Short Communications

Vertical Stresses and Displacements due to Distributed Loading by Numerical Solutions

by

K. G. Sharma*

G. Venkatappa Rao*

Introduction

The equations for stresses due to single point load on an idealized soil mass were first completely presented by Boussinesq (1885). As the load, in most civil engineering structures is actually spread over a certain area, solutions are necessary for different shapes of loaded area and different loading configurations. Several solutions are currently available either in the form of tables and charts or FEM solutions, for loads placed on layered media. Use of charts, tables, etc. is cumbersome and time consuming whereas FEM solutions are proving to be costlier due to large capacity computers needed. Simplified procedures are, therefore, called for, for evaluation of stresses, strains and displacements in soil systems which are usually assumed to be linear elastic.

An attempt is made in the present study to develop simple numerical solutions for obtaining vertical stresses, strains and displacements in a semi-infinite mass due to distributed vertical and shear loads of circular and rectangular shapes. The same are extended for multi-layered media, by using equivalent thickness criteria. The solutions developed can easily be programmed and run on a Mini/Microcomputer of a memory of about 1 K.

Development of Solutions

Calculation of Stresses and Displacements due to Distributed Normal and Horizontal Loads on a Single Layer

For a point load acting normally on the surface of a semi-infinite homogeneous, isotropic, elastic mass (Figure 1 (a)), Boussinesq's solutions for vertical normal stress and vertical deflections respectively are

$$\sigma_z = \frac{3Pz^3}{2\pi\, R^5} \qquad \ldots (1)$$

and

$$w = \frac{P\,(1+\nu)}{2\,\pi ER}\left[2\,(1-\nu)+\frac{z^2}{R^2}\right] \qquad \ldots (2)$$

---

* Assistant Professor of Civil Engineering, Indian Institute of Technology, Delhi, New Delhi-110016, INDIA.

This Communication was received in September, 1982 and is open for discussion till the end of May, 1983.

For a horizontal point load acting along the surface of a semi-infinite mass (Figure 1 (b)) Cerutti's solution (Poulos and Davis, 1974) for vertical normal stress and vertical deflections respectively are

$$\sigma_z = \frac{3Pxz^2}{2\,\pi R^5} \qquad \dots (3)$$

and

$$w = \frac{P\,(1+\nu)}{2\,\pi\,E\,R}\left[\frac{xz}{R^2} + \frac{(1-2\nu)x}{(R+z)}\right] \qquad \dots (4)$$

Equations (1) to (4) are only valid for a surface point load. However, in actual practice, surface loads are usually distributed over a certain area. Though analytical solutions for such distributed loads are available, their actual use is rather laborious and time consuming. Simpler solutions can be sought by use of numerical integration. In the present work, numerical integration of equations (1) to (4) has been carried out using the multidimensional numerical integration rules as per Abramowitz and Stegun (1965).

For a distributed load over a circular area either ninepoint integration rule or 21-point integration rule can be conveniently used. The general integration rule is

$$\int\int f\,(x, y)\;dxdy = \pi\,a^2 \sum_{i=1}^{n} w_i\,f\,(x_i, y_i) \qquad \dots (5)$$

where $f\,(x, y)$ = function to be integrated

a = radius of the circle

w_i = weighting function

x_i, y_i = co-ordinates of the integration points

n = total number of integration points

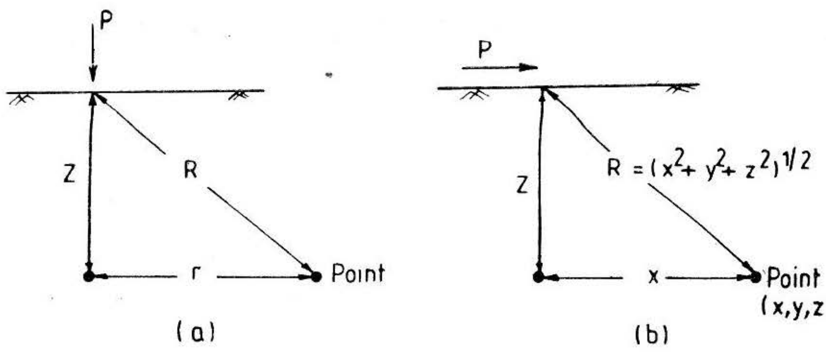

FIGURE 1 Definition Sketch for Boussinesq's solution

For the 21 point integration rule ($n = 21$) used in the present study the co-ordinates of integration points (Figure 2(a)) and the weighting functions are

x_i, y_i	w_i
$0, 0$	$1/9$
$\left(\sqrt{\frac{6-\sqrt{6}}{10}}\, a \cos \frac{2\pi k}{10},\ \sqrt{\frac{6-\sqrt{6}}{10}}\, a \text{ Sin } \frac{2\pi k}{10}\right)$	$\frac{16+\sqrt{6}}{360}$
$\left(\sqrt{\frac{6+\sqrt{6}}{10}}\, a \cos \frac{2\pi k}{10},\ \sqrt{\frac{6+\sqrt{6}}{10}}\, a \sin \frac{2\pi k}{10}\right)$	$\frac{16-\sqrt{6}}{360}$

where $k = 1, \ldots\ldots\ldots\ldots, 10.$

For a distributed load over a rectangular area, (Fig. 2(b)), the general integration rule is

$$\int\int f(x, y)\, dx\, dy = ab \sum_{i=1}^{n} w_i f(x_i, y_i) \quad \ldots (6)$$

where a and b are the lengths of the sides of the rectangle. For a nine-point ($n = 9$) integration rule, which was used for the present work, the

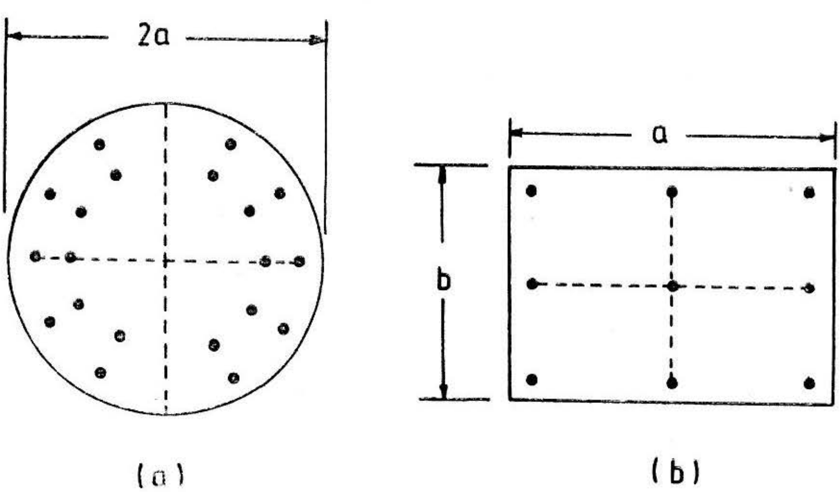

FIGURE 2 Location of points for numerical integration

co-ordinates of the integration points and the weighting functions are

(x_i, y_i)	w_i
$(0, 0)$	$16/81$
$\left(\pm \sqrt{\frac{3}{5}}\ \frac{a}{2},\ \pm \sqrt{\frac{3}{5}}\, b/2\right)$	$25/324$
$\left(0, \pm\sqrt{\frac{3}{5}}\ b/2\right)$	$10/81$
$\left(\pm \sqrt{\frac{3}{5}}\ \frac{a}{2}, 0\right)$	$10/81$

The function $f(x, y)$ in the equation (5) and (6) can be either that of stress, displacement or strain, the solutions for which is desired.

Calculation of stresses and displacements due to distributed normal load on a multi-layered media :

For obtaining stress distribution in layered, like flexible pavements a simplified method is used, where the equivalent layer approach (Odemark, 1969) is coupled with numerical integration technique, with the underlying assumption that the stresses, strains and displacements below a layer will be unchanged as long as the flexural stiffness of the layer remains constant. Thus for two layers of moduli E_1 and E_2 and thicknesses h_1 and h_2 respectively,

$$\frac{h_1^3\, E_1}{(1-\nu_1^2)} = \frac{h_2^3\, E_2}{(1-\nu_2^2)}$$

A two.layer system can be transformed to a semi-infinite space, as illustrated in Fig. 1 by replacing layer 1 with a material having the properties of the semi-infinite space and a thickness of h_e, in such a way that

$$h_e = h_1 \sqrt[3]{\frac{E_1}{E_2}\frac{(1-\nu_2^2)}{(1-\nu_1^2)}} \tag{7}$$

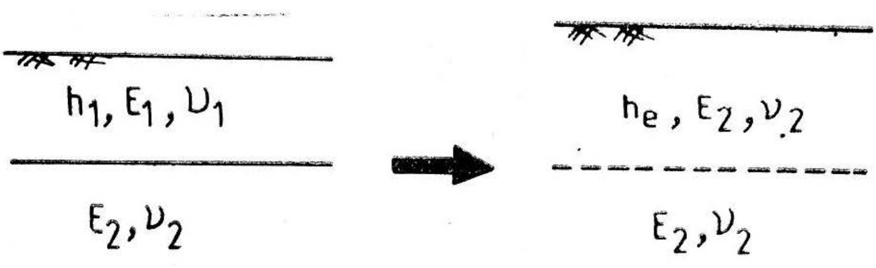

FIGURE 3 Transformation of a 2-layer system

A generalization of this principle yields the equivalent thickness of the $(n-1)$ layers above the layer n, as

$$h_{e,n} = \sum_{i=1}^{n-1} h_i \sqrt[3]{\frac{E_i \, (1-\nu_n^2)}{E_n \, (1-\nu_i^2)}} \tag{8}$$

Using the aforesaid numerical integration rules for circular and rectangular areas, computer program was developed on ICL 2960. The following cases were studied :

(*i*) Uniformly distributed vertical, and shear loads on circular area.

(*ii*) Uniformly distributed vertical and shear loads on square $(a/b = 1)$ and rectangular $(a/b = 2)$ areas.

The above cases have been analysed for normal vertical stress (σ_z), vertical displacement (w) and normal verical strain (ϵ_z) for poission's ratio $(\nu) = 0.2, 0.3, 0.4$ and 0.5, but only the results for the first two parameters have been presented herein.

Results and Discussions

As detailed in the earlier section, values of three parameters, namely, normal vertical stress, vertical displacement and normal vertical strain have been obtained using the numerical solutions developed. An attempt has been made to compare these results with available analytical solutions. Such comparison is presented in the following two steps. Firstly, for a constant value of Poisson's ratio $(\nu = 0.3)$ the three parameters were compared through tables for z/a ranging from 0 to 2 and r/a ranging from 0 to 2. In next, the effect of Poisson's ratio has been investigated for the different cases through figures.

Uniformly distributed vertical load on a circular area

The variation of normal vertical stress (σ_z/σ_o), and vertical displacement (w/a) with depth (z/a) for different values of (r/a) are presented in Tables 1 *a* and *b*. Included in these tables are the values obtained by the analytical method (Poulos and Davis, 1974) as well as those by 21-point integration rule. Comparison of the results obtained by the two methods for (z/a) upto 1.0 and for values of (r/a) from 0 to 2.0, reveals that in general there is a zone in the vicinity of the loaded area, wherein the results for the stress, displacement as well as strain do not match. The two zones are differentiated by a thick line in these as well as other tables. Whereas this zone of mismatch is relatively small for vertical displacement, it is significantly large for stress (i.e. from $(z/a) \simeq 0.6$ to 0.8. extending from $r/a = 0$ to 1.5). Outside this zone one notes perfect matching.

Uniformly distributed vertical load on a rectangular area

Table 2 presents the variation of vertical normal stress and vertical displacement beneath the corner of uniformly loaded rectangular $(a/b = 2)$ and square $(a/b = 1)$ areas. Both the analytical and numerical solutions are included in this table. Here again one notes perfect matching below a

TABLE 1*a*

Stresses due to uniformly distributed vertical load on a circular area

z/a	σ_z/σ_o					
	$r/a = 0$		1.0		2.0	
	Anal	Num.	Anal	Num.	Anal X10^{-1}	Num. X10^{-1}
0.2	0.992	4.236	0.468	1.023	0.009	0.009
0.4	0.949	1.336	0.435	0.505	0.060	0.060
0.6	0.864	0.924	0.400	0.410	—	0.161
0.8	0.756	0.766	0.366	0.367	—	0.289
1.0	0.646	0.648	0.332	0.333	0.418	0.418

TABLE 1*b*

Displacements due to uniformly distributed vertical load on a circular area for $\nu = 0.3$

z/a	$w/a \times 10^{-2}$					
	$r/a = 0$		1.0		2.0	
	Anal	Num.	Anal	Num.	Anal	Num.
0.2	1.701	2.038	1.100	1.162	0.472	0.472
0.4	1.559	1.608	1.037	1.045	0.475	0.475
0.6	1.409	1.417	0.973	0.973	—	0.478
0.8	1.265	1.267	0.909	0.909	—	0.479
1.0	1.135	1.135	0.849	0.849	0.478	0.478

depth of $z/a \simeq 0.2$ for a rectangular area for all the parameters studied and for square area this depth is around $z/a = 0.4$. The matching in the case of displacement (w/a) is excellent for all depths for both square and rectangular loaded areas.

Uniformly distributed shear load on circular area

Variations in the normal vertical stress and vertical displacement by both analytical and numerical methods are presented in Tables 3 *a* and *b* for uniformly distributed shear load on a circular area. From these tables one notices that the comparison between the values obtained for shear loading by the analytical and the the numerical methods is very similar to that observed in case of uniform vertical loading.

TABLE 2

Stresses, displacements and strains beneath the corner due to uniformly distributed vertical load on a rectangular area for $\nu = 0.3$

z/a	$a/b = 2$				$a/b = 1$			
	σ_z/σ_0		$w/b \times 10^{-2}$		σ_z/σ_0		$w/b = 10^{-2}$	
	Anal	Num.	Anal	Num.	Anal	Num.	Anal	Num.
0.0	0.250	—	0.697	0.672	0.250	—	0.500	0.499
0.2	0.244	0.269	0.635	0.642	0.249	0.293	0.481	0.488
0.4	0.218	0.223	0.562	0.563	0.240	0.252	0.446	0.447
0.5	0.200	0.201	0.526	0.526	0.232	0.235	0.428	0.428
0.6	0.182	0.182	0.493	0.492	0.223	0.223	0.409	0.409
0.8	0.148	0.148	0.432	0.432	0.200	0.199	0.372	0.372
1.0	0.120	0.120	0.381	0.381	0.175	0.175	0.338	0.338

TABLE 3*a*

Stresses due to uniformly distributed shear load on a circular area

z/a	σ_z/σ_0							
	$r/a = 0$		0.25		0.75		1.50	
	Anal	Num.	Anal	Num.	Anal	Num.	Anal	Num.
0.25	0.	0.	0.0223	0.3036	0.1595	0.2320	0.0439	0.0460
0.50	0.	0.	0.0564	0.1054	0.2201	0.2232	0.1019	0.1033
0.75	0.	0.	0.0698	0.0763	0.2028	0.2025	0.1266	0.1266
1.00	0.	0.	0.0656	0.0665	0.1688	0.1687	0.1275	0.1275
1.50	0.	0.	0.0434	0.0434	0.1071	0.1071	0.1049	0.1049
2.00	0.	0.	0.0263	0.0263	0.0670	0.0670	0.0700	0.0700

Uniformly distributed shear loading on rectangular area

In Table 4 is presented a comparison between the numerical and analytical values of normal vertical stress and vertical displacement beneath the corners of both rectangular ($a/b = 2$) and square ($a/b = 1$) areas (only the analytical values for surface displacement are available in literature).

TABLE 3b

Displacements due to uniformly distributed shear load on a circular area for $\nu = 0.3$

z/a	$w/a \times 10^{-4}$							
	$r/a = 0$		0.25		0.75		1.50	
	Anal	Num.	Anal	Num.	Anal	Num.	Anal	Num.
0.25	0.	0.	8.642	13.135	26.238	26.833	22.060	22.069
0.50	0.	0.	9.357	10.064	25.780	25.785	23.974	23.974
0.75	0.	0.	8.726	8.823	22.805	22.788	23.587	23.588
1.00	0.	0.	7.519	7.535	19.390	19.385	22.028	22.028
1.50	0.	0.	5.177	5.167	13.590	13.590	17.900	17.900
2.00	0.	0.	3.541	3.547	9.618	9.618	14.077	14.077

TABLE 4

Stresses, displacements and strains beneath the corner due to uniformly distributed shear load on a rectangular area for $\nu = 0.3$

z/a	$a/b = 2$				$a/b = 1$			
	σ_z/σ_0		$w/b \times 10^{-2}$		σ_z/σ_0		$w/b \times 10^{-2}$	
	Anal	Num.	Anal*	Num.	Anal	Num.	Anal*	Num.
0.0	0.159	0.040	0.140	—	0.159	0.025	0.094	—
0.2	0.145	0.177		0.168	0.152	0.0195		0.116
0.4	0.115	0.113		0.142	0.133	0.132		0.103
0.5	0.100	0.097		0.128	0.121	0.119		0.095
0.6	0.085	0.084		0.114	0.109	0.108		0.087
0.8	0.062	0.062		0.090	0.086	0.086		0.073
1.0	0.045	0.045		0.071	0.067	0.067		0.060

*Equations for displacement not traceable in literature.

A study of this table along with Table 2 brings out that whereas the behaviour is very similar for rectangular area, it improves in the case of a square area. Perfect matching for both square and rectangular areas is observed for all parameters below a depth of $z/a = 0.2$. In the case of a uniform vertical load such a matching was observed below $z/a = 0.4$ for square area and $z/a = 0.2$ for rectangular area.

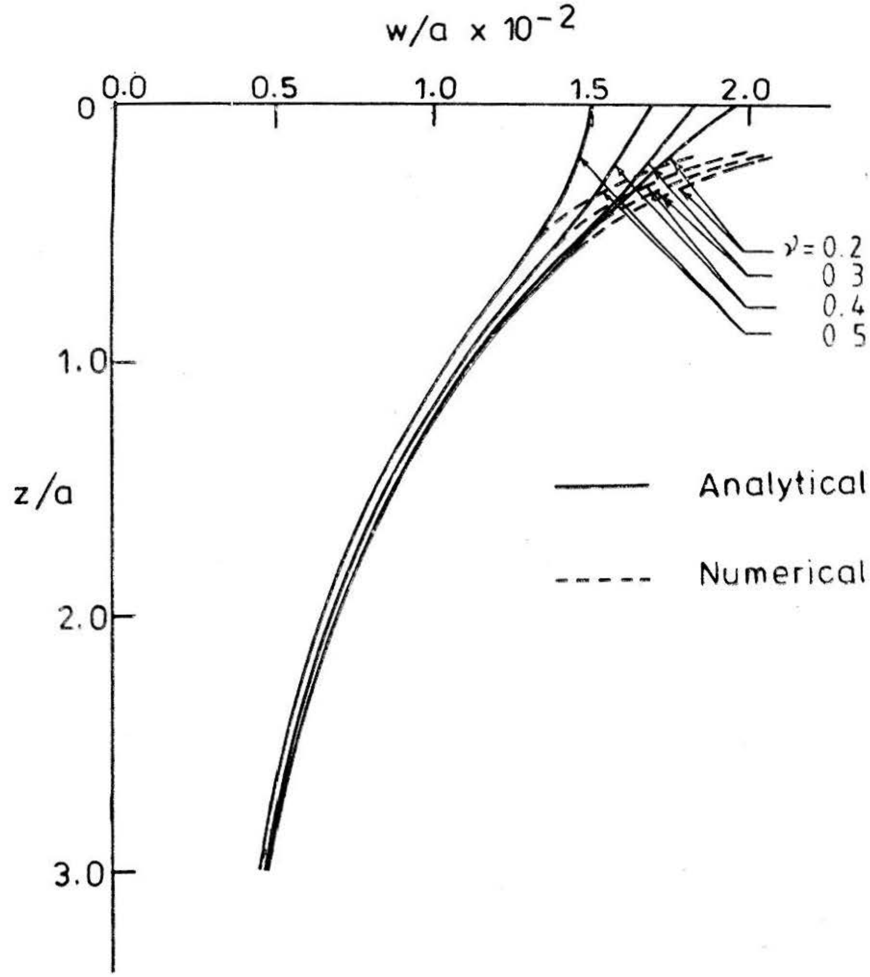

FIGURE 4 Variation of vertical displacement beneath the centre with depth for a uniformly distributed vertical load over a circular area

Influence of Poisson's ratio :

Figs. 4 and 5 present the influence of Poisson's ratio on vertical displacement for uniformly loaded circular and rectangular areas. As is common knowledge, the normal vertical stresses are not influenced by Poisson's ratio.

Fig. 4 presents the variation of displacement (w/a) below the centre of a uniformly distributed vertical load over a circular area, with depth (z/a) for values of Poisson's ratio ranging from 0.2 to 0.5. In this figure and the following ones the thick lines are those obtained by analytical method and the dotted ones are for those of the numerical technique. It is clearly noticed that below a depth of $z/a = 0.6$, there is perfect agreement between the values obtained by the two methods for all values of ν. Above this

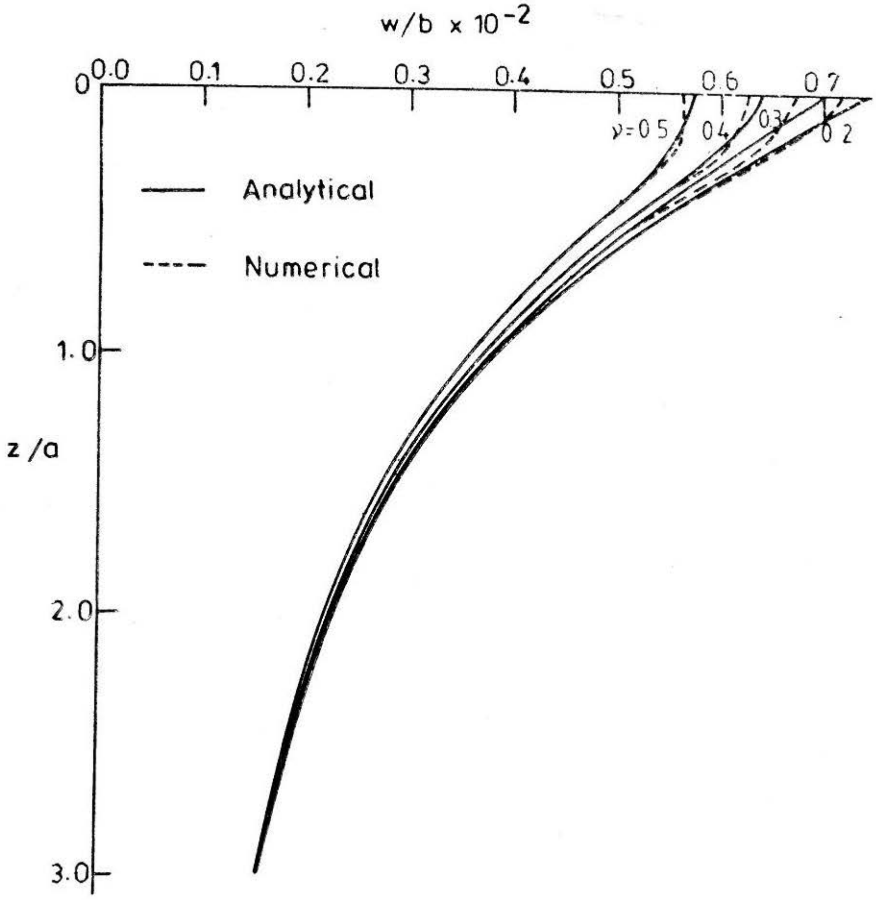

FIGURE 5a Variation of vertical displacement beneath the corner with depth for a uniformly distributed vertical load over a rectangular area

depth the values obtained by the numerical technique are generally larger.

Figs. 5(a) and (b) present ihe variation of vertical displacement beneath the corner with depth for a uniform vertical load over rectangular and square areas, respectively. In both these cases there is excellent matching for all values of ν between the results obtained by the analytical and numerical methods below $z/a = 0.4$.

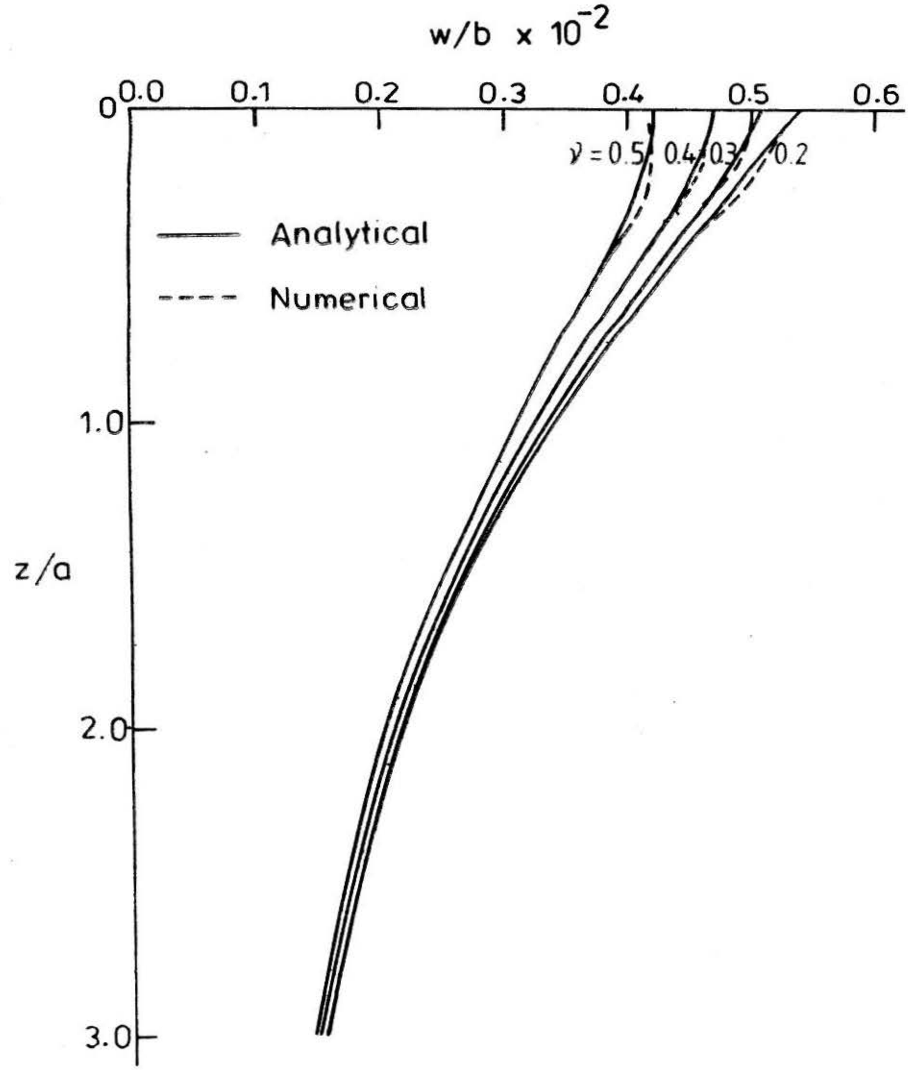

FIGURE 5b **Variation of vertical displacement beneath the corner with depth for a uniformly distributed vertical load over a square area**

Uniformly distributed circular load on three-layered media

For illustrating the results of the method presently adopted, the data used by Ullidtz and Peattie (1980) is considered, as shown in Fig. 6. Table 5 gives the results obtained by the numerical solution of the equivalent layer as well as those computed analytically by Ullidtz and Peattie (1980), by the use of the equivalent layer approach. Excellent matching is obtained at both interfaces 1 and 2, in respect of the vertical normal stress σ_z as well as vertical strain ϵ_z.

TABLE 5

Comparisons of vertical stresses and strains in 3-layered media due to uniformly distributed circular load

Interface Number	Parameter	Prediction by Ullidtz & Peattie	Prediction by Numerical Solution
1	σ_z (MP_a)	0.024	0.02395
	ϵ_z	4.93×10^{-6}	4.9335×10^{-6}
2	σ_z (MP_a)	0.012	0.01177
	ϵ_z	596×10^{-6}	595.48×10^{-6}

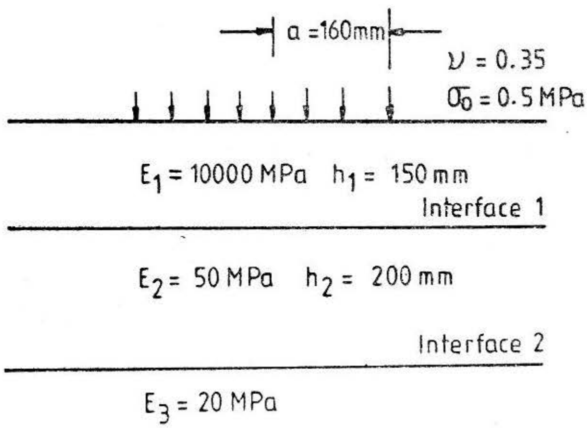

FIGURE 6 Typical uniformly distributed circular load on a 3—layer system

Conclusions

Multi-dimensional numerical integration technique has been used to obtain simple numerical solutions for stresses, strains and displacements due to distributed vertical and shear loads over rectangular and circular areas.

Except in the vicinity of loading, where singularity problems arise, the matching between the analytical solutions already available and the numerical solutions presently developed is excellent, for uniform vertical as well as shear loading for the different shapes studied.

However, when vertical stresses and strains are computed at interfaces in 3-layered media, there is excellent correspondence.

To minimise the discrepancies occuring in the stress/strain distribution descretization technique can be adopted. Such studies are in progress.

References

ABRAMOWITZ, M. AND STEGUN, I.A. (Ed) (1965), *Handbook of Mathematical Functions*, Dover Publications, New York.

ODEMARK, N. (1949), Investigations as to the Elastic properties of Soil and Design of Pavements according to the theory of elasticity, Statens Vaegrnstitut, Stock holm.

POULOS, H.G. AND DAVIS, E.H. (1974), *Elastic Solutions for Soil and Rock Mechanics*, John Wiley, New York.

ULLIDTZ, P. AND PEATTIE, K.R. (1980), Pavement Analysis by Programmable Calculators, *Proc. Am. Soc. Civ. Engrs, Jnl. Transp. Engg. Divn.*, 106: TE 5 : 581-597.

List of notations

a	= radius of circular area
$a \times b$	= size of rectangular area
E	= Young's modulus
f	= the function to be integrated
h_e	= equivalent thickness
n	= number of integrating points
P	= vertical or horizontal point load
R	$= \sqrt{x^2 + y^2 + z^2}$
r	= polar co-ordinate for circular area
w	= vertical displacement
w_i	= weighting function values of integration points
x, y, z	= cartesian co-ordinates of a point
x_i, y_i	= co-ordinates of numerical integration points
σ_o	= intensity of uniformly distributed load
σ_z	= normal vertical stress
ϵ_z	= normal vertical strain
ν	= Poisson's ratio.