Finite Element Analysis of Rafts Resting on Elastic Half Space

by

K. G. Sharma*

A. K. Nagpal*

M. K. Garg**

Introduction

All civil engineering structures are built on soil or rock bed and the live and dead loads acting on such structures are transmitted to the soil and rock through the individual footings or foundation strips and rafts. Because of this load transfer, contact pressures are developed at the interface between the foundation structure (footing, raft) and the soil. Three methods are available to determine the contact pressure :

(*i*) Conventional method, which assumes the foundation structure to be rigid and linear contact pressure distribution.

(*ii*) Subgrade reaction method which is based on Winkler's hypothesis. In this method it is assumed that the contact pressure, p, at a point is proportional to the vertical displacement, w, of that point or simply

$$p = k\,w \qquad \ldots(1)$$

in which k is the prportionality constant called modulus of subgrade reaction.

(*iii*) Elastic half space method in which soil media is considered as continuum and material behaviour is assumed to be elastic.

The simplified assumptions of conventional method can be considered satisfactory for preliminary studies only and should not be used for the analysis if important structures since displacement compatibility is not taken into account. The behaviour of both soil and rock foundations differs considerably from Winkler's hypothesis. This happens in case of a rigid raft. The subgrade reaction method results in uniform settlement and therefore, uniform contact pressure, whereas the actual contact pressure is parabolic. Elastic half space method represents the realistic behaviour of the structure.

In the present paper, finite element solution of foundations using elastic half space method has been presented assuming perfectly smooth raft base. The finite element formulation is general and applies to any

* Assistant Professor
** Former graduate student } Department of Civil Engineering, Indian Institute of Technology, New Delhi-110016, INDIA.

(*This paper was received in October 1983 and is open for discussion till the end of June 1984*).

shape of raft in plan, e.g. rectangular, circular etc. The results predicted by the method have been compared with the analytical solutions available in literature.

Elastic Half Space Method

It is a common experience that in the case of soil media, surface deflections occur not only under the loaded area but also within certain zone outside the loaded area. Boussinesq and Cerruti gave solutions (stresses, displacements) for vertical and horizontal point loads on the elastic half space, respectively. These solutions were integrated to obtain solutions for distributed loading on the elastic half space. Poulos and Davis (1974) have compiled various analytical solutions available in literature for distributed loading on elastic half space. For rafts resting on isotropic elastic half space, analytical solutions in the form of expressions and charts have been derived by Borowicka (1939) for circular rafts, by Gorbunov—Possadov and Serebrjanyi (1961) for rectangular and square rafts.

Brown (1969) and Teng (1974) used finite difference method to analyse circular and rectangular rafts. The earliest application of the finite element method to rectangular rafts resting on elastic half space is due to Cheung and Zienkiewicz (1965) and Cheung and Nag (1968). They considered 4-noded rectangular plate bending elements and assumed that the contact pressure is uniformly distributed around each node. Svec and Gladwell (1973) developed a refined triangular plate bending element resting on the elastic half space. Buragohain and Shah (1981) formulated a half space element for the analysis of rafts of any shape. In the present paper, eight noded isoparametric quadrilateral plate bending element (Hinton and Owen, 1977) has been used to represent any shape of the raft. The plate element takes into consideration shear deformations, therefore, the formulation can be applied to thick rafts as well.

Finite Element Formulation

Plate Bending Element

Isoparametric quadrilateral plate bending element (Hinton and Owen,

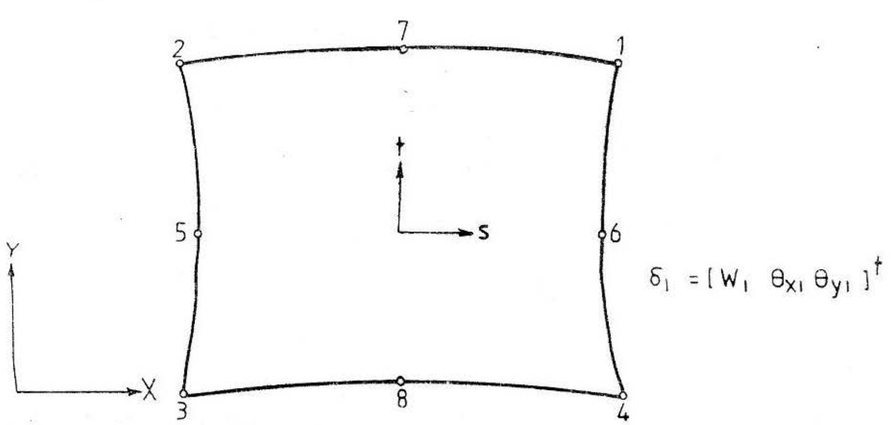


FIGURE 1 Parabolic Isoparametric Plate Bending Element.

1977) has been used for the present analysis. This element consists of eight nodes (Fig 1). The degrees of freedom at each node are lateral deflection w, two average rotations θ_x and θ_y (Fig 2).

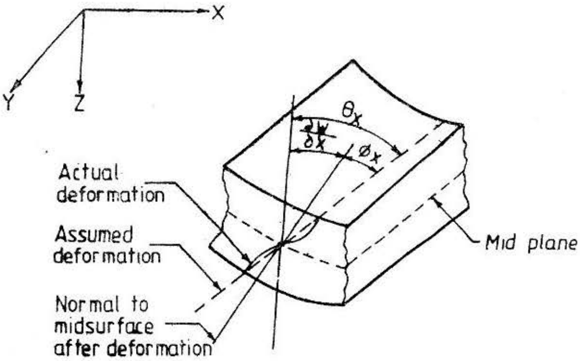


FIGURE 2 Deformation of the Cross-Section of Plate.

The displacement vector $\{d\}$ at a point is expressed as

$$\{d\} = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} + \phi_x \\ \dfrac{\partial w}{\partial y} + \phi_y \end{Bmatrix} \tag{2}$$

and

$$\{\phi\} = \begin{Bmatrix} -\phi_x \\ -\phi_y \end{Bmatrix} \quad \ldots(3)$$

In Equation (3), ϕ_x and ϕ_y represent the average shear deformations in x and y directions, respectively.

The displacement variation over the element is defined in terms of the nodal displacement components by the following expression :

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{8} [N_i] \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \quad \ldots(4)$$

$$= \sum_{i=1}^{8} [N_i] \{d_i\}$$

where $[N_i] = N_i [I_3]$

and $[I_3]$ is the 3×3 identity matrix and N_i is the shape function for the node 'i'.

The strain matrix which relates strain components to nodal displacement vector is given by the relation

$$\{\epsilon\} = [B]\{d\} \qquad \ldots(5)$$

where

$\{\epsilon\}$ is the strain vector,

$\{d\}$ is the displacement vector, and

$\{B\}$ is the element strain matrix.

Equation (5) cane be written as

$$\{\epsilon\} = \sum_{i=1}^{8} [B_i]\{d_i\} \qquad \ldots(6)$$

where

$$[B_i] = \left[\frac{B_{fi}}{B_{si}}\right] = \begin{bmatrix} 0 & -\frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial N_i}{\partial y} \\ 0 & -\frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} \\ \hline \frac{\partial N_i}{\partial x} & -N_i & 0 \\ \frac{\partial N_j}{\partial y} & 0 & -N_i \end{bmatrix} \qquad \ldots(7)$$

in which

$[B_{fi}]$ = Strain matrix associated with bending deformations,

$[B_{si}]$ = Strain matrix associated with shear deformations, and

$$\{d_i\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix}$$ is the displacement vector at node 'i'.

The stress-strain relationship for an isotropic elastic material may be written in the form

$$\{\sigma\} = [D]\{\epsilon\} \qquad \ldots(8)$$

where $\{\sigma\}$ is the stress vector and $[D]$ in the matrix of the elastic constants given by

$$[D] = \begin{bmatrix} \frac{E_p h^3}{12(1-\nu_p^2)} & \frac{E_p h^3}{12(1-\nu_p^2)} & 0 & 0 & 0 \\ \frac{E_p h^3}{12(1-\nu_p^2)} & \frac{E_p h^3}{12(1-\nu_p^2)} & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu_p) E_p h^3}{24(1-\nu_p^2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{E_p h}{2.4(1+\nu_p)} & 0 \\ 0 & 0 & 0 & 0 & \frac{E_p h}{2.4(1+\nu_p)} \end{bmatrix} \quad (9)$$

where E_p, ν_p are Young's Modulus and Poisson's ratio of plate and h is its thickness.

The plate element stiffness matrix is then obtained as

$$[K_p] = \int \int [B]^T [D] [B] \, h \, dx \, dy \qquad ...(10)$$

The integration of Equation (10) is carried out numerically (Zienkiewicz, 1977). The stiffness matrices for each element are assembled to get overall (global) plate bending stiffness matrix.

Soil Stiffness Matrix

The contact pressure distribution in an element is represented in terms of shape functions as

$$p(x, y) = \sum_{i=1}^{8} N_i p_i \qquad ...(11)$$

where N_i are shape functions and p_i are the nodal contact pressures.

By integrating Boussinesq solution for a point load, vertical deflection at node (x_1, y_1) due to the contact pressure $p(x, y)$ on an element area can be written as

$$(x_1, y_1) = \frac{(1-\nu_s^2)}{\pi E_s} \int \int \frac{p(x, y) \, dx \, dy}{\sqrt{(x-x_1)^2 + (y-y_1)^2}} \qquad ...(12)$$

where E_s, ν_s are the Young's modulus and Poisson's ratio of soil respectively.

Substituting for $p(x, y)$ from Equation (11) into Equation (12), we get

$$w(x_1, y_1) = \frac{(1-\nu_s^2)}{\pi E_s} \int_{-1}^{1} \int_{-1}^{1} \frac{[N] \, ds \, dt \, [J]}{\sqrt{(x-x_1)^2 + (y-y_1)^2}} \{p\} \qquad ...(13)$$

where (x, y) are the cartesian coordinates of Gauss points, $[J]$ is the determinant of Jacobian matrix and (s, t) are the local coordinates.

For all the nodes constituting the whole raft, Equation (13) takes the form

$$\{W\} = \sum_{n} \frac{(1-\nu_s^2)}{\pi E_s} \int_{-1}^{1} \int_{-1}^{1} \frac{[N]\, ds\, dt\, [J]}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \{p\} \qquad \ldots(14)$$

Where n is the total number of elements in the raft and $\{W\}$ is the global nodal vertical displacement vector.

In matrix form Equation (14) can be written as

$$\{W\} = [\bar{N}]\{P\} \qquad \ldots(15)$$

where $\{P\}$ is global nodal contact pressure vector.

The load vecor $\{f\}$ due to the contact pressures can be written as

$$\{f\} = \iint [N]^T p(x, y)\, dx\, dy \qquad \ldots(16)$$

substituting for $p(x, y)$ from Equation (11) we get

$$\{f\} = \iint [N]^T [N]\, dx\, dy\, \{p\} \qquad \ldots(17)$$

Assembling Equation (17) for all the elements, one obtains

$$\{F\} = [A]\,[P] \qquad \ldots(18)$$

whers $\{F\}$ is the global nodal force vector.

Solving for $\{P\}$ from Equation (15)

$$\{P\} = [\bar{N}]^{-1}\{W\}$$

Premultiplying both sides by $[A]$, we get

$$[A]\,\{P\} = [A]\,[\bar{N}]^{-1}\{W\}$$

or

$$\{F\} = [K_s]\,\{W\} \qquad \ldots(19)$$

where $[K_s] = [A]\,[\bar{N}]^{-1}$ is the required global soil stiffness matrix using elastic half space method.

Combined Stiffness Formulation

The soil stiffness matrix $[K_s]$ is to be combined with that of the raft, $[K_p]$. Let $\{R\}$ be the vector of externally applied nodal loads on the raft, then $(\{R\}-\{F\})$ is the effective externally applied load on the raft at the nodal points, i.e,

$$\{R\}-\{F\} = [K_p]\,\{d\} \qquad \ldots(20)$$

Substituting for $\{F\}$ from Equation (19) and rearranging the terms we get

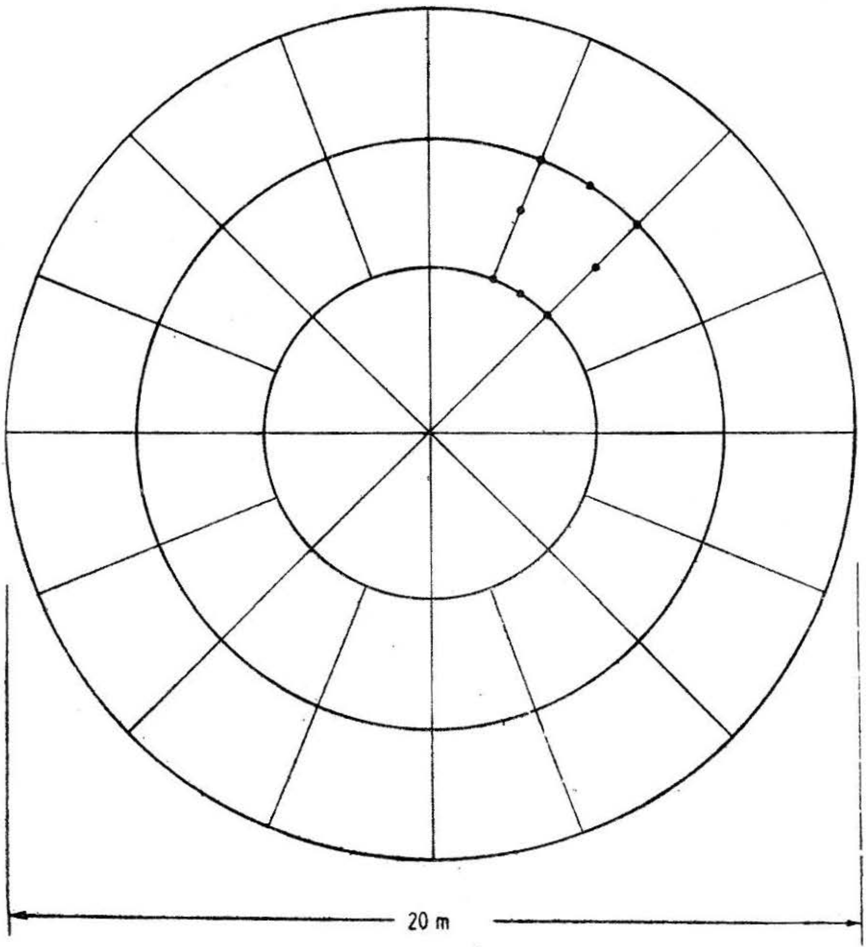
$$[K_p]\{d\}+[K_s]\{W\}=\{R\} \qquad ...(21)$$

Since the vertical displacement vector $\{W\}$ is also included in $\{d\}$ both the terms on left hand side can be combined together to give

$$[K]\{d\} = \{R\} \qquad ...(22)$$

where $[K]$ is the over all stiffness matrix taking into account both the raft and soil stiffness.

Once the nodal displacements are calculated using Equation (22), stresses (bending moments and shear forces) in the raft and contact pressures are computed using Equiations (6), (8) and (15).



$E_p = 2 \times 10^6\ t/m^2$, $\nu_p = 0.3$ Number of elements = 40

$E_s = 4000\ t/m^2$, $\nu_s = 0.15$ Number of nodes = 137

FIGURE 3 Finite Element Discretization for Circular Raft

Results and Discussion

A computer program has been developed on ICL 2960 based on the finite element formulation presented in the last section. The program is capable of handling rafts of any shape. Material properties, thickness of raft can be varied from element to element. Rafts with cut-outs can also be analysed.

For the numerical integration of Equation (14), it has been found that 4×4 Gaussian quadrature rule gives sufficiently accurate results. Equation (10) was integrated using 2×2 Gaussian quadrature rule (Hinton and Owen, 1977).

Using the program, rectangular soil areas subjected to uniformly distributed loading were analysed. The results were compared with the analytical (Poulos and Davis, 1974) and it was found that both the results are very close (Garg, 1983) with maximum error in displacements of 0.44 per cent.

A circular raft of different rigidities resting on soil and subjected to uniformly distributed loading was analysed to obtain displacements, contact pressures, bending moments. The raft is of 20m diameter and is discretized into 40 elements as shown in Fig 3. For the soil stiffness matrix, all the forty elements are needed. The material properties used were as follows :

Plate : $E_p = 2\times10^6$ t/m^2

$\nu_p = 0.15$

Soil : $E_s = 4000$ t/m^2

$\nu_s = 0.15$

The rigidity of circular raft is expressed by a factor, K (Brown,

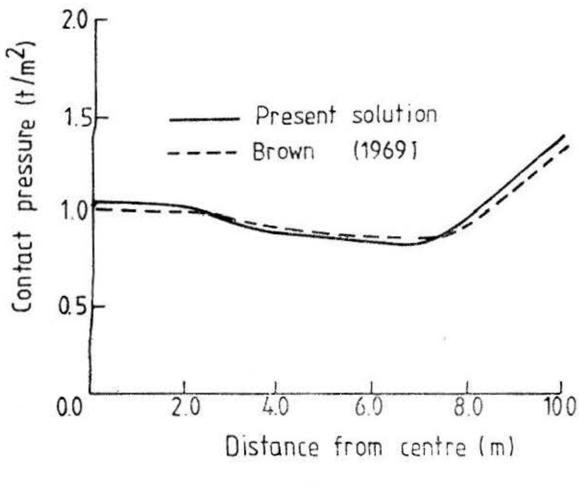


FIGURE 4 Contact Pressure Variation for Uniformly Loaded Circular Flexible Raft (K = 0.1).

1969) as

$$K = \frac{E_p}{E_s} (1-\nu_s^2) \left(\frac{h}{a}\right)^3 \qquad \ldots(23)$$

where a is the radius of the raft.

For the present analysis, solutions were obtained for three values of K, viz. 0.1 (corresponding to flexible raft), 1.0 and 10.0 (corresponding to rigid raft). The results have been compared with those given by Brown (1969). In Table I are presented vertical displacements at the centre of raft by the finite element method and by analytical method for three values of K. It is seen that both the results are close with maximum error of 2.86 per cent.

Contact pressure variations have been plotted in Figs. 4 and 5 for $K = 0.1$ and 1.0, respectively. Analytical results have also been plotted in these figures. Both analytical and numerical solutions compare very well.

TABLE 1

Displacements at the Centre of the Circular Raft

K	Displacement at Centre in cm. Analytical method	Displacement at Centre in cm. Present method	Percentage error
0.1	0.4789	0.4652	2.86
1.0	0.4054	0.3989	1.60
10.0	0.3736	0.3643	2.49

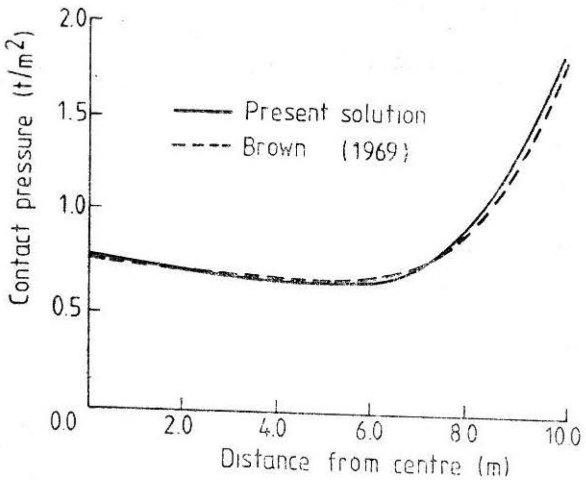


FIGURE 5 Contact Pressure Variation for Uniformly Loaded Circular Rigid Raft (K = 1.0).

Radial and tangential moments have been plotted in Figs. 6 and 7 for $K = 0.1$ and in Figs. 8 and 9 for $K = 1.0$, respectively. Again the two solutions match closely.

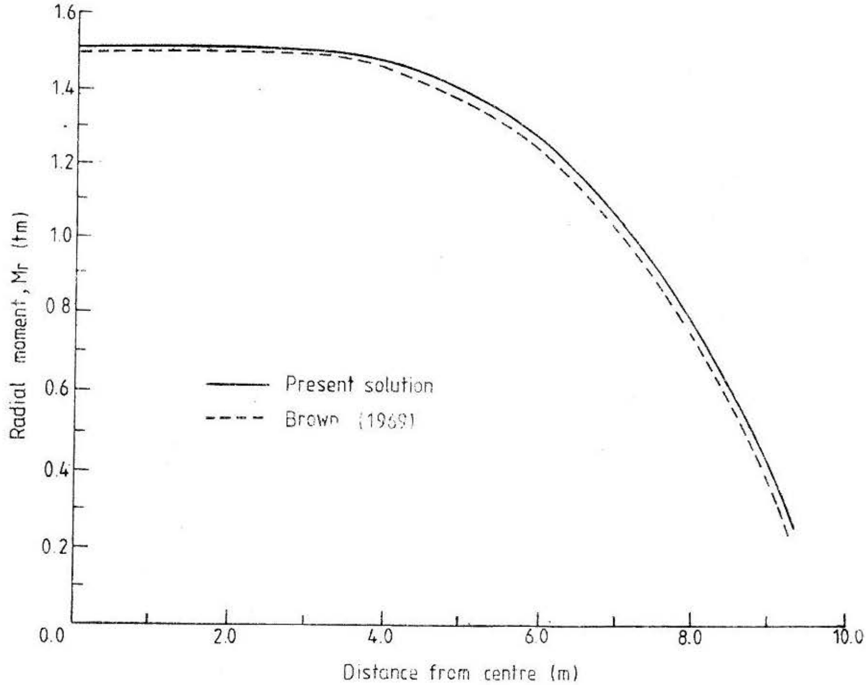


FIGURE 6 Radial Moment (M_r) Variation for Uniformly Loaded Circular Flexible Raft (K = 0.1).

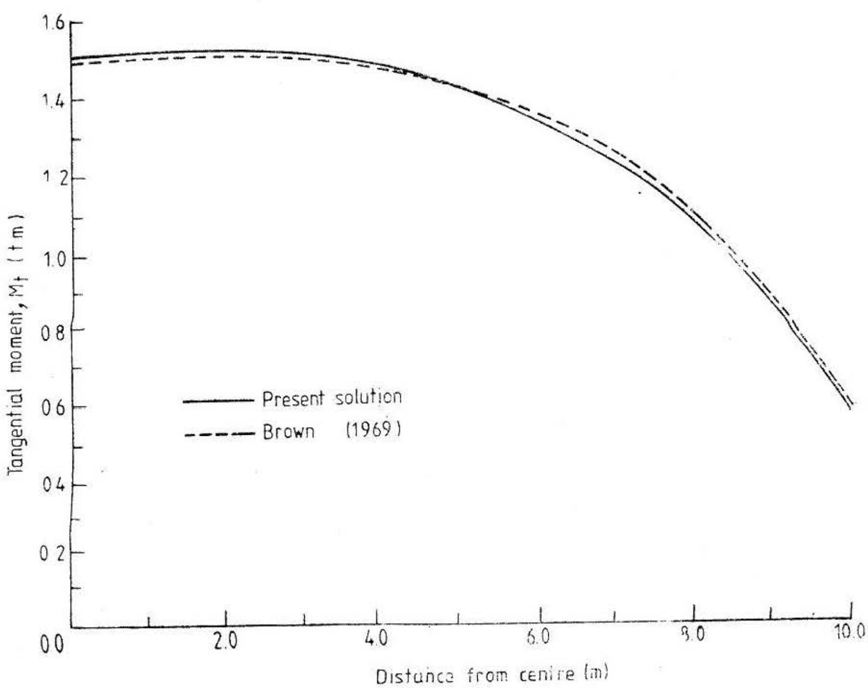


FIGURE 7 Tangential Moment (M_t) Variation for Uniformly Loaded Circular Flexible Raft (K = 0.1).

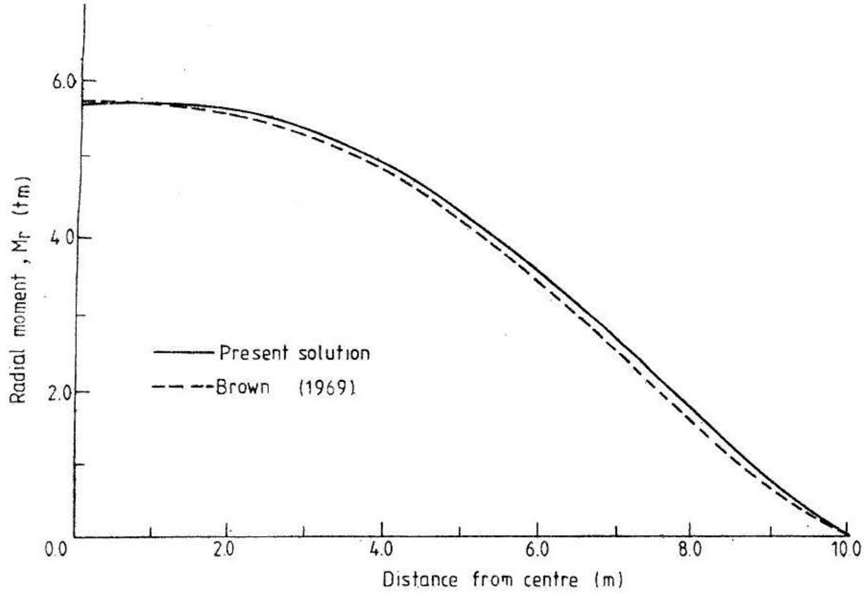


FIGURE 8 Radial Moment (M_r) Variation for Unformly Loaded Circular Rigid Raft (K = 1.0).

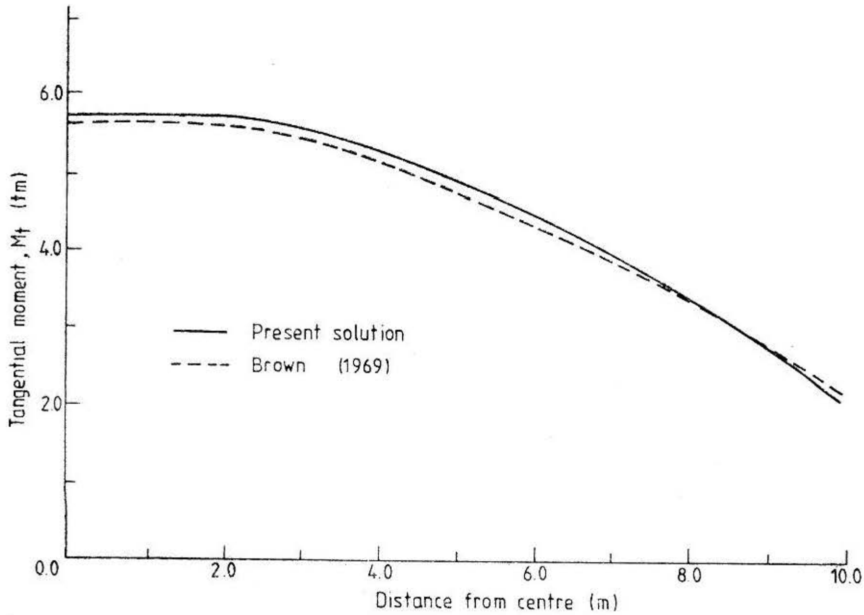


FIGURE 9 Tangential Moment (M_t) Variation for Uniformly Loaded Circular Rigid Raft (K = 1.0).

Conclusions

The finite element formulation for rafts resting on elastic half space has been presented in this paper. The formulation applies to any shape of the raft. The numerical results compare very well with the analytical results available in the literature.

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