Eqs. 17, 18) are coefficients of the terms of energy density function (Eq. 5) containing I_3. It can also be noted that α_7' is a material parameter (Eq. 19), and α_7 ($= \alpha_7'\, b_0$) is a function of the initial strain state b_0; this signifies that the present model accounts for the effects of initial state.

Substitution of Eqs. (17), (18) and (19) into Eqs. (11) and (14) will yield modified stress-strain relation for simple shear and triaxial conditions respectively.

Simple Shear Condition

$$\triangle\sigma_1 = -\beta_1\varepsilon - \beta_2\varepsilon^3 \quad \ldots(20a)$$

$$\triangle\sigma_2 = \beta_1\varepsilon + \beta_2 \quad \ldots(20b)$$

$$\triangle\sigma_3 = 0\ ;\ \triangle\sigma_{ij} = 0,\ i \neq j \quad \ldots(20c)$$

where

$$\beta_1 = (\alpha_2 - 9\alpha_4\, b_0^2 - 3\alpha_6\, b_0^2) \quad \ldots(20d)$$

$$\beta_2 = 2\alpha_6 \quad \ldots(20e)$$

Triaxial Shear Condition

$$\triangle\sigma_1 = -\beta_3\varepsilon - \beta_4\varepsilon^3 \quad \ldots(21a)$$

$$\triangle\sigma_2 = \tfrac{1}{2}\beta_3\varepsilon + \tfrac{1}{2}\beta_4\varepsilon^3 \quad \ldots(21b)$$

$$\triangle\sigma_3 = \triangle\sigma_2\ ;\ \triangle\sigma_{ij} = 0,\ i \neq j \quad \ldots(21c)$$

where

$$\beta_3 = \alpha_2 - (9\alpha_4 + 3\alpha_6)b_0^2 \quad \ldots(21d)$$

$$\beta_4 = \tfrac{3}{2}\,\alpha_6 \quad \ldots(21e)$$

The corresponding relations for a conventional triaxial compression ($\triangle\varepsilon_1 = \varepsilon_1$; $\triangle\varepsilon_2 = \triangle\varepsilon_3 = \varepsilon_2$; $\triangle\varepsilon_{ij} = 0,\ i \neq j$) can be written as

$$\triangle\sigma_1 = -\beta_5\varepsilon_1 - \beta_6\varepsilon_2 - \beta_7\,\varepsilon_1^2 - \beta_8\varepsilon_1\varepsilon_2 - \beta_8\,\varepsilon_2^2 - \beta_9\,\varepsilon_1^3 - \beta_{10}\,\varepsilon_1^2\,\varepsilon_2 - \beta_{11}\,\varepsilon_1\,\varepsilon_2^2 - \beta_{12}\,\varepsilon_2^3 \quad \ldots(22a)$$

$$\triangle\sigma_2 = \triangle\sigma_3 = \beta_{13}\varepsilon_1 - \beta_{14}\varepsilon_2 - \beta_7\varepsilon_1^2 - \beta_8\varepsilon_1\varepsilon_2 - \beta_8\,\varepsilon_2^2 - \tfrac{1}{8}\beta_{10}\varepsilon_1^3 - \tfrac{1}{2}\beta_{11}\,\varepsilon_1^2\,\varepsilon_2 - \beta_{15}\,\varepsilon_1\,\varepsilon_2^2 - \beta_{16}\,\varepsilon_2^3 \quad \ldots(22b)$$

$$\Delta\sigma_{ij} = 0,\ i \neq j \qquad \ldots(22c)$$

where

$$\beta_5 = \{(2\alpha_1+\alpha_2)+(108\alpha_3-6\alpha_4-5\alpha_6)\ b_0^2\} \qquad \ldots(22d)$$

$$\beta_6 = \{4\alpha_1+(216\alpha_3+6\alpha_4-4\alpha_6)\ b_0^2\} \qquad \ldots(22e)$$

$$\beta_7 = (36\alpha_3+3\alpha_4)\ b_0 \qquad \ldots(22f)$$

$$\beta_8 = (144\alpha_3+12\alpha_4)\ b_0 \qquad \ldots(22g)$$

$$\beta_9 = (4\alpha_3+2\alpha_4+\alpha_6) \qquad \ldots(22h)$$

$$\beta_{10} = (24\alpha_3+6\alpha_4) \qquad \ldots(22i)$$

$$\beta_{11} = (48\alpha_3+6\alpha_4+2\alpha_6) \qquad \ldots(22j)$$

$$\beta_{12} = (32\alpha_3+4\alpha_4) \qquad \ldots(22k)$$

$$\beta_{13} = \{2\alpha_1+(108\alpha_3+3\alpha_4-2\alpha_6)\ b_0^2\} \qquad \ldots(22l)$$

$$\beta_{14} = \{(4\alpha_1+\alpha_2)+(216\alpha_3-3\alpha_4-7\alpha_6)\ b_0^2\} \qquad \ldots(22m)$$

$$\beta_{15} = (48\alpha_3+6\alpha_4) \qquad \ldots(22n)$$

$$\beta_{16} = (32\alpha_3+8\alpha_4+2\alpha_6) \qquad \ldots(22o)$$

Note that Eqs. (22) can be specialized to one dimensional compression condition and triaxial shear condition by putting $\varepsilon_2 = 0$ and $\varepsilon_2 = -\frac{1}{2}\varepsilon_1$, respectively. For hydrostatic compression condition, the equivalent stress-strain relation is

$$\sigma_1 = \sigma_2 = \sigma_3 = -(6\alpha_1+\alpha_2)\ b_0 - (108\alpha_3 - \alpha_4 - 6\alpha_6)\ b_0^3 \qquad \ldots(23)$$

Dilatancy

To study the dilatant behavior of granular material, volumetric strain ($\varepsilon_v = \varepsilon_1^1 + \varepsilon_2^1 + \varepsilon_3^1$) is frequently plotted against deviatoric stress $\sigma_1^1 - \sigma_3^1$). For the case of conventional triaxial compression condition, expressions for these quantities can be obtained using Eqs. (22) and following the steps shown in Eq. (11).

$$\varepsilon_v = -3b_0 - \varepsilon_1 - 2\varepsilon_2 \qquad \ldots(24)$$

$$(\sigma_1-\sigma_2) = -\{\alpha_2-(9\alpha_4+3\alpha_6)\ b_0^2\}\ (\varepsilon_1-\varepsilon_2) - \{(\alpha_4+\alpha_6)\ (\varepsilon_1-\varepsilon_2)^3\}$$

$$-\{(3\alpha_4+\alpha_6)\ (2\varepsilon_1\varepsilon_2+\ \varepsilon_2^2)\ (\varepsilon_1-\varepsilon_2\} \qquad \ldots(25)$$

Eqs. (24) and (25) are used subsequently in predicting the dilatant behaviour of Ottawa sand. It should be noted that compressive strain has been designated as a negative quantity in the formulation, therefore appropriate sign of ε_l must be used in numerical evaluation of Eq. (25).

Comments

The major weaknesses of the formulation outlined above are that it accounts for only monotonic loading paths and assumes isotropic state of initial stress and homogeneous material. Very often a soil medium is non-homogeneous and state of stress is anisotropic in nature. None the less, from the comparison of experimentally observed and model predicted response presented subsequently it is evidenced that the proposed model is capable of representing some important behaviours (e.g., dilatancy and effects of initial state) of granular materials; Although the reduction of the number of material constants is based primarily on simple and triaxial shear conditions, predictions for other stress paths are also fairly reasonable.

From Eq. (8) it can be noted that the parameters α_1 and α_2 are associated with the elastic properties of a material. The remaining parameters (α_3, α_4 and α_6) account for (material) nonlinearity. The constant α_7 which is dependent on α_4 and α_6 accounts for the initial state of stress.

Determination of Material Parameters

The five material parameters (α_1, α_2, α_3, α_4 and α_6) which define the stress-strain relation for a granular material can be determined from test data obtained from appropriate laboratory tests and using a least square collocation curve fitting procedure (Ayala and Arboleda (1976), Desai and Siriwardane (1984) and Ko and Masson (1976)). In order to obtain meaningful values of the material constants it is important to include test data from as many different paths as possible in evaluating the parameters. Various commonly used stress paths are depicted in Fig. 2.

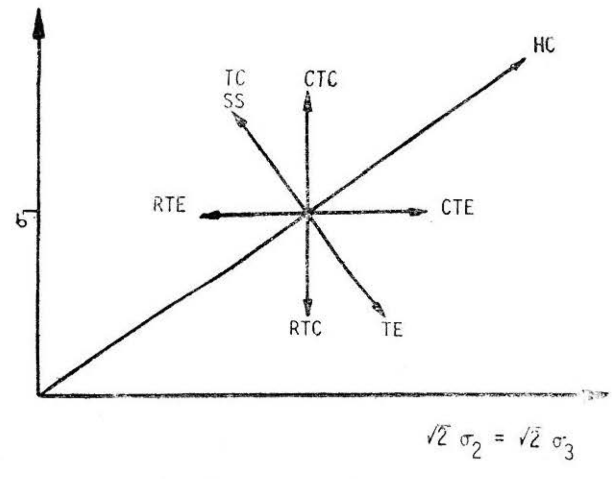


FIGURE 2 Various Stress Paths

In the present paper, the laboratory test data originally presented by Chang et. al., (1967) are used to determine the material constants. These tests are strain controlled tests and were performed using dry Ottawa sand (passing ASTM sieve No. 20 but retaining on ASTM sieve No. 40) in a multiaxial test cell (Chang et. al. (1967) and Ko et. al. (1967a, b)). The initial void ratio was kept constant at 0.52 for all tests. A detailed description of the test device, testing procedure and results, is given by Ko et. al., (1967a, b).

Parameters are determined in two steps. In the first step, incremental stress-strain relations for a give stress path (e.g., Eq. 20 for simple shear) are evaluated at number of discrete points to get an overdetermined system of simultaneous equation in terms of β_i (e.g. β_1 and β_2 for Eq. 12); these equations are solved using a least square method to obtain the values of β_i. This step is repeated for all the stress paths to be included in the evaluation of parameters. In the second step, explicit expressions for β_1's are written in terms of α_i's (e,g. β_5 is given by Eq. 22d) and equated with the corresponding values obtained in the previous step. This gives another set of equations in terms of α_1, α_2, α_3, α_4 and α_6 which is solved by least square method.

It should be noted that in the least square method used here, we can assign different weights for different points as well as for different stress paths to reflect their relative importance. Thus, the values of the parameters are not unique and depend upon the weights used in the procedure. Since the behavior of granular material is usually predominant in shear, more weights can be assigned to the shear tests data. A general computer program was written to evaluate the parameters. Different weights were tried to get the best possible predictions of various stress paths. The final values of the material parameters for the Ottawa sand are

$$\begin{aligned}
\alpha_1 &= 172.0 \text{ kg/cm}^2 \text{ (2446.3 psi)} \\
\alpha_2 &= 346.3 \text{ kg/cm}^2\text{/(4926.2 psi)} \\
\alpha_3 &= 137\times10^5 \text{ kg/cm}^2 \text{ (}1948.6\times10^5 \text{ psi)} \\
\alpha_4 &= 348.5\times10^2 \text{ kg/cm}^2 \text{ (}4957.7\times10^2 \text{ psi)} \\
\alpha_6 &= -219.5\times10^4 \text{ kg/cm}^2 \text{ (}-3122.5\times10^4 \text{ psi)}
\end{aligned} \qquad \ldots(26)$$

Comparison of Experimental and Predicted Curves

Knowing the material constants α_1, α_2, α_3, α_4, and α_6, the stress-strain response for any stress path can be predicted by using the specialized incremental form of Eq. (7) (see e.g. Eq. 21 for triaxial shear condition). For several selected stress paths, the comparisons of the predicted stress-strain response with the corresponding experimental curves are presented in this section.

For hydrostatic compression, the predicted and experimental values are approximately the same (Fig. 3) and very little difference can be detected. A reasonable agreement between observed and predicted values is obtained for the triaxial shear stress path also (Figs. 4 and 5). However, it should be noted that here the predicted curves deviate from the experimental curves after strain equal to 0.75×10^{-2}. This is due to the assumed cubic relationship between stress and strain in the present model. It is possible to adjust the error at higher strain levels by assigning suitable weights to

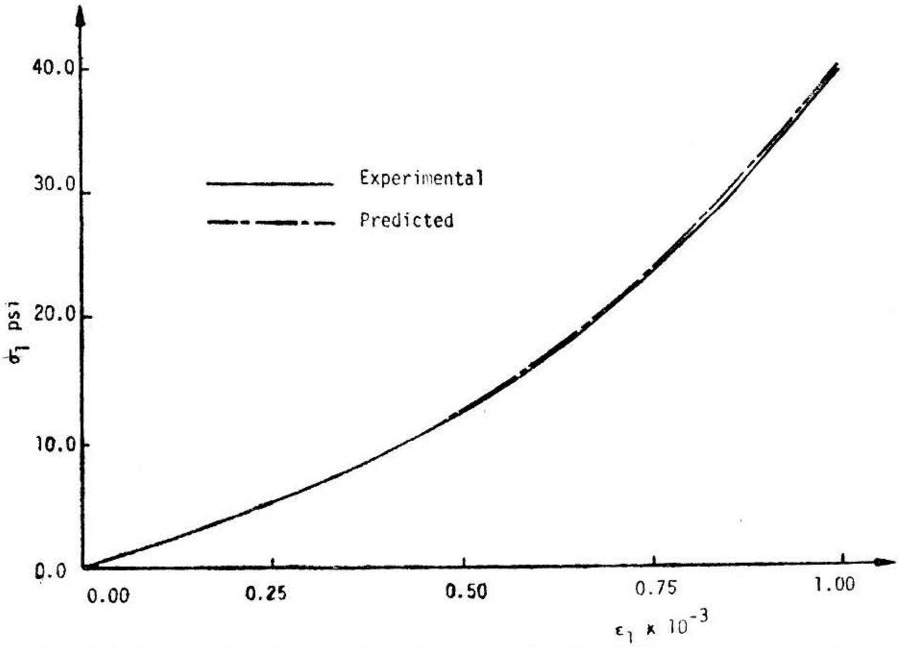


FIGURE 3 Comparison Between Experimental and Predicted Curves for Hydrostatic Compression Test (1 kg/cm² = 14.2 psi)

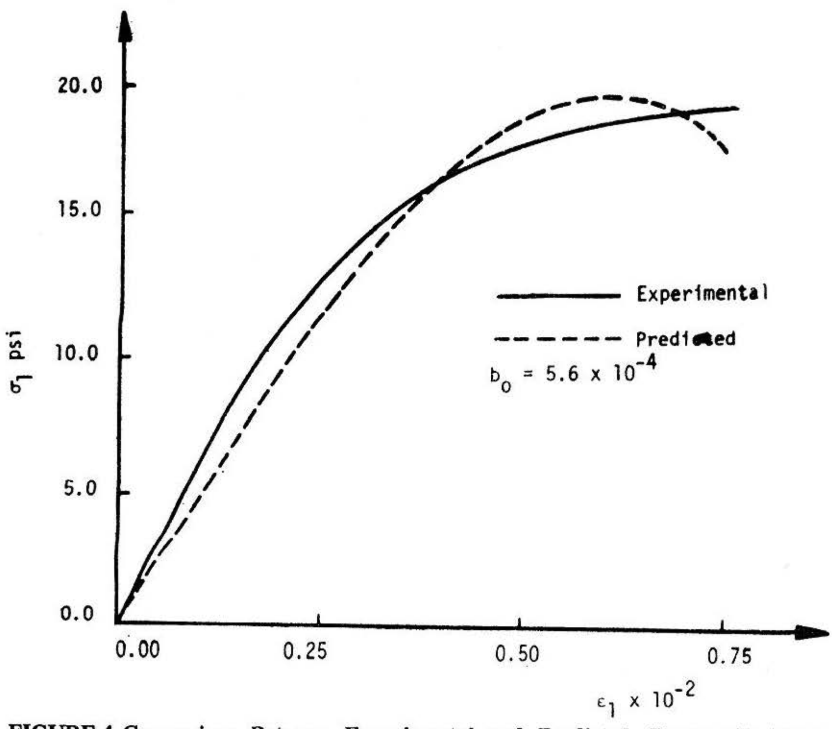


FIGURE 4 Comparison Between Experimental and Predicted Curves of Triaxial Shear Test for σ_1 (1 kg/cm² = 14.2 psi)

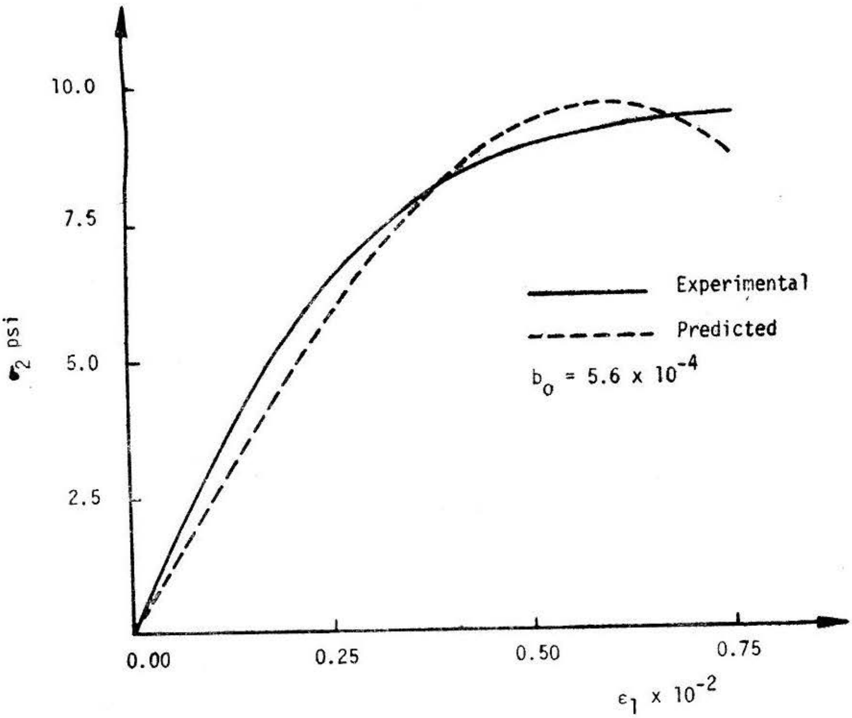


FIGURE 5 Comparison Between Experimental and Predicted Curves for Triaxial Shear Test for σ_2 (1 kg/cm² = 14.2 psi)

different points in the least square fitting procedure. However, this will result in a comparatively poor prediction at low strain levels. Figures 6 to 9 show that for other stress paths (one dimensional confined compression and simple shear) also the predicted and experimental values are in fairly good agreement.

Prediction of Dilatancy

To demonstrate the applicability of any constitutive model, it is not sufficient to show reasonable prediction of the stress paths which are used in the determination of material parameters; the model should also be able to predict other stress paths which are not included in the evaluation of parameters. For this purpose, the dilatan behavior of Ottawa sand is predicted using α_I's (Eq. 26) in Eqs. (24) and (25). The comparison between the predicted and experimental curves is shown in Fig. 10. The experimental curve for Ottawa sand was presented by Ko et al., (7). It is seen that the present model predicts dilatant behaviour of the material with sufficient engineering accuracy, at least qualitatively. Since the dilatant behaviour of granular material is one of the most important aspectst the present model is expected to be well suited and successful in representing, deformability characteristics of such materials.

It should be noted that because the proposed model assumes cubic stress-strain relation, no matter how the material constants are chosen, the nature of predicted stress-strain response (third order) remains unchanged.

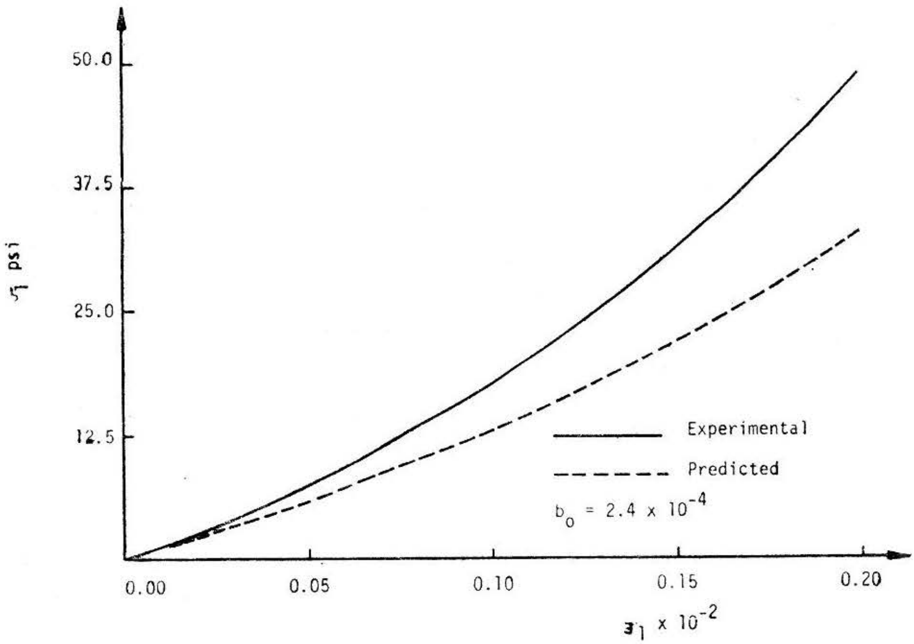


FIGURE 6 Comparison Between Experimental and Predicted Curves for One Dimensional Confined Compression Test for σ_1 (1 kg/cm² = 14.2 psi)

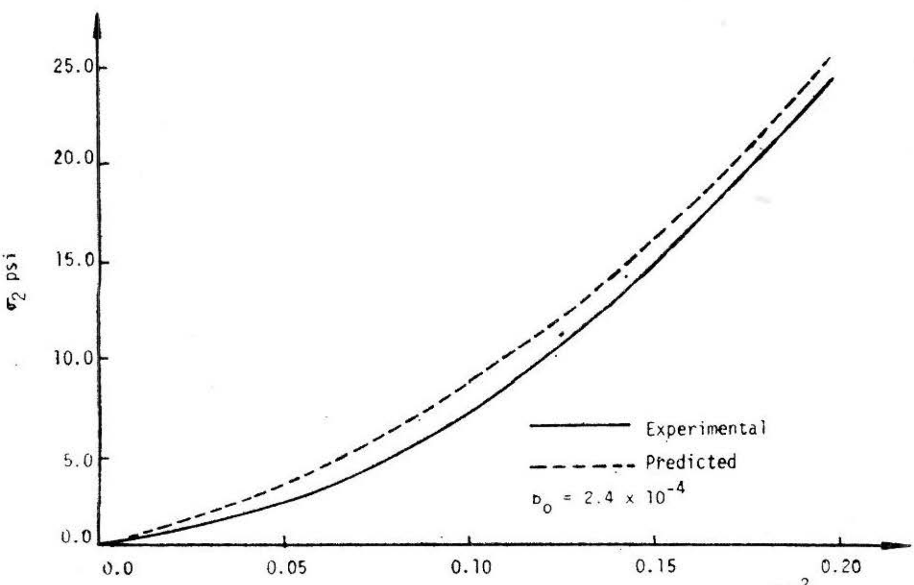


FIGURE 7 Comparison Between Experimental and Predicted Curves for One Dimensional Confined Compression Test for σ_2 (1 kg/cm² = 14.2 psi)

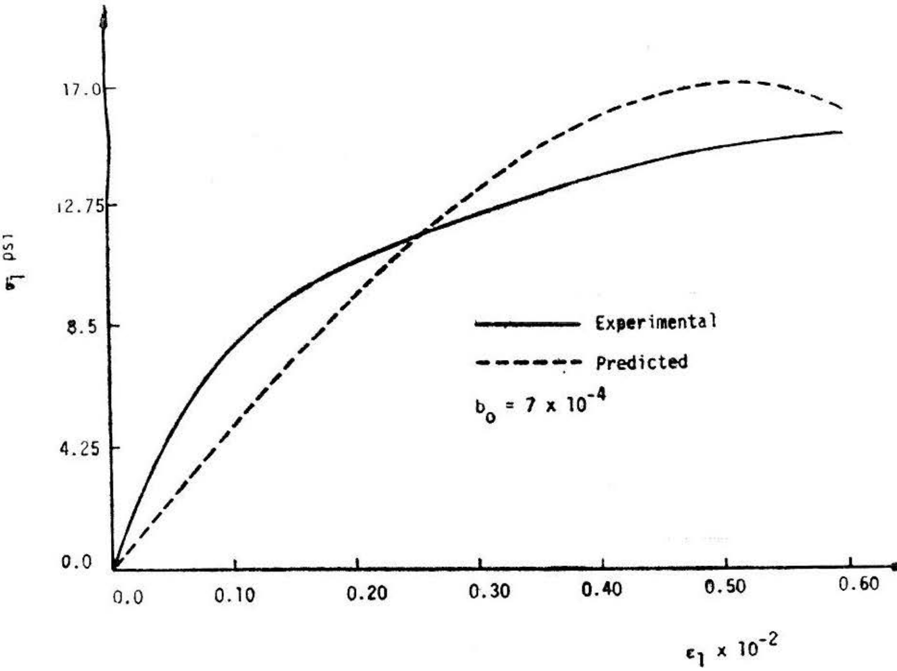


FIGURE 8 Comparison Between Experimental and Predicted Curves for Simple Shear Test for σ_1 1 kg/cm² = 14.2 psi)

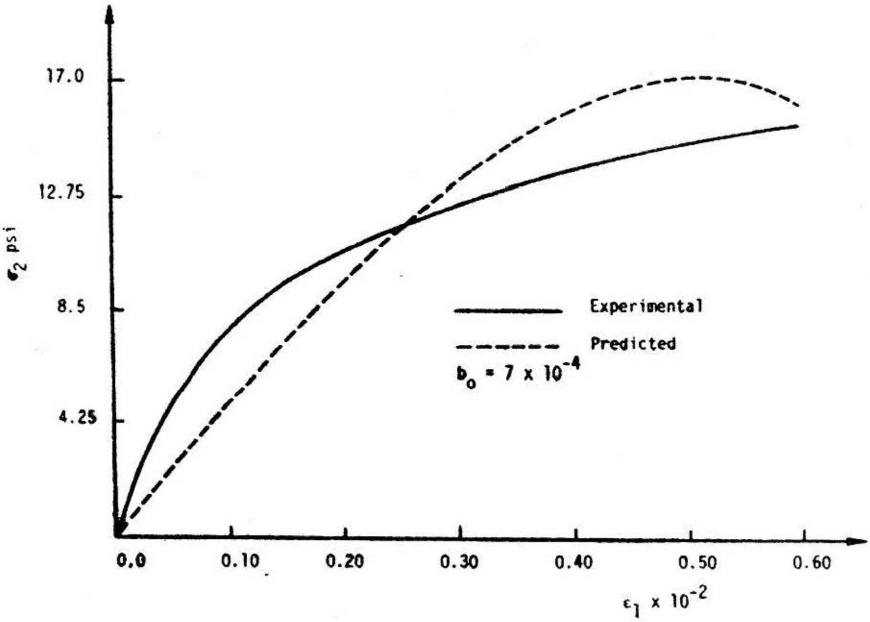


FIGURE 9 Comparison Between Experimental and Predicted Curves for Simple Shear Test for σ_2 (1 kg/cm² = 14.2 psi)

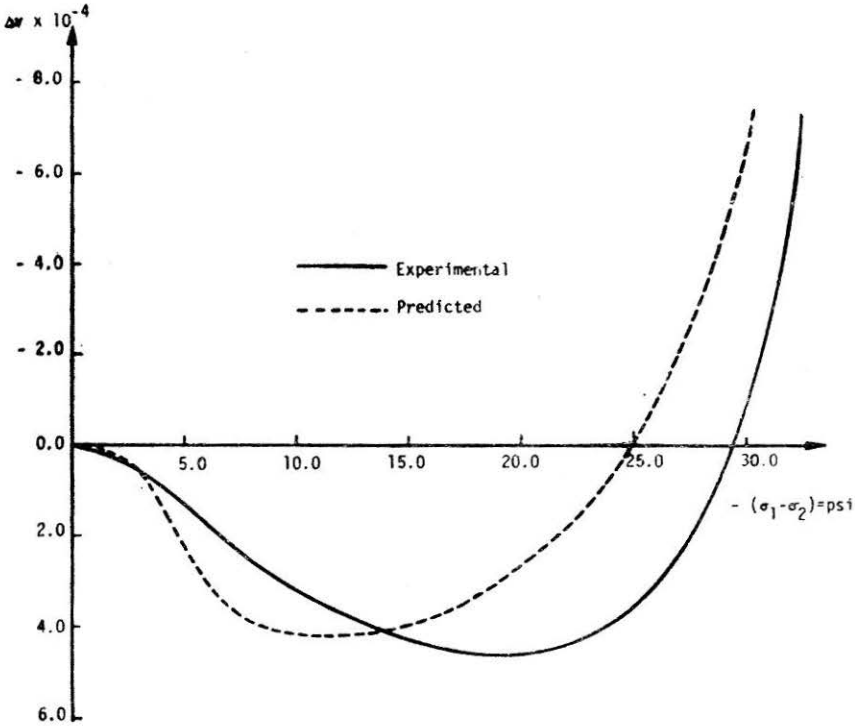


FIGURE 10 Prediction of Dilantancy for Ottawa Sand
(1 kg/cm² = 14.2 psi)

Thus, this model may not be very accurate for certain stress paths. Saleeb and Chen (1980) pointed out that a third order hyperelastic model is very sensitive to the change in material constants associated with the higher order terms in stress-strain relation (Eq. 7), especially for proportional loading paths and at high stress levels. This is because the "cubic" stress-strain curve either exhibits inflection points or shows much softer behavior than experimentally observed response. To demonstrate whether the reduction of number of material constants from nine to five will improve this situation, is presently under investigation.

Effect of Initial State

The stress-strain behaviour of any granular soil may be significantly affected by the initial state or confining pressure. With increased confining pressure, soil usually becomes more stiff and, as a result, can sustain more load to reach a given level of strain. This important aspect of material behaviour is included in the present model by incorporating the term b_o (initial strain) in the formulation. The qualitative demonstration of the influence of b_o on the stress-strain relation of granular material is shown in Figs. 11 to 14. It is seen that for given strain level, the stress increases with increased b_o, i.e., the material becomes stiffer which is consistent with observed behaviour.

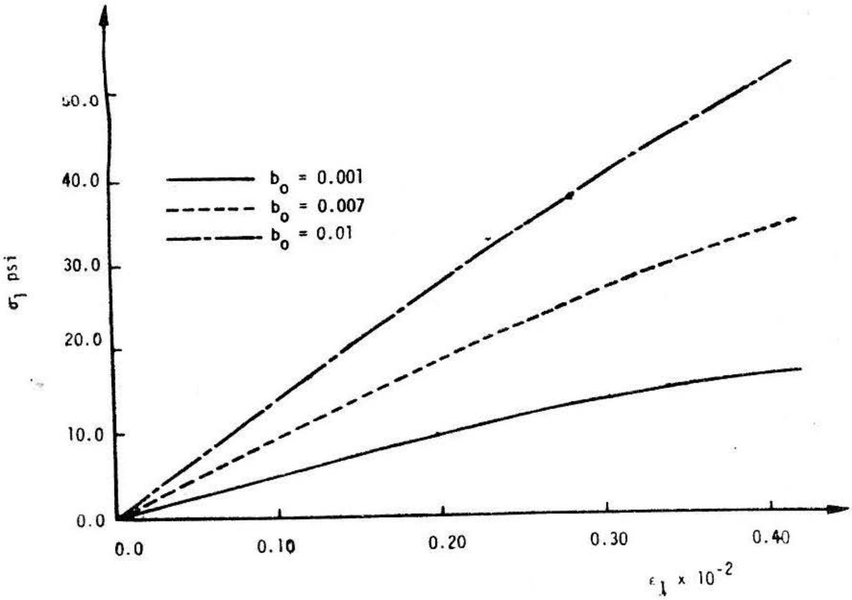


FIGURE 11 Effect of Initial State (b_0) Under Simple Shear for σ_1 (1 kg/cm² = 14.2 psi)

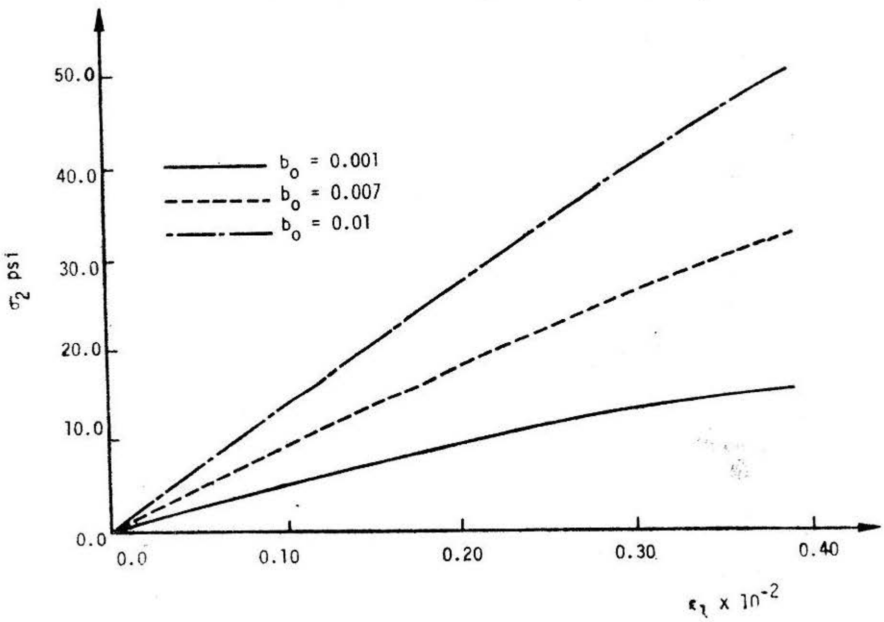


FIGURE 12 Effect of Initial State (b_0) Under Shear for σ_2 (1 kg/cm² = 14.2 psi)

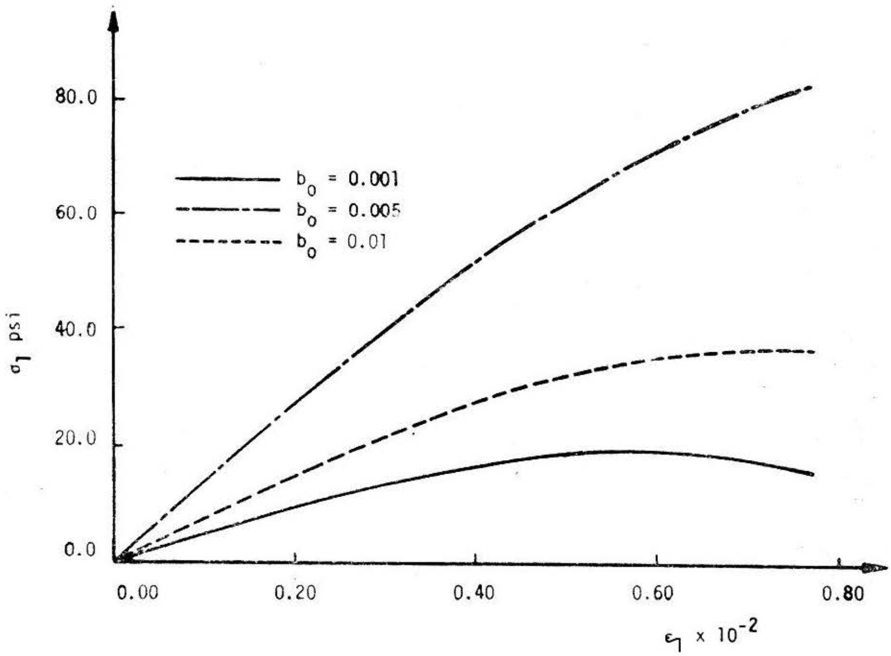


FIGURE 13 Effect of Initial State (b_0) Under Triaxial Shear for σ_1 (1 kg/cm² = 14.2 psi)

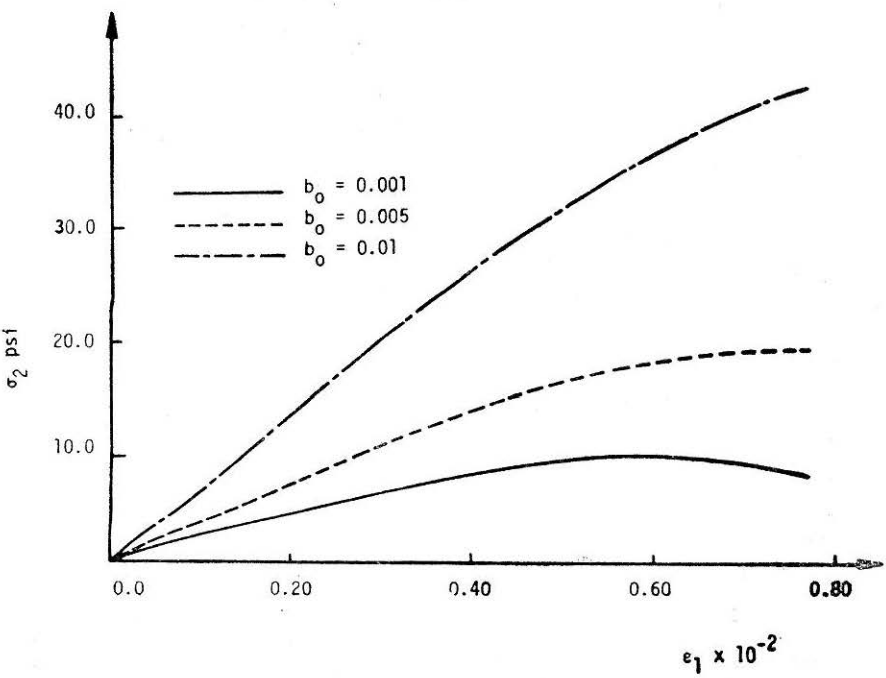


FIGURE 14 Effect of Initial State (b_0) Under Triaxial Shear for σ_2 (1 kg/cm² = 14.2 psi)

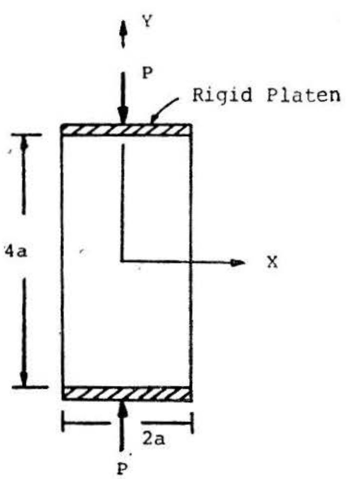
Boundary Value Problem

In the preceding section the efficiency of the proposed hyperelastic model in predicting stress-strain response of granular materials for various stress paths is demonstrated. To verify its practical applicability, this model has been implemented in a finite element computer program capable of solving soil-structure interaction problems with two-dimensional idealizations. Typical results of a boundary value problem are presented in this section.

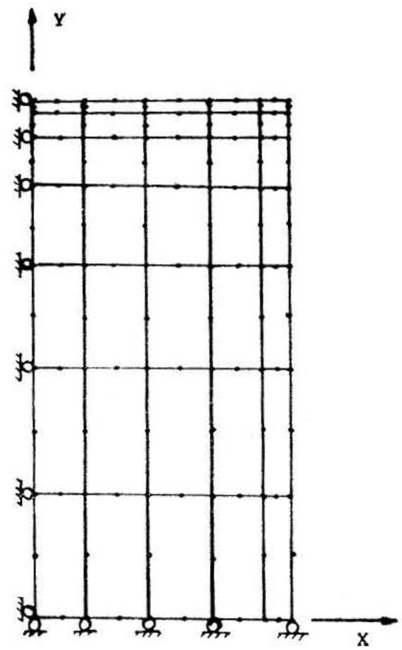
Figure 15(*a*) depicts plane strain idealization of a Conventional Triaxial (CT) test. Chang et. al., (1967) solved a similar problem to illustrate application of their second order hyperelastic model. It may be noted that the plane strain model is probably not a very accurate representation of the CT test. However, because the objective here is to test the essential features of the constitutive model, plane strain representation should suffice. The finite element mesh used in the analysis is shown in Fig. 15(*b*). The advantage of symmetry is utilized in constructing the mesh.

A solution of the aforementioned boundary value problem will require specialization of Eq. (7) for plane strain condition. A similar approach as outlined previously for triaxial shear test is adopted to accomplish this task.

The loads are applied in increments. First the specimen is subjected to a uniform hydrostatic strain state, that is $\varepsilon_{11} = \varepsilon_{22} = 0.00068$ and $\varepsilon_{ij} = 0$ for $i \neq j$, then the load P is applied to the rigid platen in ten equal



(a) Schematic of Conventional Triaxial Test (Confining Pressure not shown)



(b) Finite Element Mesh

FIGURE 15 Plane Strain Idealization of Conventional Triaxial Test

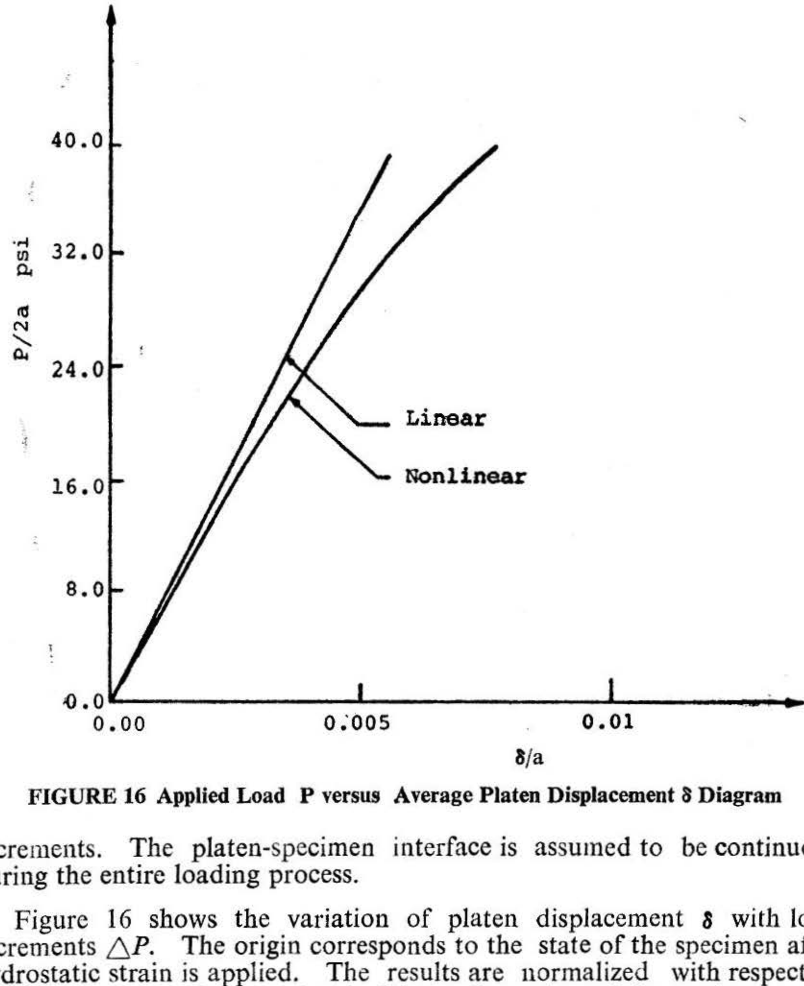


FIGURE 16 Applied Load P versus Average Platen Displacement δ Diagram

increments. The platen-specimen interface is assumed to be continuous during the entire loading process.

Figure 16 shows the variation of platen displacement δ with load increments $\triangle P$. The origin corresponds to the state of the specimen after hydrostatic strain is applied. The results are normalized with respect to constant "a" (Fig. 15) associated with the dimension of the specimen analyzed. As can be expected, in Fig. 16 the effect of nonlinearity is seen to be increasingly predominant at higher load level. For instance, at $P/2a = 16$, the ratio of platen displacements for nonlinear and linear material behaviour is 1.102. For $P/2a = 32$, this ratio is 1.148.

Qualitatively, a representative response of granular material under monotonic loading situation is observed in this figure.

Conclusions

A third order hyperelastic constitutive model is presented for granular material. The model is derived from an assumed fourth order energy density function including eight constants. These eight constants are reduced to five independent material parameters by imposing certain conditions observed in the laboratory.

The parameters α_1, α_2, α_3, α_4 and α_6 are derived for Ottawa sand using the test data reported by Chang et. al., (1967) and a least square curve fitting procedure. The predicted stress-strain relationships are compared with the corresponding experimental values for a number of stress paths and reasonable agreements are observed for all the stress paths. In addition, dilatant behavior of the material is predicted and compared with experimental data. A reasonable agreement is observed over the entire range.

This model also includes the effect of initial state on the material behaviour. The study of the effect of the initial state qualitatively shows that the material becomes stiffer with increased b_0 which is consistent with observed behaviour.

The model is implemented in a finite element computer program based on plane strain idealization. For illustration, response of dry Ottawa sand in a Conventional Triaxial test is predicted. Numerical results show that the proposed model can successfully represent some essential features of granular soils under monotonic loading.

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