Dynamic Response of Rigid Circular Surface Footings

by

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Introduction

In a simplified approach to the rigourous elastic halfspace approach, for the behaviour of a rigid circular surface footing (Figure 1) acted upon by a dynamic excitation force, the elastic half-space model is replaced by an equivalent model, represented by a mass, a spring and a dashpot, which is then used as the basic model for analysing the motion of the footing. The expressions for the equivalent spring constant (K), the equivalent damping factor (D) and a dimensionless mass ratio (B), for the various modes of vibration of the footing are given in Table 1. A perusal of these expressions reveals that a change in the dimensions of a footing alters its values of B, K ond D. Consequently this modifies the response of the footing at resonance and other operating frequencies. A quantification of such variations in the response of the footings has been attempted in this paper. The approach is based on the lumped—parameter models for the elastic half-space solutions.

Dynamic Response of Footings

The difference in the dynamic response of a footing arises fundamentally from a change in its value of mass ratio, B. The mass ratio can be affected in different ways, by a change in radius alone with the mass of the system remaining the same, by a change in the mass without any change in its radius or by a change both in the mass and the radius. The study here, considers the effect of change in the radius and the mass of the footing, separately. Using the principle of superposition, the approach could be used when there is a change both in the radius and mass of the footing.

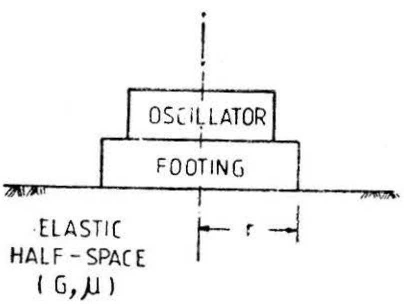

FIGURE 1 Rigid circular surface footing

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This paper was received in December 1981, and is open for discussion till the end of December 1982.

TABLE 1

Expressions for Mass Ratio, Spring Constant and Damping Factor

Mode of vibration	Mass ratio B_i	Spring constant K_i	Damping factor D_i
Vertical	$B_z = \frac{(1-\mu)}{4} \frac{m}{\rho r^3}$	$K_z = \frac{4\,Gr}{1-\mu}$	$D_z = \frac{0.425}{\sqrt{B_z}}$
Sliding	$B_x = \frac{(7-8\mu)}{32\,(1-\mu)} \frac{m}{\rho r^3}$	$K_x = \frac{32\,(1-\mu)\,Gr}{7-8\,\mu}$	$D_x = \frac{0.288}{\sqrt{B_x}}$
Torsional	$B_\theta = \frac{I_\theta}{\rho r^5}$	$K_\theta = \frac{16}{3}\,Gr^3$	$D_\theta = \frac{0.5}{1+2\,B_\theta}$
Rocking	$B_\psi = \frac{3\,(1-\mu)}{8} \frac{I_\psi}{\rho r^5}$	$K_\psi = \frac{8\,Gr^3}{3\,(1-\mu)}$	$D_\psi = \frac{0.15}{(1-B_\psi)\sqrt{B_\psi}}$

Consider two footings, say 1 and 2, with radius and mass of r_1 and m_1, and r_2 and m_2 respectively. Following four case have been investigated for vertical, sliding, torsional, and rocking modes of vibration.

Case—A : Constant force (or moment) excitation and the mass of the system is constant.

$r_1 \neq r_2$, $m_1 = m_2$

Case—B : Rotating mass excitation and the mass of the system is constant $r_1 \neq r_2$; $m_1 = m_2$

Case—C : Constant force (or moment) excitation and radius is constant $r_1 = r_2$; $m_1 \neq m_2$

Case—D: Rotating mass excitation and radius is constant $r_1 = r_2$; $m_1 \neq m_2$

Results and Discussions

Expressions for ratio of amplitude of displacement and that of resonant frequency of footing 2 to that of 1 have been developed using appropriate results reported in literature (Richart et al, 1970). Appendix I briefly describes how a typical expression is arrived at. Table 2,3,4, and 5 present the results respectively for vertical, sliding, torsional, and rocking modes of vibration. In Tables 2 to 5, Q_{ij} represents :

the quantity Q — which can take values like A, amplitude of motion; B, mass ratio ; ω, frequency,

for i-th mode vibration — i taking values like z, x, θ and ψ for vertical horizontal, torisonal, and rocking modes of vibration respectively,

TABLE 2

Vertical Vibration

Case A

$$R_{1z} = X_{1z} X_z^{4/3} \quad \text{... (1)}$$

$$R_{2z} = X_z^{1/3} \sqrt{\frac{(1-Y_z)^2+(0.7225\ Y_z/B_{z_1})}{(1-Y_z X_z^{1/3})^2+(0.7225\ Y_z X_z^{1/3}/B_{z_2})}} \quad \text{... (2)}$$

$$R_{3z} = 1/X_{2z} X_z^{2/3} \quad \text{... (3)}$$

Case B

$$R_{1z} = X_z X_{1z} \quad \text{... (4)}$$

R_{2z} = Same as Equation 2

$$R_{3z} = X_{2z}\ X_z^{1/3} \quad \text{... (5)}$$

Case C

R_{1z} = Same as Equation 4

$$R_{2z} = \sqrt{\frac{(1-Y_z)^2+(0.7225\ Y_z/B_{z_1})}{(1-X_z\ Y_z)^2+(0.7225\ X_z\ Y_z/B_{z_2})}} \quad \text{... (6)}$$

$$R_{3z} = 1/X_z X_{2z} \quad \text{... (7)}$$

Case D

$$R_{1z} = X_{1z} \quad \text{... (8)}$$

R_{2z} = Same as Equation 6

$$R_{3z} = X_{2z} \quad \text{... (9)}$$

Note :

$X_z = B_{z_2}/B_{z1}$

$Y_z = (\omega/\omega_{z_1n})^2$

$R_{1z} = A_{z_2m}/A_{z_1m}$

$R_{2z} = A_{z_2}/A_{z_1}$

$R_{3z} = \omega_{z_2m}/\omega_{z_1m}$

$X_{1z} = [(B_{z_1}-0.18)/(B_{z_2}-0.18)]^{1/2}$

$X_{2z} = [(B_{z_1}-0.36125)/B_{z_2}-0.36125)]^{1/2}$

TABLE 3

Sliding Vibration

Case A

$$R_{1x} = X_{1x} \; X_x^{4/3} \qquad \dots (10)$$

$$R_{2x} = X_x^{1/3} \sqrt{\frac{(1-Y_x)^2+(0.332\;Y_x/B_{x1})}{(1-Y_x\,X_x^{1/3})^2+0.332\;Y_x\;X_x^{1/3}/B_{x2}}} \qquad \dots (11)$$

$$R_{3x} = 1/(X_{2x}\;X_x^{2/3}) \qquad \dots (12)$$

Case B

$$R_{1x} = X_x\,X_{1x} \qquad \dots (13)$$

$R_{2x} =$ Same as Equation 11

$$R_{3x} = X_{2x}\;X_x^{1/3} \qquad \dots (14)$$

Case C

$R_{1x} =$ Same as Equation 13

$$R_{2x} = \sqrt{\frac{(1-Y_x)^2+(0.332\;Y_x/B_{x1})}{(1-Y_x\,X_x)^2+(0.332\;Y_x\,X_x/B_{x2})}} \qquad \dots (15)$$

$$R_{3x} = 1/X_x\,X_{2x} \qquad \dots (16)$$

Case D

$$R_{1x} = X_{1x} \qquad \dots (17)$$

$R_{2x} =$ Same as Equation 15

$$R_{3x} = X_{2x} \qquad \dots (18)$$

Note :

$X_x = B_{x2}/B_{x1}$

$Y_x = (\omega/\omega_{x1n})^2$

$R_{1x} = A_{x2m}/A_{x1m}$

$R_{2x} = A_{x2}/A_{x1}$

$R_{3x} = \omega_{x2m}/\omega_{x1m}$

$X_{1x} = [(B_{x1}-0.083)/(B_{x2}-0.083)]^{1/2}$

$X_{2x} = [(B_{x1}-0.166)/(B_{x2}-0.166)]^{1/2}$

TABLE 4

Torsional Vibration

Case A

$$R_{1\theta} = X_\theta \; X_{1\theta} \; X_{4\theta} / X_{3\theta} \qquad \dots (19)$$

$$R_{2\theta} = X_\theta \sqrt{\frac{(1-Y_\theta)^2 + (Y_\theta / Y_{3\theta})}{(1-Y_\theta \; X_\theta^{1/3})^2 + (Y_\theta \; X_\theta^{1/3} / X_{4\theta})}} \qquad \dots (20)$$

$$R_{3\theta} = (X_{3\theta} / X_{4\theta})^{1/2} \; (1/X_{2\theta} \; X_\theta^{1/6}) \qquad \dots (21)$$

Case B

$$R_{1\theta} = X_{1\theta} \; X_{4\theta} \; X_\theta^{2/3} / X_{3\theta} \qquad \dots (22)$$

$R_{2\theta}$ = Same as Equation 20

$$R_{3\theta} = (X_{4\theta} / X_{3\theta})^{1/2} \; (X_{2\theta} / X_\theta^{1/6}) \qquad \dots (23)$$

Case C

$$R_{1\theta} = X_{1\theta} \; X_{4\theta} / X_{3\theta} \qquad \dots (24)$$

$$R_{2\theta} = \sqrt{\frac{(1-Y_\theta)^2 + (Y_\theta / X_{3\theta})}{(1-X_\theta \; Y_\theta) + (X_\theta \; Y_\theta / X_{4\theta})}} \qquad \dots (25)$$

$$R_{3\theta} = (X_{3\theta} / X_\theta \; X_{4\theta})^{1/2} \; (1/X_{2\theta}) \qquad \dots (26)$$

Case D

$$R_{1\theta} = X_{1\theta} \; X_{4\theta} / X_\theta \; X_{3\theta} \qquad \dots (27)$$

$R_{2\theta}$ = Same as Equation 25

$$R_{3\theta} = X_{2\theta} \; (X_{4\theta} / X_\theta \; X_{3\theta})^{1/2} \qquad \dots (28)$$

Note :

$X_\theta = B_{\theta 2} / B_{\theta 1}$; $Y_\theta = (\omega / \omega_{\theta 1n})^2$

$R_{1\theta} = A_{\theta 2m} / A_{\theta 1m}$; $R_{2\theta} = A_{\theta 2} / A_{\theta 1}$; $R_{3\theta} = \omega_{\theta 2m} / \omega_{\theta 1m}$

$X_{1\theta} = [(X_{3\theta} - 0.25)/(X_{4\theta} - 0.25)]^{1/2}$

$X_{2\theta} = [(X_{3\theta} - 0.5)/(X_{4\theta} - 0.5)]^{1/2}$

$X_{3\theta} = (1+2 \; B_{\theta 1})^2$; $X_{4\theta} = (1+2 \; B_{\theta 2})^2$

TABLE 5

Rocking Vibration

Case A

$$R_{1\psi} = R^3 \ X_\psi \ X_{1\psi} \ X_{4\psi}/X_{3\psi} \quad \dots (29)$$

$$R_{2\psi} = R^3 \sqrt{\frac{(1-Y_\psi)^2+(0.09\,Y_\psi/X_{5\psi})}{[1-(X_\psi \ Y_\psi/R^2)]^2+(0.09\,X_\psi \ Y_\psi/X_{6\psi}\,R^2)}} \quad \dots (30)$$

$$R_{3\psi} = R\,(X_{5\psi}/X_\psi \ X_{6\psi})^{1/2}/X_{2\psi} \quad \dots (31)$$

Case B

$$R_{1\psi} = R^5 \ X_{1\psi} \ X_{4\psi}/X_{3\psi} \quad \dots (32)$$

$R_{2\psi}$ = Same as Equation 30

$$R_{3\psi} = R\,(X_{6\psi}/X_\psi \ X_{5\psi})^{1/2} \quad \dots (33)$$

Case C

$$R_{1\psi} = X_\psi \ X_{1\psi} \ X_{4\psi}/X_{3\psi} \quad \dots (34)$$

$$R_{2\psi} = \sqrt{\frac{(1-Y_\psi)^2+(0.09\,Y_\psi/X_{5\psi})}{(1-X_\psi \ Y_\psi)^2+(0.09\,X_\psi \ Y_\psi/X_{6\psi})}} \quad \dots (35)$$

$$R_{3\psi} = [(X_{5\psi}/X_\psi \ X_{6\psi})^{1/2}]/X_{2\psi} \quad \dots (36)$$

Case D

$$R_{1\psi} = X_{1\psi} \ X_{4\psi}/X_{3\psi} \quad \dots (37)$$

$R_{2\psi}$ = Same as Equation 35

$$R_{3\psi} = X_{2\psi}\,(X_{6\psi}/X_\psi \ X_{5\psi})^{1/2} \quad \dots (38)$$

Note :

$X_\psi = B_{\psi 2}/B_{\psi 1}$; $Y_\psi = (\omega/\omega_{\psi 1n})^2$

$R_{1\psi} = A_{\psi 2m}/A_{\psi 1m}$

$R_{2\psi} = A_{\psi 2}/A_{\psi 1}$

$R_{3\psi} = \omega_{\psi 2m}/\omega_{\psi 1m}$

$X_{1\psi} = [(X_{3\psi} \ B_{\psi 1} - 0.0225)/(X_{4\psi} \ B_{\psi 2} - 0.0225)]^{1/2}$

$X_{2\psi} = [(X_{5\psi} - 0.045)/(X_{6\psi} - 0.045)]^{1/2}$

$X_{3\psi} = (1+B_{\psi 1})^2$; $X_{4\psi} = (1+B_{\psi 2})^2$

$X_{5\psi} = (1+B_{\psi 1}^{3/2})^2$; $X_{6\psi} = (1+B_{\psi 2}^{3/2})^2$

$R = (r_1/r_2)$

for j-th footing— j-taking values of either 1 or 2. A_{ijm} corresponds to the resonant amplitude of motion, ω_{ijm} to the resonant frequency and ω_{ijn} the natural frequency.

For example :

B_{x_1} = indicates mass ratio of footing 1 in sliding mode of vibration.

A_{z_2m} = indicates the amplitude of displacement of footing 2 in vertical mode of vibration during resonance. The corresponding resonant frequency is $\omega_{z\,2m}$.

A_{z_2} = indicates the amplitude of displacement of footing 2 in vertical mode of vibration during frequencies other than resonance.

A comprehensive list of notations is included in Appendix II.

Figure 2 shows the variation of R_{1z} with B_{z_2} for different values of B_{z_1}, for CASE—A (i.e. Equation 1). A replot of the same data as a variation of R_{1z} with B_{z_2}/B_{z_1} (where $B_{z_2} \geqslant B_{z_1}$) suggests the following generalised relationship for any combination of B_{z_1} and B_{z_2} :

$$R_{1z} = 0.66\,(B_{z_2}/B_{z_1}) + 0.34 \qquad \text{...(39)}$$

for CASE—A where $B_{z_1} \geqslant B_{z_1}$

Figure 3 shows the variation of R_{2z} with ω/ω_{z_1n} for different combinations of B_{z_1} and B_{z_2}, for CASE-A (and also for CASE-B). The peak point in the curves are obtained by differentiating Equation 2 with respect to ω/ω_{z_1n}, setting the resulting expression equal to zero and then solving it by trial and error. The values ω/ω_{z_1n} at which the ratio R_{2z} tends to attain a value of unity is obtained by setting Equation 2 equal to unity and solving for the values of ω/ω_{z_1n}.

For *CASE—A* the variation of R_{3z} with B_{z_2} (Equation 3) for two different values of B_{z_1} is shown in Figure 4.

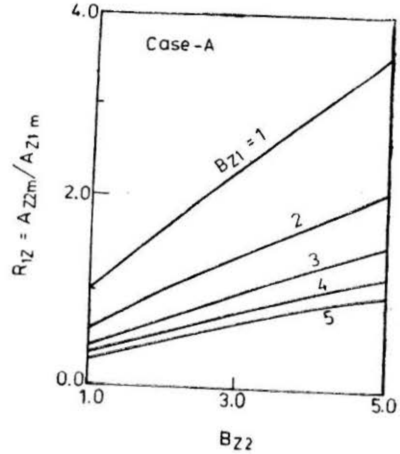

FIGURE 2 Variation of R_{1z} with B_{z2} for Case A

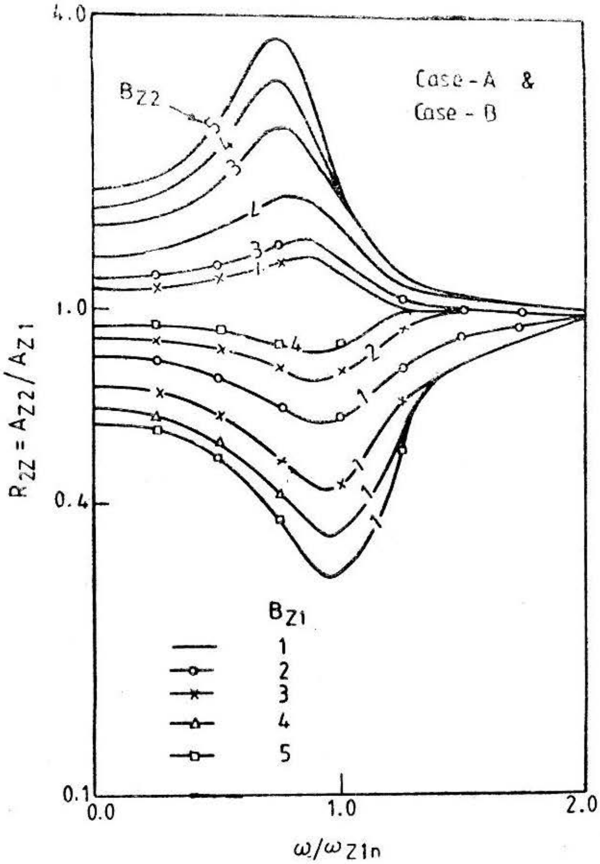

FIGURE 3 **Variation of R_{2z} with ω/ω_{z1n} for Case A and B**

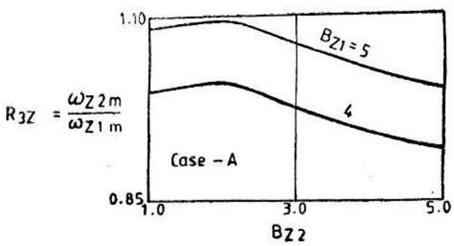

FIGURE 4 **Variation of R_{3z} with B_{z2} for Case A**

Figures similar to 2,3, and 4 can be developed for *CASE—B*, *C* and *D* also, for vertical vibration and for all cases for the other three modes of vibration. The following discussions are based on a perusal of all such figures which have been made but not presented here.

It may also be noted that in the case of rocking vibrations, for *CASE—B*, the equations in Table 5 are valid only when the axis of horizontal force causing rocking is at the same vertical distance from the base for both the footings 1 and 2. However, if as a result of change in B_1 the vertical distances are different, say Z_1 and Z_2 for footings 1 and 2 respectively, then the Equations for *CASE—B* in Table 5 should be multiplied by (Z_1/Z_2).

Resonant amplitude of motion

The results show that except in the case of torsional mode of vibration for *CASE—D*, in all other cases and for all modes of vibration, the resonant amplitude of motion (displacement or rotation) increases as the corresponding mass ratio, *B*, of the footing increases and vice-versa. In *CASE—A* and *CASE—B* (where $m_1 = m_2$) the increase in the value of mass ratio is brought about by a reduction in the radius of the footing. This reduces the values of both the damping factor and the spring constant. The effect of each is to increase the amplitude of motion. It can thus also be stated that increased contact pressure increases the resonant amplitude of motion of a footing. It is further seen that this effect is remarkable for rotational modes of vibration than for translational modes. This may be attributed to the large decrease in damping in rotational modes of vibration.

For torsional mode of vibration for *CASE—D* the value of area moment of inertia, I_θ, increases as the value of B_θ increases. This causes the decrease in the resonant amplitude of the footing.

Amplitude of motion at conditions other than resonance

Examination of Figure 3 indicates as though the amplitude of motion increases if the mass ratio of the footing increases, for all frequencies of excitation force. But this is not true always, even for vertical vibration. For example, consider Figure 5 which depicts the variation $A_{\theta 2}/A_{\theta 1}$ with $\omega/\omega_{\theta 1n}$ for *CASE—C* (and also *CASE—D*). In this case it is evident that the increase in mass ratio results in an increase in the amplitude of rotation only up to a certain frequency after which the amplitude of rotation becomes less than that for the footing with lower mass ratio. Hereagain, the effect of change in mass ratio on the response of footings is more significant in rotational modes of vibration than in translational vibrations.

Resonant frequency

For all modes of vibration and for all cases the resonant frequency decreases as the value of the corresponding mass ratio increases.

Comparison with experimental observations

Experimental data on vertical vibration, for *CASE—B* and *CASE—D*, have been collected from literature (Novak 1970, Anandakrishnan and Krishnaswamy 1973, Sridharan and Raman 1977). Using appropriate equations the R_{1z} values and R_{3z} values have been computed. Figures 6 and 7 show the comparison of these theoretically predicted values with

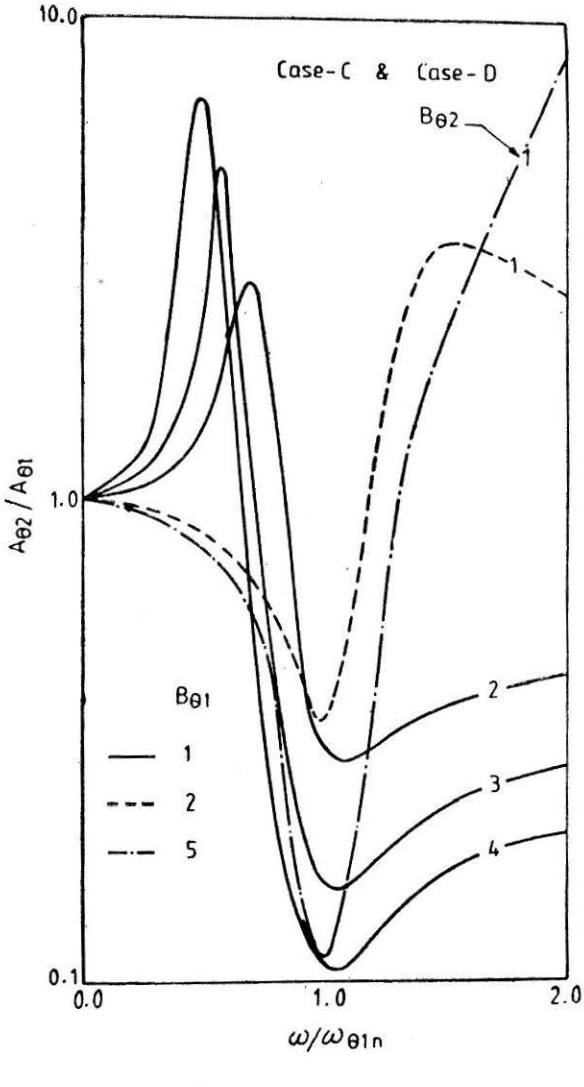

FIGURE 5 Variation of $A_{\theta 2}/A_{\theta 1}$ with $\omega/\omega_{\theta 1n}$ for Cases C and D

those of actually measured values. The agreement between theory and experiments is good in the case of resonant displacements but not as much so in the case of resonant frequencies. Qualitatively, the increase in the mass ratio decreases the resonant frequency but its actual value is more than what is predicted by the theory.

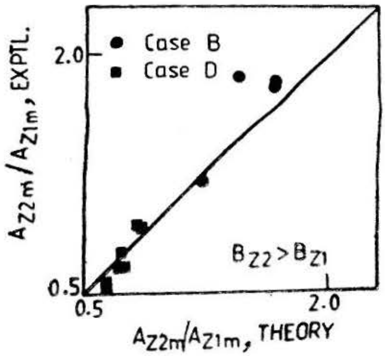

FIGURE 6 **Comparison of theoretical and experimental ratios of resonant vertical displacements**

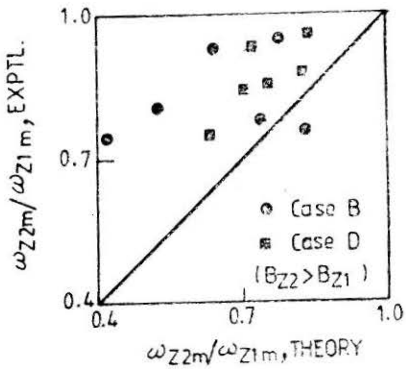

FIGURE 7 **Comparison of theoretical and experimental ratios of resonant frequencies**

Experiments show (Sridharan and Raman. 1977) little damping compared to Lysmer's theoretical value. Hence, direct use of elastic half-space model will predict higher than actual displacements. This limitation can be overcome now by considering the ratio of displacements.

Conclusions

Based on elastic half-space model, mathematical expressions to quantify the variation in the dynamic response of footings have been developed. The following are the main conclusions from the study.

1. Except in the case of torsional mode of vibration for *CASE—D* in all other cases and for all modes of vibration the resonant amplitudc of motion (displacement or rotation) increases as the corresponding mass ratio, *B*, of the footing increases. This is more significant for rotational modes of vibration than for translational modes.

2. The amplitude of motion during conditions other than resonance, in many instances increases with an increase in the mass ratio only

up to a certain frequency. Beyond this the amplitude of motion becomes less than that for the footing with lower mass ratio. Hereagain, the effect is comparatively more significant for rotational modes of vibration than for translational modes.

3. For all modes of vibration and for all cases the resonant frequency decreases as the value of the corresponding mass ratio increases.

4. Comparison with experimental results for vertical vibration indicates that the theoretical quantification agrees very well in the case of resonant displacements than in the case of resonant frequencies.

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Appendix—I

Comparison of resonant amplitude of vertical displacement—'CASE—A'

From fundamental theory the resonant amplitude of displacement of footing 1 and 2 are given by Equation 40 and 41, respectively (Richart, et. al. 1970)

$$A_{z_1m} = \frac{Q_o\,(1-\mu)}{4\,G\,r_1} \quad \frac{B_{z_1}}{0.85\,\sqrt{B_{z_1}-0.18}} \qquad \ldots(40)$$

$$A_{z_2m} = \frac{Q_o\,(1-\mu)}{4\,G\,r_2} \quad \frac{B_{z_2}}{0.85\,\sqrt{B_{z_2}-0.18}} \qquad \ldots(41)$$

From the definition of mass ratio ;

$$r_1/r_2 = (B_{z_2}/B_{z_1})^{1/3} \qquad \text{since } m_1 = m_2$$

It can be now shown that

$$A_{z_2m}/A_{z_1m} = (B_{z_2}/B_{z_1})^{4/3}\,\{\,(B_{z_1}-0.18)/(B_{z_2}-0.18)\,\}^{1/2}$$

$$\text{i.e. } R_{1z} = X_{1z}\;X_z^{\;4/3} \qquad \ldots(42)$$

Appendix—II Notations

A_{ij} = amplitude of motion of footing j in i-th mode of vibration

A_{ijm} = resonant amplitude of motion of footing j in i-th mode of vibration

B_{ij} = mass ratio of footing j in i-th mode of vibration

D_i = damping factor in i-th mode of vibration

G = shear modulus of soil

K_i = spring constant in i-th mode of vibration

m_j = mass of footing j

r_j = radius of footing j

Q_o = amplitude of constant force excitation

ω_{ijm} = resonant frequency of footing j in i-th mode of vibration

ω_{ijn} = natural frequency of footing j in i-th mode of vibration

$i = x$ = sliding vibration

$i = z$ = vertical vibration

$i = \theta$ = torsional vibration

$i = \psi$ = rocking vibration

μ = Poisson's ratio of soil