Stresses And Displacements In Nonlinear Nonhomogeneous Soil Media

by

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Introduction

The stress distribution and displacement under shallow foundations are generally determined as the corresponding distribution in a linear, homogeneous, isotopic elastic medium with boundary condition approximating those of the actual problem of interest. Whereas the soil medium usually departs from such idealised behaviour. Hence it should be generally treated as non-linear, non-homogeneous and anisotropic. There are solutions available for some very simple cases (Michell 1900, Frohlich 1934, Wolf 1935, Westergaard 1938, Holl 1940, Burmister 1947, Taylor 1948, Burmister 1956, Klein 1956, Barden 1963, Gibson 1967, Huang 1968, Gibson and Sills 1971, Bushan and Haley 1976, Babu Shankar 1977). At an advanced level, the finite element method incorporating more realistic conditions has been used successfully (Duncan and Chang 1970, Burland, Sills and Gibson 1973, Carrier and Christian 1973). But most solutious are too complex for routine use in design office. Evidently there is a need for better and simpler solutions which take into account the departures in the behaviour of soil from the idealised behaviour.

Hruban (1958) examined the effect of non-linearity and non-homogeneity for a point load acting on the surface of a half-space (Figure 1). He concluded from field observation that the actual curves representing the ratio of the deflection to the applied load corrospond approximately to the non-linear relation.

$$\epsilon_i = (\sigma_i/c)^n \qquad \ldots(1)$$

where, c = a constant, characteristic of the material

$$\epsilon_i = \frac{1}{\sqrt{2}} \frac{m}{(m+1)} \left[(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_3-\epsilon_1)^2\right]^{\frac{1}{2}}$$

$$\sigma_i = \frac{1}{\sqrt{2}} \left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{\frac{1}{2}}$$

$\epsilon_1, \epsilon_2, \epsilon_3,$ = principal strains

$\sigma_1, \sigma_2, \sigma_3$ = principal stresses

n = a constant

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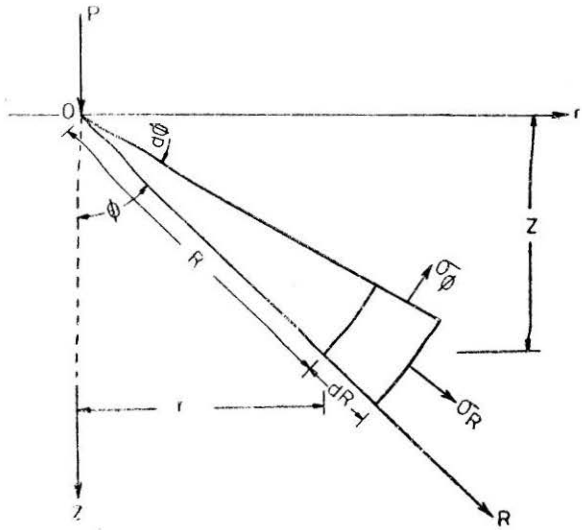


FIGURE 1 Case of axial symmetry

m = reciprocal of Poisson's ratio $= 1/\mu$

Taking the above non-linear relation into account, Hruban derived the expression for a radial stress σ_R and for displacement u and v in the R and ϕ direction respectively, for axially symmetric case for point loading. He considered in his solutions different values of n, different types of non-linearity and different values of m. Table 1 gives the Hruban's relations between ϵ_i, m, u, v and σ_R. The particular case of $n = 1$, Equation 1 represents Hooke's law, c being the Young's modulus of elasticity. With that the well known expression given at Case No. 1 in Table 1 is obtained. In consequence of pressure produced by the weight of the soil itself, the modulus of deformation of natural deposits increases sometimes with increasing depths. Case No. 6 and 7 of Table 1 give solutions for such a type of non-homogeneity.

If the modulus of elasticity, LF, is not constant but is a function of co-ordinates, i.e. if $LF = f(R, \phi)$, the solutions, as far as a simple radial distribution is possible, can be obtained, Hruban obtained solutions for stresses and displacements for some cases of heterogeneity which are reported in Table 2.

Considering the necessity for simple solutions which take into account some of the departures from idealised behaviour of soil, an attempt has been made in this paper to develop simple enough expression for computing stresses and displacements based on the work of Hruban. The expressions for vertical stresses and displacements due to concentrated and distributed loading have been developed separately for various cases of non-linearity and variable modulus of elasticity. The investigations and the results are presented in the following sections.

Vertical Stress Below Loaded Areas

In order to find expressions for vertical for stress (σ_z), it is desirable to transform the expressions for stresses and displacements in spherical co-

TABLE 1

Hruban's Solutions for Stresses and Displacements Assuming Parabolic Deformation Law (Nonlinear Case)

$$\varepsilon_i = \left(\frac{\sigma_i}{c}\right)^n, \ (E,\ c,\ d,\ m\text{—constants})$$

Case No.	ε_i	m	u	v	σ_R
(1)	(2)	(3)	(4)	(5)	(6)
1	$\frac{\sigma_i}{E}$	2	$\frac{3P}{2\pi}\frac{\cos\varphi}{ER}$	$\frac{-3P\sin\varphi}{4\pi ER}$	$\frac{3P\cos\varphi}{3\pi R^2}$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{1}{2R^2}\left(\frac{4P}{3\pi c}\right)^{3/2}\cos\varphi$	$\frac{-1}{6R^2}\left(\frac{4P}{3\pi c}\right)^{3/2}\sin\varphi$	$\frac{4P}{3\pi R^2}\cos^{2/3}\varphi$
3	$\left(\frac{\sigma_i}{c}\right)^2$	3	$\frac{P^2}{3\pi^2 c^2 R^3}$	0	$\frac{P}{\pi R^2}$
4	$\left(\frac{\sigma_i}{c}\right)^2$	4	$\frac{1}{3R^3}\left(\frac{5P}{4\pi c}\right)^2\cos\varphi$	$\frac{-1}{12R^2}\left(\frac{5P}{4\pi c}\right)^2\sin\varphi$	$\frac{5P}{4\pi R^2}\sqrt{\cos\varphi}$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{1}{4R^4}\left(\frac{P}{\pi c}\right)^{5/2}$	0	$\frac{P}{\pi R^2}$
6	$\left(\frac{\sigma_i^{3/2}}{c}\right)^2\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\frac{2}{7}\left(\frac{9P}{8\pi c}\right)^2\left(\frac{d}{R^7}\right)^{\frac{1}{2}}$	0	$\frac{9P}{8\pi R^2}\cos^{1/4}\varphi$
7	$\left(\frac{\sigma_i}{c}\right)\left(\frac{d}{z}\right)^{1/2}$	3,5	$\frac{2}{5}\left(\frac{3P}{2\pi c}\right)^{3/2}\left(\frac{d}{R^5}\right)^{\frac{1}{2}}\cos\varphi$	$\frac{-4}{35}\left(\frac{3P}{2\pi c}\right)^{3/2}\left(\frac{d}{R^5}\right)^{\frac{1}{2}}\sin\varphi$	$\frac{3P\cos\varphi}{2\pi R^2}$

TABLE 2

Hruban's Solutions for Stresses and Displacements for Variable Modulus of Elasticity

(E_0, f, m—constants)

Case No.	$LF=F(R, \varphi)$	m	u	v	σ_R
(1)	(2)	(3)	(4)	(5)	(6)
1	$E_0\,R$	2	$\frac{P}{2\pi E_0\,R^2}$	0	$\frac{P}{\pi\,R^2}$
2	$E_0\,R$	3	$\frac{3P\cos\varphi}{4\pi E_0\,R^2}$	$\frac{-P\sin\varphi}{4\pi E_0\,R^2}$	$\frac{3P\cos\varphi}{2\pi\,R^2}$
3	$\frac{E_0 \cdot z}{(f+z)}$	2	$\frac{3P(f+2z)}{4\pi E_0\,R^2}$	$\frac{-3P\sin\varphi}{4\pi E_0\,R}$	$\frac{3P\cos\varphi}{2\pi\,R^2}$
4	$E_0\sqrt{z}$	2.5	$\frac{7P\cos\varphi}{6\pi E_0\,R^{3/2}}$	$\frac{-7P\sin\varphi}{15\pi E_6\,R^{3/2}}$	$\frac{7P\cos^{3/2}\varphi}{4\pi\,R^2}$
5	$E_0\cos\varphi$	2	$\frac{2P\cos\varphi}{\pi E_0\,R}$	$\frac{-P\sin\varphi}{\pi E_0\,R}$	$\frac{2P\cos^2\varphi}{\pi\,R^2}$
6	$E_0\,z\cos\varphi$	2	$\frac{P}{\pi E_0\,R^2}$	0	$\frac{2P\cos^2\varphi}{\pi\,R^2}$
7	$E_0\,z\cos\varphi$	3	$\frac{5P\cos\varphi}{4\pi E_0\,R^2}$	$\frac{-5P\sin\varphi}{12\pi E_0\,R^2}$	$\frac{5P\cos^3\varphi}{2\pi\,R^2}$

ordinate system given in Tables 1 and 2 into cylindrical co-ordinate system, The expressions for σ_z for concentrated point loading for the case of non-linearity and variable modulus of elasticity are presented in Tables 3 and 4 respectively. These Tables indicate that for the chosen cases of non-linearity and non-homogeneity the expressions for σ_R and σ_z are still found to be linear functions of external load P. The expression for the vertical stresses for concentrated point loading are now integrated over the loaded areas to find the vertical stress at any point within the medium. Solutions have been obtained for a number of cases which are explained below.

Stresses in Nonlinear Medium (Table 3)

(*i*) The expression for total vertical stress $\sigma_{z_{NCU}}$ at a point on the axis of loading at a depth z below the surface of a circular loaded area of radius A with a uniform load intensity of q per unit area is given as

$$\sigma_{z_{NCU}} = q\ I_{z_{NCU}} \qquad ...(2)$$

TABLE 3

Modified Hruban's Solutions for Vertical Normal Stress (σ_z) due to Concentrated Point Loading for Parabolic Deformation Law (Nonlinear Case) Given in Table 1

Case No.	ε_i	m	σ_R	σ_z
(1)	(2)	(3)	(4)	(5)
1	$\frac{\sigma_i}{E}$	2	$\frac{3P}{2\pi R^2}\cos\varphi$	$\frac{3P}{2\pi}\frac{z^3}{R^5}$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{4P}{2\pi R^2}\cos^{2/3}\varphi$	$\frac{4P}{3\pi}\frac{z^{8/3}}{R^{14/3}}$
3	$\left(\frac{\sigma_i}{c}\right)^2$	3	$\frac{P}{\pi R^2}$	$\frac{P}{\pi}\cdot\frac{z^2}{R^4}$
5	$\left(\frac{\sigma_i}{c}\right)^2$	4	$\frac{5P}{4\pi R^2}\sqrt{\cos\varphi}$	$\frac{5P}{4\pi}\cdot\frac{z^{5/2}}{R^{9/2}}$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{P}{\pi R^2}$	$\frac{P}{\pi}\cdot\frac{z^2}{R^4}$
6	$\left(\frac{\sigma_i}{c}\right)\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\frac{9P}{8\pi R^2}\cos^{1/4}\varphi$	$\frac{9P}{8\pi}\cdot\frac{z^{9/4}}{R^{17/4}}$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\frac{3P}{3\pi R^2}\cos\varphi$	$\frac{3P\,z^3}{2\pi\,R^5}$

where $I_{z_{NCU}}$ = dimensionless influence factor for vertical stress under the centre of uniformly loaded circular area in nonlinear medium, a function of the dimensionless parameter $L\ (= A/Z)$

Table 5 presents the expressions for the influence factors for the different cases.

(*ii*) The case of parabolic loading on a circular area has been investigated. The expression for the loading is given as

$$q_r = q_{max}\left[1-\left(\frac{r}{A}\right)^2\right] \qquad \ldots(3)$$

where q_r = intensity of loading at distance r from the centre of circular loaded area and

q_{max} = maximum intensity of loading at the centre of the circular loaded area,

TABLE 4

Modified Hruban's Solutions for Vertical Normal Stress (σ_z) due to Concentrated Point Loading for Variable Modulus of Elasticity Given in Table 2

Case No.	$LF = F(R, \varphi)$	m	σ_R	σ_z
(1)	(2)	(3)	(4)	(5)
1	$E_o R$	2	$\frac{P}{\pi R^2}$	$\frac{P z^2}{\pi R^4}$
2	$E_o R$	3	$\frac{3P \cos\varphi}{2\pi R^2}$	$\frac{3P z^3}{2\pi R^5}$
3	$\frac{E_o . z}{(f+z)}$	2	$\frac{3P \cos\varphi}{2\pi R^2}$	$\frac{3P z^3}{2\pi R^5}$
4	$E_o \sqrt{z}$	2.5	$\frac{7P}{4\pi R^2} \cos^{3/2}\varphi$	$\frac{7P}{4\pi} \frac{z^{7/2}}{R^{11/2}}$
5	$E_o \cos\varphi$	2	$\frac{2P}{\pi R^2} \cos^2\varphi$	$\frac{2P z^4}{\pi R^6}$
6	$E_o z \cos\varphi$	2	$\frac{2P}{\pi R^2} \cos^2\varphi$	$\frac{2P z^4}{\pi R^6}$
7	$E_o z \cos\varphi$	3	$\frac{5P}{2\pi R^2} \cos^3\varphi$	$\frac{5P}{2\pi} \frac{z^5}{R^7}$

The vertical stress $\sigma_{z_{NCP}}$ at a point at a depth z below the centre of the circular area is given by the expression

$$\sigma_{z_{NCP}} = q \ I_{z_{NCP}} \qquad \ldots(4)$$

where q = the mean load intensity on the loaded area = $q_{max}/2$

$I_{z_{NCP}}$ = dimensionless influence factor for vertical stress under the centre of a circular area with parabolic loading in a nonlinear medium.

Table 6 gives the expressions for the influence factors for the different cases.

(*iii*) The vertical stress $\sigma_{z_{NUR}}$ at a depth z below the corner of a uniformly loaded rectangular area is given by the expression,

$$\sigma_{z_{NUR}} = q \ I_{z_{NUR}} \qquad \ldots(5)$$

TABLE 5

Influence Factors for Vertical Stress ($\sigma_{z_{NCU}}$) Under the Centre of a Uniformly Loaded Circular Area for Nonlinear Cases Given in Table 3

Case No.	ε.	m	$I_{z_{NCU}}$
(1)	(2)	(3)	(4)
1	$\frac{\sigma_i}{E}$	2	$\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\left[1-\frac{1}{(1+L^2)^{4/3}}\right]$
3	$\left(\frac{\sigma_i}{c}\right)^{2}$	3	$\left[1-\frac{1}{(1+L^2)}\right]$
4	$\left(\frac{\sigma_i}{c}\right)^{2}$	4	$\left[1-\frac{1}{(1+L^2)^{5/4}}\right]$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\left[1-\frac{1}{(1+L^2)}\right]$
6	$\left(\frac{\sigma_i}{c}\right)^{2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\left[1-\frac{1}{(1+L^2)^{9/8}}\right]$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$

Note : $\sigma_{z_{NCU}} = q\, I_{z_{NCU}}$

where $I_{z_{NUR}}$ = dimensionless influence factor for vertical stress under the corner of a uniformly loaded rectangular area, a function of dimensionless parameters M ($= a/b$) and N ($= z/b$).

The expressions for $I_{z_{NUR}}$ for the different cases are shown in Table 7.

Stresses in Nonhomogeneous Medium (*Table 4*)

The expression for vertical stresses σ_z at any point vertically below the centre of a loaded circular area for all cases of variable modulus of elasticity given in Table 4 can be obtained as,

$$\sigma_{z_{VCU}} = q\, I_{z_{VCU}} \text{ (for uniform loading)} \qquad ...(6)$$

TABLE 6

Influence Factors for Vertical Stress ($\sigma_{z_{NCP}}$) Under the Centre of a Circular Area with Parabolic Loading for Nonlinear Cases Given in Table 3

Case No.	ε_i	m	$I_{z_{NCP}}$
(1)	(2)	(3)	(4)
1	$\frac{\sigma_i}{E}$	2	$2\left[1-\frac{1}{(1+L^2)^{3/2}}\right]-\frac{6}{L^2}\left[1-\frac{1}{(1+L^2)^{1/2}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$2\left[1-\frac{1}{(1+L^2)^{4/3}}\right]-\frac{8}{L^2}\left[1-\frac{1}{(1+L^2)^{1/3}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{4/3}}\right]$
3	$\left(\frac{\sigma_i}{c}\right)^{2}$	3	$2\left[1-\frac{1}{(1+L^2)}\right]-\frac{2}{L^2}\log_e(1+L^2)+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)}\right]$
4	$\left(\frac{\sigma_i}{c}\right)^{2}$	4	$2\left[1-\frac{1}{(1+L^2)^{5/4}}\right]-\frac{10}{L^2}\left[1-\frac{1}{(1+L^2)^{1/4}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{5/4}}\right]$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$2\left[1-\frac{1}{(1+L^2)}\right]-\frac{2}{L^2}\log_e(1+L^2)+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)}\right]$
6	$\left(\frac{\sigma_i}{c}\right)^{2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$2\left[1-\frac{1}{(1+L^2)^{9/8}}\right]-\frac{18}{L^2}\left[1-\frac{1}{(1+L^2)^{1/8}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{9/8}}\right]$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$2\left[1-\frac{1}{(1+L^2)^{3/2}}\right]-\frac{6}{L^2}\left[1-\frac{1}{(1+L^2)^{1/2}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$

Note : $\sigma_{z_{NCP}} = q\, I_{z_{NCP}}$

TABLE 7

Influence Factors for Vertical Stress ($\sigma_{z_{NRU}}$) Under the Corner of a Uniformly Loaded Rectangular Area for Nonlinear Cases Given in Table 3

Case No.	ε_i	m	$I_{z_{NRU}}$
(1)	(2)	(3)	(4)
1	$\frac{\sigma_i}{E}$	2	$\frac{3}{2\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dX}{(X^2+Y^2+1)^{5/2}}$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{4}{3\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{7/3}}$
3	$\left(\frac{\sigma_i}{c}\right)^{2}$	3	$\frac{1}{2\pi}\left[\frac{1}{(1+N^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{M}{(1+N^2)^{\frac{1}{2}}}\right\} + \frac{M}{(M^2+N^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{1}{(M^2+N^2)^{\frac{1}{2}}}\right\}\right.$
4	$\left(\frac{\sigma_i}{c}\right)^{2}$	4	$\frac{5}{4\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{9/4}}$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{1}{2\pi}\left[\frac{1}{(1+N^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{M}{(1+N^2)^{\frac{1}{2}}}\right\} + \frac{M}{(M^2+N^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{1}{(M^2+N^2)^{\frac{1}{2}}}\right\}\right]$
6	$\left(\frac{\sigma_i}{c}\right)^{2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\frac{9}{8\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{17/8}}$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{\frac{1}{2}}$	3,5	$\frac{3}{2\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{5/2}}$

Note : $\sigma_{z_{NRU}} = q\, I_{z_{NRU}}$

$$\sigma_{z_{VCP}} = q\, I_{z_{VCP}} \text{ (for parabolic loading)} \quad \ldots(7)$$

where $I_{z_{VCU}}$ and $I_{z_{VCP}}$ are dimensionless influence factors. The expressions for these are presented in Table 8.

The expression for the vertical stress $\sigma_{z_{VRU}}$ below the corner of a uniformly loaded rectangular area in terms of a dimensionless influence factor $I_{z_{VRU}}$ can be obtained as

$$\sigma_{z_{VRU}} = q\, I_{z_{VRU}} \quad \ldots(8)$$

Table 9 presents the expressions for $I_{z_{VRU}}$ for the different cases.

Vertical Displacement Below Loaded Areas

The expressions for vertical displacement ρ_z for concentrated point loading for the cases of non-linearity and variable modulus of elasticity are presented in Tables 10 and 11, respectively. The solutions developed for the various cases using these expressions are explained below.

Displacement in Nonlinear Medium (Table 10)

In the expressions for vertical displacement given in Table 10, the term P is raised to some power (index). Therefore direct integration is not possible in these cases. The method used in the analysis is explained by Shahi (1980). This method has also been used to obtain the vertical displacement under the centre of uniformly loaded circular area for Case No. 1 (linear case) of Table 10 and the same result as given by Egorov (1958) has been obtained proving the correctness of the approach.

The vertical displacement $\rho_{z_{NCU}}$ at a point at depth z below the centre of a uniformly loaded circular area can be expressed in terms of a dimensionless influence factor $K_{z_{NCU}}$. The expression for $\rho_{z_{NCU}}$ and $K_{z_{NCU}}$ for the various cases are presented in Table 12.

Displacement in Nonhomogeneous Medium (Table 11)

The displacement in all these cases is a linear function of the load and is as such amenable for direct integration. The expressions for vertical displacement below the centre of a circular loaded area are developed in terms of the dimensionless influence factor $K_{z_{VCU}}$ for uniform loading and $K_{z_{VCP}}$ for parabolic loading respectively. Table 13 gives the expressions for displacement and the corresponding influence factor.

TABLE 8

Influence Factors for Vertical Stress Under the Centre of a Circular Loaded Area for Uniformly Distributed ($\sigma_{z_{VCU}}$) and Parabolic Loading ($\sigma_{z_{VCP}}$) Considering Variable Modulus of Elasticity for Cases Given in Table 4

Case No.	$LE = F(R, \varphi)$	m	$I_{z_{VCU}}$	$I_{z_{VCP}}$
(1)	(2)	(3)	(4)	(5)
1	$E_o\, R$	2	$\left[1-\frac{1}{(1+L^2)}\right]$	$2\left[1-\frac{1}{(1+L^2)}\right]-\frac{2}{L^2}\log_e(1+L^2)+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)}\right]$
2	$E_o\, R$	3	$\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{3/2}}\right]-\frac{6}{L^2}\left[1-\frac{1}{(1+L^2)^{1/2}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$
3	$\frac{E_o\, z}{(f+z)}$	2	$\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{3/2}}\right]-\frac{6}{L^2}\left[1-\frac{1}{(1+L^2)^{1/2}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{3/2}}\right]$
4	$E_o\sqrt{z}$	2.5	$\left[1-\frac{1}{(1+L^2)^{7/4}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{7/4}}\right]-\frac{14}{3L^2}\left[1-\frac{1}{(1+L^2)^{3/4}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{7/4}}\right]$
5	$E_o \cos\varphi$	2	$\left[1-\frac{1}{(1+L^2)^{2}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{2}}\right]-\frac{4}{L^2}\left[1-\frac{1}{(1+L^2)}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{2}}\right]$
6	$E_o\, z \cos\varphi$	2	$\left[1-\frac{1}{(1+L^2)^{2}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{2}}\right]-\frac{4}{L^2}\left[1-\frac{1}{(1+L^2)}\right]-\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{2}}\right]$
7	$E_o\, z \cos\varphi$	3	$\left[1-\frac{1}{(1+L^2)^{5/2}}\right]$	$2\left[1-\frac{1}{(1+L^2)^{5/2}}\right]-\frac{10}{3L^2}\left[1-\frac{1}{(1+L^2)^{3/2}}\right]+\frac{2}{L^2}\left[1-\frac{1}{(1+L^2)^{5/2}}\right]$

Note : $\sigma_{z_{VCU}} = q\, I_{z_{VCU}}$ *and* $\sigma_{z_{VCP}} = q\, I_{z_{VCP}}$

TABLE 9

Influence Factors for Vertical Stress ($\sigma_{z_{VRU}}$) Under the Corner of a Uniformly Loaded Rectangular Area Considering Variable Modulus of Elasticity for Cases Given in Table 4

Case No.	$LF=F(R, \varphi)$	m	$I_{z_{VRU}}$
(1)	(2)	(3)	(4)
1	$E_o\, R$	2	$\frac{1}{2\pi}\left[\frac{1}{(1+N^2)^{1/2}}\tan^{-1}\left\{\frac{M}{(1+N^2)^{1/2}}\right\}+\frac{M}{(M^2+N^2)^{1/2}}\tan^{-1}\left\{\frac{1}{(M^2+N^2)^{1/2}}\right\}\right]$
2	$E_o\, R$	3	$\frac{3}{2\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{5/2}}$
3	$\frac{E_o\, R}{(f+z)}$	2	$\frac{3}{2\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{5/2}}$
4	$E_o\sqrt{z}$	2.5	$\frac{7}{4\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{11/4}}$
5	$E_o\cos\varphi$	2	$\frac{2}{\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{3}}$
6	$E_o\, z\cos\varphi$	2	$\frac{2}{\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{3}}$
7	$E_o\, z\cos\varphi$	3	$\frac{5}{2\pi}\int_0^{M/N}\int_0^{1/N}\frac{dX\,dY}{(X^2+Y^2+1)^{7/2}}$

Note : $\sigma_{z_{VRU}} = q\, I_{z_{VRU}}$

TABLE 10

Modified Hruban's Solutions for Vertical Displacement (ρ_z) due to Concentrated Point Loading for Nonlinear Cases Given in Table 1

Case No.	ε_i	m	u	v	$\rho_z = (u \cos \varphi - v \sin \varphi)$
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
1	$\frac{\sigma_i}{E}$	2	$\frac{3P \cos \varphi}{2\pi ER}$	$\frac{-3P \sin\varphi}{4\pi ER}$	$\frac{3P z^2}{4\pi ER^3} + \frac{3P}{4\pi ER}$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{1}{2R^2}\left(\frac{4P}{3\pi c}\right)^{3/2} \cos \varphi$	$-\frac{\sin \varphi}{6R^2}\left(\frac{4P}{3\pi c}\right)^{3/2}$	$\frac{1}{3R^4}\left(\frac{4P}{3\pi c}\right)^{3/2} z^2 + \frac{1}{6R^2}\left(\frac{4P}{3\pi c}\right)^{3/2}$
3	$\left(\frac{\sigma_i}{c}\right)^2$	3	$\frac{P^2}{3\pi^2 c^2 R^3}$	0	$\frac{P^2 z}{3\pi^2 c^2 R^4}$
4	$\left(\frac{\sigma_i}{c}\right)^2$	4	$\frac{1}{3R^3}\left(\frac{5P}{4\pi c}\right)^2 \cos \varphi$	$\frac{-1}{12 R^3}\left(\frac{5P}{4\pi c}\right)^2 \sin \varphi$	$\frac{1}{4R^5}\left(\frac{5P}{4\pi c}\right)^2 z^2 + \frac{1}{12 R^3}\left(\frac{5P}{4\pi c}\right)^2$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{1}{4R^4}\left(\frac{P}{\pi c}\right)^{5/2}$	0	$\frac{z}{4R^5}\left(\frac{P}{\pi c}\right)^{5/2}$
6	$\left(\frac{\sigma_i^2}{c}\right)\left(\frac{d}{z}\right)^{1/2}$	3, 5	$\frac{2}{7}\left(\frac{9P^2}{8\pi c}\right)\left(\frac{d}{R^7}\right)^{1/2}$	0	$-\frac{2}{7}\left(\frac{9P}{8\pi c}\right)^2\left(\frac{d}{R^7}-\right)^{1/2} \frac{z}{R}$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2} \times \left(\frac{d}{z}\right)^{1/2}$	3, 5	$\frac{2}{5}\left(\frac{3P}{2\pi c}\right)^{3/2} \times \left(\frac{d}{R^5}\right)^{1/2} \cos \varphi$	$\frac{-4}{35}\left(\frac{3P}{2\pi c}\right)^{3/2} \times \left(\frac{d}{R^5}\right)^{1/2} \sin \varphi$	$\frac{2}{7R^2}\left(\frac{3P}{2\pi c}\right)^{3/2}\left(\frac{d}{R^5}\right)^{1/2} z^2 + \frac{4}{35}\left(\frac{3P}{2\pi c}\right)^{3/2}\left(\frac{d}{R^5}\right)^{1/2}$

TABLE 11

Modified Hruban's Solutions for Vertical Displacement Due to Concentrated Point Loading for Variable Modulus of Elasticity Given in Table 2

Case No.	$LF = F(R, \varphi)$	m	u	v	$\rho_z = (u \cos \varphi - v \sin \varphi)$
(1)	(2)	(3)	(4)	(5)	(6)
1	$E_0\ R$	2	$\frac{P}{2\pi\ E_0\ R^2}$	0	$\frac{Pz}{2\pi\ E_0\ R^3}$
2	$E_0\ R$	3	$\frac{3P \cos \varphi}{4\pi\ E_0\ R^2}$	$\frac{-P \sin \varphi}{4\pi\ E_0\ R^2}$	$\frac{Pz^2}{4\pi\ E_0\ R^4} + \frac{P}{4\pi\ E_0\ R^2}$
3	$\frac{E_0\ z}{(f+z)}$	2	$\frac{3P\ (f+2z)}{4\pi\ E_0\ R^2}$	$\frac{-3P \sin \varphi}{4\pi\ E_0\ R}$	$\frac{3P\ (f+z)z}{4\pi\ E_0\ R^3} + \frac{3P}{4\pi\ E_0\ R}$
4	$E_0 \sqrt{z}$	2.5	$\frac{7P \cos \varphi}{6\pi\ E_0\ (R)^{3/2}}$	$\frac{-7P \sin \varphi}{15\pi\ E_0\ (R)^{3/2}}$	$\frac{7P\ z^2}{10\pi\ E_0\ R^{7/2}} + \frac{7P}{15\pi\ E_0\ R^{3/2}}$
5	$E_0 \cos \varphi$	2	$\frac{2P \cos \varphi}{\pi E_0\ R}$	$\frac{-P \sin \varphi}{\pi\ E_0\ R}$	$\frac{P\ z^2}{\pi\ E_0\ R^3} + \frac{P}{\pi\ E_0 R}$
6	$E_0\ z \cos \varphi$	2	$\frac{P}{\pi\ E_0\ R^2}$	0	$\frac{Pz}{\pi\ E_0\ R^3}$
7	$E_0\ z \cos \varphi$	3	$\frac{5P \cos \varphi}{4\pi\ E_0\ R^2}$	$\frac{-5P \sin \varphi}{12\pi\ E_0\ R^2}$	$\frac{5P\ z^2}{6\pi\ E_0\ R^4} + \frac{5P}{12\pi\ E_0\ R^2}$

TABLE 12

Expressions and Influence Factors for Vertical Displacement Under the Centre of a Uniformly Loaded Circular Area for Nonlinear Cases Given in Table 1

Case No.	ε_i	m	$\rho_{z_{NCU}}$	$K_{z_{NCU}}$
(1)	(2)	(3)	(4)	(5)
1	$\frac{\sigma_i}{E}$	2	$\frac{q}{E}\ A\ K_{z_{NCU}}$	$\frac{3L}{2(1+L^2)^{1/2}}$
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{q^{3/2}}{c^{3/2}}\ A\ K_{z_{NCU}}$	$\int_0^L \frac{4}{3L^2(1+L^2)^2}[\{(1+L^2)^{4/3}-1\}^{3/2}-(1+L^2)\{(1+L^2)^{4/3}-1\}^{1/2} \times\{(1+L^2)^{1/3}-1\}]dL$
3	$\left(\frac{\sigma_i}{c}\right)$	3	$\frac{q^2}{c^2}\ A\ K_{z_{NCU}}$	$\int_0^L \frac{1}{3(1+L^2)^2}[4L^2-(1+L^2)\log_e(1+L^2)]dL$
4	$\left(\frac{\sigma_i}{c}\right)^2$	4	$\frac{q^2}{c^2}\ A\ K_{z_{NCU}}$	$\int_0^L \frac{5}{4L^2(1+L^2)^{5/2}}[\{(1+L^2)^{5/4}-1\}^2(1+L^2)\{(1+L^2)^{5/4}-1\} \times\{(1+L^2)^{1/4}-1\}]\,dL$
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{q^{5/2}}{c^{5/2}}\ A\ K_{z_{NCU}}$	$\int_0^L \frac{L}{(1+L^2)^{5/2}}[5L^2-(1+L^2)\log_e(1+L^2)]\,dL$

(Continued)

TABLE 12—(*Continued*)

(1)	(2)	(3)	(4)	(5)
6	$\left(\frac{\sigma_i}{c}\right)^2\left(\frac{d}{z}\right)^{1/2}$	3,5	$\frac{q^2}{c^2}(dA)^{1/2}K_{z_{NCU}}$	(i) $\int_0^L \frac{1}{3L^{3/2}(1+L^2)^{9/4}}[4\{(1+L^2)^{9/8}-1\}^2-9(1+L^2)\{(1+L^2)^{9/8}-1\} \times\{(1+L^2)^{1/8}-1\}]\,dL$ for $m=3$ (ii) $\int_0^L \frac{1}{5L^{3/2}(1+L^2)^{9/4}}[6\{(1+L^2)^{9/8}-1\}^2-9(1+L^2)\{(1+L^2)^{9/8}-1\} \times\{(1+L^2)^{1/8}-1\}]\,dL$ for $m=5$
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{1/2}$	3,5	$\frac{q^{3/2}}{c^{3/2}}(dA)^{1/2}K_{z_{NCU}}$	(i) $\int_0^L \frac{1}{3L^{3/2}(1+L^2)^{9/4}}[4\{1+L^2)^{3/2}-1\}^{3/2}-3(1+L^2)\{1+L^2)^{3/2}-1\}^{1/2}\{(1+L^2)^{1/2}-1\}]\,dL$ for $m=3$ (ii) $\int_0^L \frac{1}{5L^{3/2}(1+L^2)^{9/4}}[6\{(1+L^2)^{3/2}-1\}^{3/2}-3(1+L^2)\{(1+L^2)^{3/2}-1\}^{1/2}\{(1+L^2)^{1/2}-1\}]\,dL$ for $m=5$

Note : For surface settlement the upper limit L in the integral is replaced by ∞.

TABLE 13

Expressions and Influence Factors for Vertical Displacement Under the Centre of a Circular Area for Uniformly Distributed and Parabolic Loading Considering Variable Modulus of Elasticity Given in Table 2

Case No.	$LF = F(R, \varphi)$	m	$\rho_{z_{VCU}}$	$K_{z_{VCU}}$	$\rho_{z_{VCP}}$	$K_{z_{VCP}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	$E_0 R$	2	$\frac{q}{E_0} K_{z_{VCU}}$	$\left[1 - \frac{1}{(1+L^2)^{1/2}}\right]$	$\frac{q}{E_0} K_{z_{VCP}}$	$2\left[1 - \frac{1}{(1+L^2)^{1/2}}\right] - \frac{2}{L^2}\left[(1+L^2)^{1/2} - 1\right] + \frac{2}{L^2}\left[1 - \frac{1}{(1+L^2)^{1/2}}\right]$
2	$E_0 R$	3	$\frac{q}{E_0} K_{z_{VCU}}$	$\frac{1}{2}\left[1 - \frac{1}{(1+L^2)} + \log e\,(1+L^2)^{1/2}\right]$	$\frac{q}{E_0} K_{z_{VCP}}$	$\left[1 - \frac{1}{(1+L^2)} + \log_e (1+L^2)^{1/2}\right] - \left[(1/2\,L^2) \log_e (1+L^2) - \frac{1}{(1+L^2)} + \frac{1}{2}\right]$
3	$\frac{E_0 z}{(f+z)}$	2	$\frac{d}{F_0} f K_{z_{VCU}}$	$\frac{3[(1+L^2)^{1/2}+(GL^2-1)]}{2(1+L^2)^{1/2}}$	$\frac{q}{E_0} f K_{z_{VCP}}$	$\frac{3[(1+L^2)^{1/2}+(GL^2-1)]}{(1+L^2)^{1/2}} - [3L^2+6-6(1+L^2)^{1/2} - 4G(1+L^2)^{1/2}+3G+G(1+L^2)^2]^2$ where $G = z/f$

(Continued)

TABLF 13—(*Continued*)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
4	$E_0 \sqrt{z}$	2.5	$\frac{q}{E_0} A^{1/2} K_{z_{VCU}}$	$\frac{14 [(1+2L^2)-(1+L^2)^{3/4}]}{15 L^{1/2} (1+L^2)^{3/4}}$	$\frac{q A^{1/2}}{E_0} K_{z_{VCP}}$	$\frac{28 [(1+2L^2)-(1+L^2)^{3/4}]}{15L^{1/2} (1+L^2)^{3/4}} - \frac{28[2L^4+9L^2+12-12(1+L^2)^{3/4}]}{75 L^{5/2} (1+L^2)^{3/4}}$
5	$E_0 \cos \varphi$	2	$\frac{q}{E_0} AK_{z_{VCU}}$	$\frac{2L}{(1+L^2)^{1/2}}$	$\frac{q}{E_0} AK_{z_{VCP}}$	$\frac{4L}{(1+L^2)^{1/2}} - \frac{4}{3L^3 (1+L^2)^{1/2}} \times [(1+L^2)^2-4 (1+L^2)^{1/3}+3]$
6	$E_0 z \cos \varphi$	2	$\frac{q}{E_0} K_{z_{VCU}}$	$2\left[1-\frac{1}{(1+L^2)^{1/2}}\right]$	$\frac{q}{E_0} K_{z_{VCP}}$	$4\left[1-\frac{1}{(1+L^2)^{1/2}}\right] - \frac{4}{L^2 (1+L^2)^{1/2}} [(2+L^2) -2 (1+L^2)^{1/2}]$
7	$E_0 z \cos \varphi$	3	$\frac{q}{E_0} K_{z_{VCU}}$	$\frac{5[2L^2+(1+L^2)\log_e(1+L^2)]}{12(1+L^2)}$	$\frac{q}{E_0} K_{z_{VCP}}$	$\frac{5 [2 L^2+(1+L^2) \log_e (1+L^2)]}{6 (1+L^2)} - \frac{5}{6L^2 (1+L^2)} [(1+L^2) \log_e (1+L^2) +L^2 (L^2-1)]$

Similarly in the case of displacement below the corner of a uniformly loaded rectangular area, the displacement is expressed in terms a dimensionless influence factor $K_{z_{VRU}}$. Table 14 presents these expressions.

Results

Tables 1 to 14 present a number of expressions which will aid in the determination of vertical stress and settlement below certain loaded areas. For a convenient use of these expressions, it will be desirable to present the results in the form of charts for a range of parameters. The values of the influence factors have been computed using ICL 1909 computer and the results are given in the form of graphs by Shahi (1980). Where the influence factors could not be obtained in closed form solutions, numerical integration using Simpson's one-third rule has been adopted. Only a few typical results are presented here in Figures 2 to 10.

Discussions

Vertical Stresses

It can be shown that the expressions for vertical stresses presented in Tables 3 and 4 take the form of Frohlich's equation for particular values of stress concentration factor κ. Frohlich's equation is as follows:

$$\sigma_z = \frac{P}{z^2}\,\frac{\kappa}{2\pi}\,\frac{1}{1+(r/z)^2} \qquad \ldots(9)$$

Where κ = Frohlich's stress distribution factor. The values of κ corresponding to the different cases of non-linearity and variable modulus of elasticity are showd in Table 15 and Table 16 respectively. κ varies from 2 to 3 in non-linear cases and 2 to 5 in the case of variable modulus of elasticity. In Frohlich's solution $\kappa = 3$ corresponds to the case of a linear medium. In Tables 15 and 16 the expression for σ_z due to Hruban also correspond to that of a linear medium when $\kappa = 3$ (cases 1 and 7 of Table 15 and Cases 2 and 3 of Table 16). This observation suggests the following:-

(*a*) The solution developed for cases of nonlinearity and variable modulus of elasticity can be used in the case of anisotropic soils also.

(*b*) The vertical stress distribution for a state of non-linearity would be identical to another state of variable modulus of elasticity or to another state of linear elastisity (or vice-versa). Sometimes a combination of these is also possible. For example, the stress distribution for case 6 of Table 15 which is a case of non-linearity cum non-homogeneity, will be identical to the stress pattern in a anisotropic soil medium for which $\kappa = 2.25$.

A perusal of the various figures indicate the following:

(*a*) The different relationships considered by Hruban for non-linear cases indicate that there is little likelihood of κ being greater than 3. In such cases the stresses below the loaded area are less than that given by Boussinesq's solution. (e.g. Figure 2).

TABLE 14

Expressions and Influence Factors for Vertical Displacement Under the Corner of a Uniformly Loaded Rectangular Area Considering Variable Modulus of Elasticity Given in Table 2

Case No.	$LF=F(R, \varphi)$	m	$\rho_{z_{VRU}}$	$K_{z_{VRU}}$
(1)	(2)	(3)	(3)	(5)
1	$E_o R$	2	$\frac{q}{E_0} K_{z_{VRU}}$	$\frac{1}{2\pi}\int_0^{M/N}\int_0^{1/N} \frac{dX\,dY}{(X^2+Y^2+1)^{3/2}}$
2	$E_o R$	3	$\frac{q}{E_0} K_{z_{VRU}}$	$\frac{1}{4\pi}\int_0^{M/N}\int_0^{1/N} \frac{[2+(X^2+Y^2+1)]}{(X^2+Y^2+1)^2}\,dX\,dY$
3	$\frac{E_o z}{(f+z)}$	2	$\frac{q}{E_0} f K_{z_{VRU}}$	$\frac{3}{4\pi}\int_0^{M/N}\int_0^{1/N} \frac{[1+G(X^2+Y^2+2)]}{(X^2+Y^2+1)^{3/2}}\,dX\,dY$
4	$E_o \sqrt{z}$	2.5	$\frac{q}{E_0} b^{1/2} K_{z_{VRU}}$	$\frac{1}{30\pi}\int_0^{M/N}\int_0^{1/N} \frac{N^{1/2}[21+14(X^2+Y^2+1)]}{(X^2+Y^2+1)^{7/4}}\,dX\,dY$

5	$E_o \cos\varphi$	2	$\frac{q}{E_0} b\, K_{z_{VRU}}$	$\frac{1}{\pi} \int_0^{M/N} \int_0^{1/N} \frac{N[1+(X^2+Y^2+1)]\, dX\, dY}{(X^2+Y^2+1)^{3/2}}$
6	$E_o\, z \cos\varphi$	2	$\frac{q}{E_0} K_{z_{VRU}}$	$\frac{1}{\pi} \int_0^{M/N} \int_0^{1/N} \frac{dX\, dY}{(X^2+Y^2+1)^{3/2}}$
7	$E\, z_o \cos\varphi$	3	$\frac{q}{E_0} K_{z_{VRU}}$	$\frac{1}{12\pi} \int_0^{M/N} \int_0^{1/N} \frac{[10+5(X^2+Y^2+1)]}{(X^2+Y^2+1)^2}\, dX\, dY$

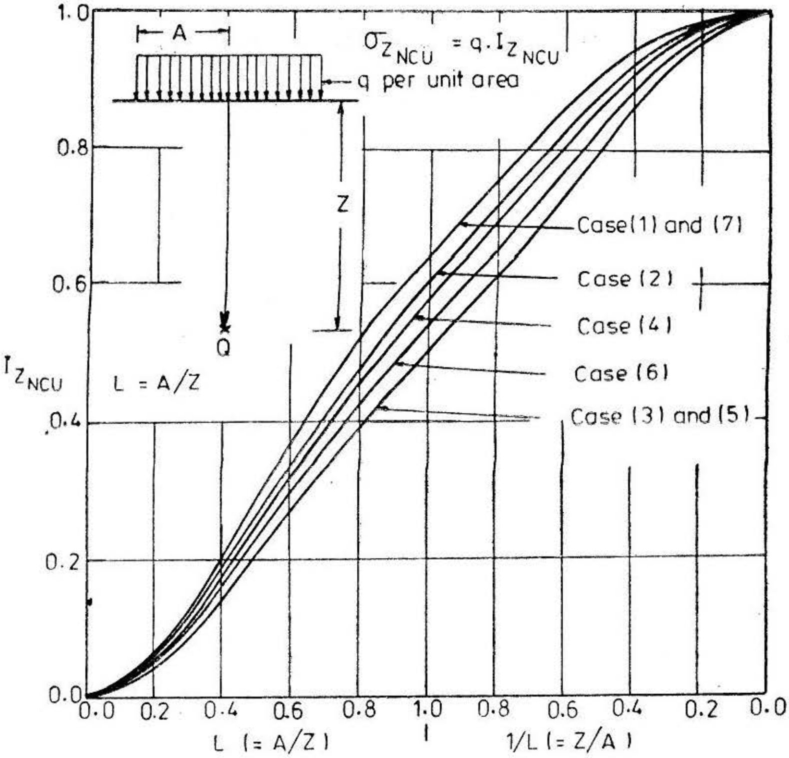


FIGURE 2 Influence factor for vertical stress under the centre of a uniformly loaded circular area for nonlinear cases given in Table 3

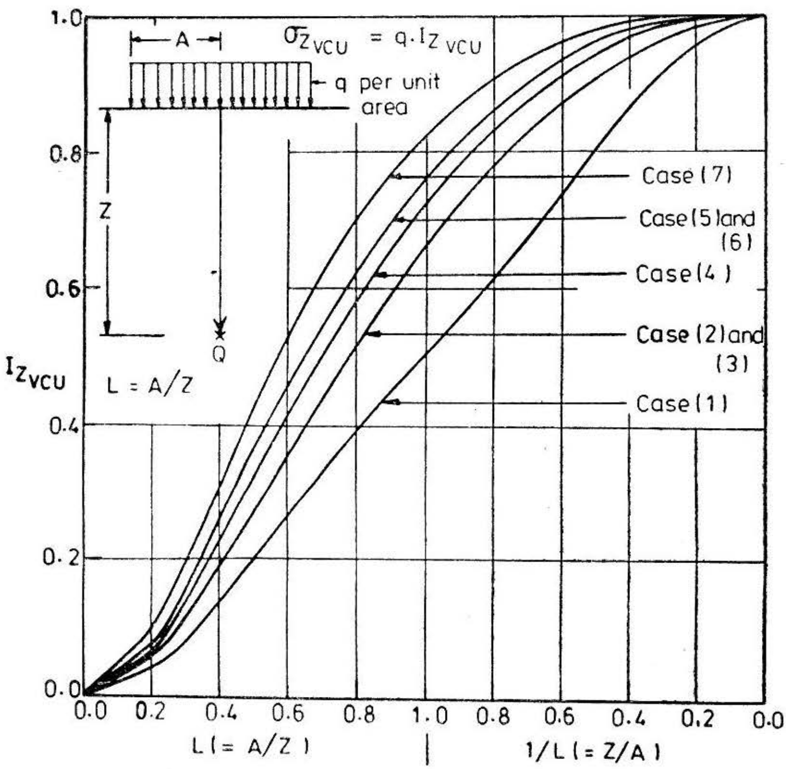


FIGURE 3 Influence factor for vertical stress under the centre of a uniformly loaded circular area for cases of variable modulus of elasticity given in Table 4.

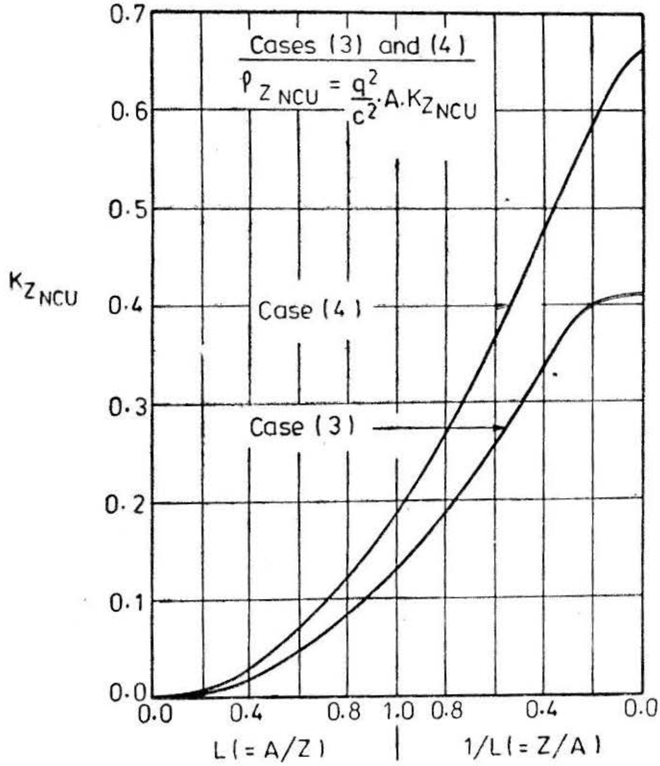
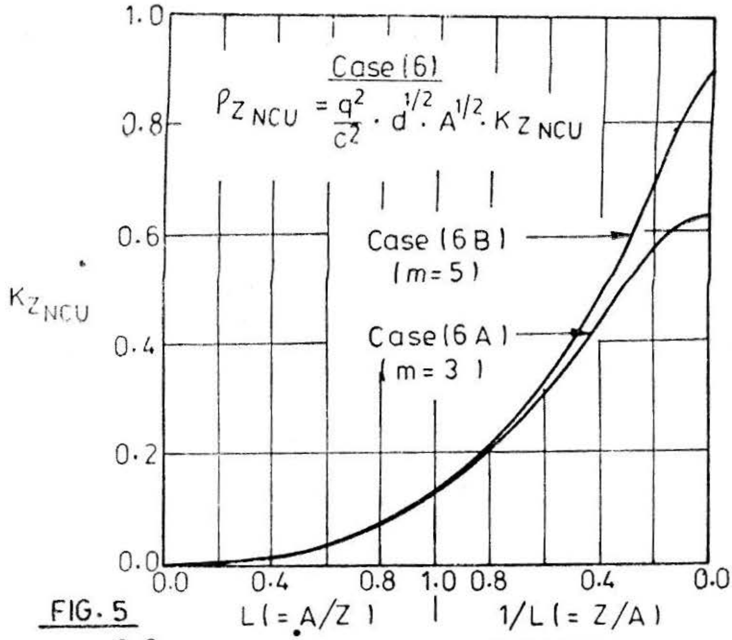


FIGURE 4 Influence factor for vertical displacement under the centre of a uniformly loaded circular area for nonlinear cases given in Table 10

(*b*) The vertical stress below a loaded area on a non-homoenous medium can be either greater than or less than that given by Boussinesq's solution (e.g. Figure 3), depending on the value of κ being more than 3 or less than 3 respectively. However, the type of expressions chosen by Hruban for *LF* (i.e. variable modulus) are generally for increasing modulus with depth and for *almost incompressible soil.* For such type of soil the vertical stress is more than that given by Boussinesq's solution. But a particular type of non-homogeneity which also includes non-linearity (case 6 of Table 15, Figure 2) gives stresses lower than those given by Boussinesq's equation.

With $m = 3$ and deformation modulus varying linearly with depth Klein presented solution for vertical stress below the centre of a circular footing. The comparison of Klein's influence factors with those of Cases 2 and 7 of Table 8 (modulus increasing linearly along radial direction and $m = 3$) is presented in Table 17. Among these three, Case 2 is for the least compressible medium and Case 7 is for the most compressible, with Klein's medium being in between cases 2 and 7 in compressibility. From the influence factors listed in Table 17, it is evident that the stresses transmitted in the medium depend on the deformability of the medium. Lesser the compressibility, lesser are the stresses.

FIG. 5



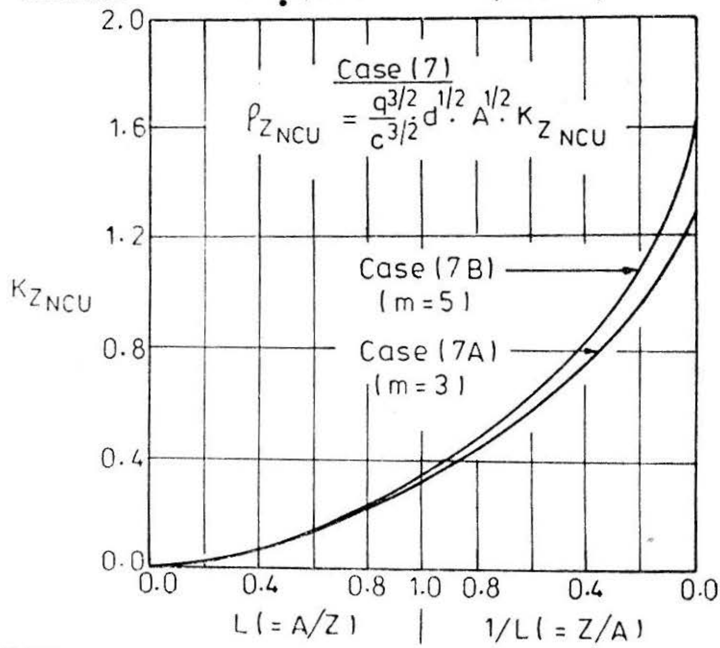


FIGURE 6 Influence factor for vertical displacement under the centre of a uniformly loaded circular area for nonlinear cases given in Table 10

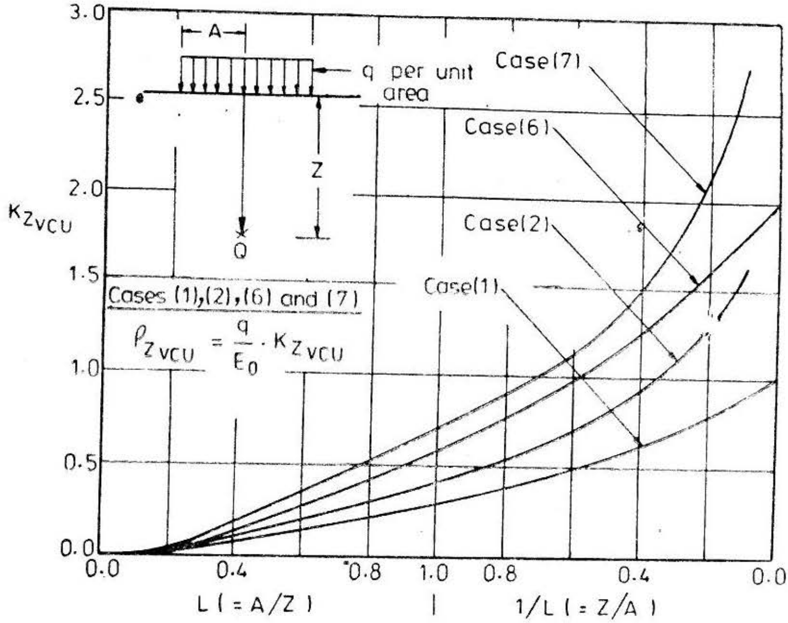


FIGURE 7 Influence factor for vertical displacement under the centre of a uniformly loaded circular area for cases of variable modulus of elasticity given in Table 11

TABLE 15

Values of Frohlich's Stress Concentration Factor for Vertical Stress due to Concentrated Point Loading for Nonlinear Cases Given in Table 3

Case No.	ε_i	m	σ_z	Frohlich stress concentration factor (κ)
(1)	(2)	(3)	(4)	(5)
1	$\frac{\sigma_i}{E}$	2	$\frac{3P\,z^3}{2\pi\,R^5}$	3
2	$\left(\frac{\sigma_i}{c}\right)^{3/2}$	3	$\frac{4P\,z^{8/3}}{3\pi\,R^{14/3}}$	2.67
3	$\left(\frac{\sigma_i}{c}\right)^2$	3	$\frac{P\,z^2}{\pi\,R^4}$	2
4	$\left(\frac{\sigma_i}{c}\right)^2$	4	$\frac{5P\,z^{5/2}}{4\pi\,R^{9/2}}$	2.5
5	$\left(\frac{\sigma_i}{c}\right)^{5/2}$	4	$\frac{P\,z^2}{\pi\,R^4}$	2
6	$\left(\frac{\sigma_i}{c}\right)^2\left(\frac{d}{z}\right)^{1/2}$	3,5	$\frac{9P\,z^{9/4}}{8\pi\,R^{17/4}}$	2.25
7	$\left(\frac{\sigma_i}{c}\right)^{3/2}\left(\frac{d}{z}\right)^{1/2}$	3,5	$\frac{3P\,z^3}{2\pi\,R^5}$	3

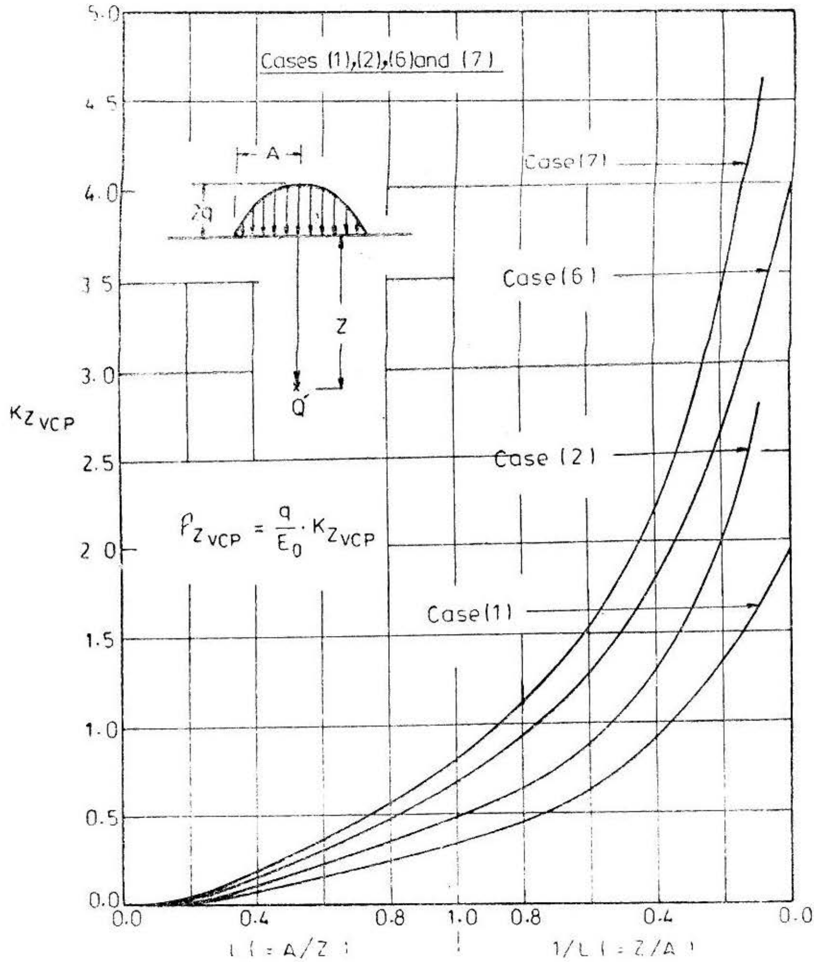


FIGURE 8 Influence factor for vertical displacement under the centre of a parabolically loaded circular area for cases of variable modulus of elasticity given in Table 11

Expressions in Table 5, 6 and 7 for cases of nonlinearity indicate that a change in the value of m or Poisson's ratio has no effect on vertical stresses (Cases 6 and 7). But when the medium is non-homogeneous (Cases 1 and 2 and Cases 6 and 7 of Tables 8 and 9) the value of Poisson's ratio influences the vertical stresses. It can be observed from numerical calculations that a decrease in Poisson's ratio increases the vertical stress.

Vertical Displacements

From Figures 4, 5 and 6 it can be seen that an increase in the value of m or a decrease in Poisson's ratio for the same case of non-linearity causes an increase in displacement.

The results in case of non-homogeneous medium reported in Tables 13 and 14 for Cases 1,2,6 and 7 are noteworthy. These are cases in

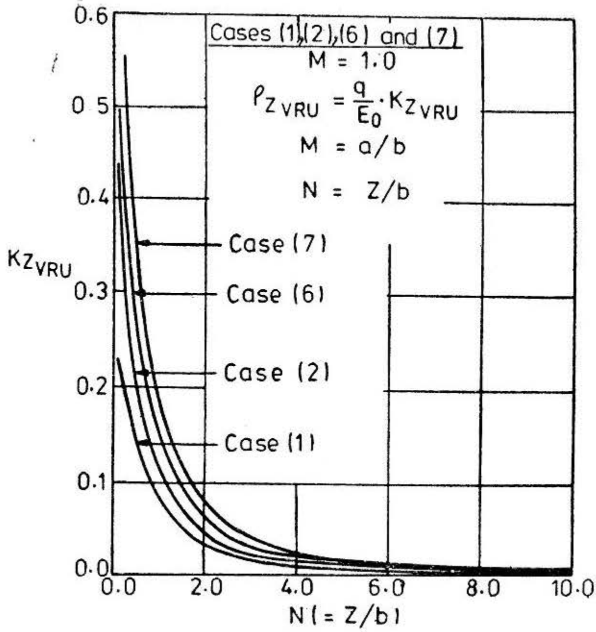


FIGURE 9 Influence factor for vertical displacement under the corner of a uniformly loaded rectangular area for case of variable modulus of elasticity given in Table 11.

TABLE 16

Values of Frohlich's Stress Concentration Factor for Vertical Stress due to Concentrated Point Loading for Variable Modulus of Elasticity Given in Table 4

Case No.	$LF = F(R, \varphi)$	m	σ_z	Frohlich's stress concentration factor (κ)
(1)	(2)	(3)	(4)	(5)
1	$E_0\, R$	2	$\frac{P\, z^2}{\pi\, R^4}$	2
2	$E_0\, R$	3	$\frac{3P\, z^3}{2\pi\, R^5}$	3
3	$\frac{E_0\, z}{(f+z)}$	2	$\frac{3P\, z^3}{2\pi\, R^5}$	3
4	$E\sqrt{z}$	2.5	$\frac{7P\, z^{7/2}}{4\pi\, R^{11/2}}$	3.5
5	$E_0 \cos \varphi$	2	$\frac{2P\, z^4}{\pi\, R^6}$	4
6	$E_0\, z \cos \varphi$	2	$\frac{2P\, z^4}{\pi\, R^6}$	4
7	$E_0\, z \cos \varphi$	3	$\frac{5P\, z^5}{2\pi\, R^7}$	5

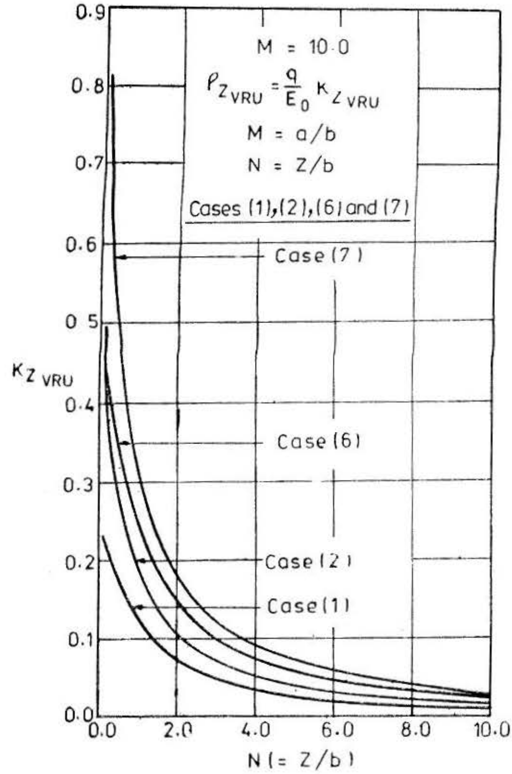


FIGURE 10 Influence factor for vertical displacement under the corner of uniformly loaded rectangular area for cases of variable modulus of elasticity given in Table 11.

TABLE 17

Comparison of Klein's Influence Factors for Vertical Stress Under the Centre of a Uniformly Loaded Circular Area with those of Cases 2 and 7 of Variable Modulus of Elasticity Given in Table 8

Sl. No.	A/z	$I_{z_{VCU}}$ by Klein's solution ($E = E_0 z$)	$I_{z_{VCU}}$ by case 2 of Table 8 ($E = E_0 R$)	$I_{z_{VCU}}$ by case 7 of Table 8 ($E = E_0 \cos\varphi$)
(1)	(2)	(3)	(4)	(5)
1	∞	1.000	1.000	1.000
2	4	0.996	0.985	0.999
3	2	0.960	0.910	0.982
4	1	0.750	0.646	0.823
5	0.5	0.360	0.284	0.427
6	0.2	0.077	0.057	0.093

which modulus of deformation increases linearly along the radial directions. It can be noted that the displacement is independent of the size of the foundation in these cases.

The comparison of the influence factors for Cases 1,2, 6 and 7 shown in Figures 7,8,9 and 10 reveals the following. Cases 1 and 2 have identical expression for modulus of deformation, with the latter having a higher value of m than the former. The same is true for the combination of Cases 6 and 7. Because of the higher value of m the latter cases show a higher displacement than the former. Between the two combinations of cases, Cases 6 and 7 have a higher displacement than Cases 1 and 2. This is because, the modulus increases at a slower rate for Cases 6 and 7 than for Cases 1 and 2.

Conclusions

Hruban's solutions for stresses and displacement for point loading in non-linear and non-homogeneous soil medium have been extended to circular loaded areas with uniform and parabolic loading and to rectangular areas with uniform loading. The expressions for vertical stresses and displacements in all cases are expressed in terms of non-dimensional influence factor which is again a function of dimensionless parameters. To facilitate a convenient use of the different expressions and also for a better appreciation of the results, the influence factors are presented in the form graphs for a useful range of variation of the parameters. It has been demonstrated that the solutions for stresses developed herein for cases of non-linearity and variable modulus of elasticity can be used for anisotropic soils also. The vertical stress and displacement are shown to depend upon the value of Poisson's ratio. Lower the Poisson's ratio, higher will be the vertical stress and the associated vertical displacement. Similarly a decrease in deformation modulus also causes an increase in vertical stress at any point.

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Notations

A = radius of the circular loaded area

a = half the length of rectangular loaded area

b = half the breadth of rectangular loaded area

c, d = constants, characteristic of the material

Eo = Young's modulus of elasticity at surface

E = modulus of deformation = σ_c / ϵ_i

f = a constant

Iz_{NCP} = influence factor for vertical stress under the centre of a circular area for parabolic loading for nonlinear cases

Iz_{NCU} = influence factor for vertical stress under the centre of a uniformly loaded circular area for nonlinear cases

Iz_{NRU} = influence factor for vertical stress under the corner of a uniformly loaded rectangular area for nonlinear cases

Iz_{VCP} = influence factor for vertical stress under the centre of a circular area for parabolic loading considering variable modulus of elasticity

Iz_{VCU} = influence factor for vertical stress under the centre of a uniformly loaded circular area for variable modulus of elasticity

Iz_{VRU} = influence factor for vertical stress under the corner of a uniformly loaded rectangular area for variable modulus of elasticity

Kz_{NCU} = influence factor for vertical displacement below the centre of a uniformly loaded circular area for non-linear cases

Kz_{VCP} = influence factor for vertical displacement below the centre of a circular area with parabolic loading considering variable modulus of elasticity

Kz_{VCU} = influence factor for vertical displacement below the centre of a uniformly loaded circular area for variable modulus of elasticity

Kz_{VRU} = influence factor for vertical displacement below the corner of a uniformly loaded rectangular area for variable modulus of elasticity

L = dimensionless parameter, A/z

M = dimensionless parameter, a/b

m = reciprocal of Poisson's ration, $1/\mu$

N = dimensionless parameter, z/b

n = a constant

P = concentrated vertical load

q = average intensity of loading

q_r = intensity of loading at distance r from the centre of a circular area for parabolic loading

q_{max} = maximum intensity of loading at the centre of a circular area for parabolic loading ($=2q$)

R, φ, ψ = spherical co-ordinates

r, θ, z = cylindrical co-ordinates

u, v = *displacements in R-and φ-directions* respectively, for axially symmetric case due to point loading

ϵ_i = $(\sigma_i/c)^n$

ρ_z = vertical displacement due to point loading

ρz_{NCU} = vertical displacement below the centre of a uniformly loaded circular area for non-linear cases

ρz_{VCP} = vertical displacement below the centre of a circular area with parabolic loading considering variable mouduls of elasticity

ρz_{VCU} = vertical displacement below the centre of a uniformly loaded circular area for variable modulus of elasticity

ρz_{VRU} = vertical displacement under the corner of a uniformly loaded rectangular area for variable modulus of elasticity

σ_z = vertical stress due to point loading

σz_{NCP} = vertical stress under the centre of a circular area for parabolic loading for non-linear cases

σz_{NCU} = vertical stress under the centre of a uniformly loaded circular area for non-linear cases

σz_{NRU} = vertical stress under the corner of a uniformly loaded rectangular area for non-linear cases

σz_{VCP} = vertical stress under the centre of a circular area for parabolic loading considering variable modulus of elasticity

σz_{VCU} = vertical stress under the centre of a uniformly loaded circular area for variable modulus of elasticity

σz_{VRU} = vertical stress under the corner of a uniformly loaded rectangular area for variable modulus of elasticity

K = Frohlich's stress concentration factor