Sequential Unconstrained Minimization In The Optimal Lower Bound Bearing Capacity Analysis

by

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Introduction

In the limit equilibrium solution or the upper and lower bound limit analysis of stability problems in geotechnical engineering, the problem is one of finding the extrema of a function subjected to some constraints. Many contemporary problems of design and analysis involve not only equality constraints but also inequalities. It has been realized that the mathematical problems that arose in their study stretched the limits of conventional analysis, and require methods, such as, mathematical programming techniques for their successful treatment. The classical techniques of Calculus and Calculus of Variations are occasionally valuable in these new areas, but are clearly limited in their range and versatility.

As with the use of any type of analytic or numerical techniques within the context of complex problem solving, the focus for discussion falls not only on the various techniques available for the analysis but also on the art of how such mathematical procedures are applied. Large scale systems may present considerable problems in terms of the number of decision variables and objectives. These issues must be acknowledged and addressed in a straight forward manner with proper attention paid to the particular important aspects of a given problem. No one procedure or series of procedure will be the pinacea that solves all problems to the last details. Optimization is a useful tool for design and analysis but its successful application depends to a great extent on how it is used. While optimization theory is well established for well defined systems with specific objectives and models its successful application is problem oriented. These methods are in extensive use in structural and other branches of engineering. Recently some attempts have been made in this direction in geotechnical engineering (Horn, 1960; Wu and Kraft, 1970; Lysmer, 1970; Potchman and Kolesnichenko, 1972; Krugman and Krizek, 1973; Gioda and Donato, 1979). Some studies had been undertaken by the authors, to explore the strength and limitations of these techniques in analysing some stability problems in geotechnical engineering and their findings were reported (Basudhar, 1976; Basudhar et al, 1978, 1979a, 1979b). The study

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reported herein, pertains to the application of sequential unconstrained minimization technique (SUMT) to isolate the optimal lower bound solution of bearing capacity problems.

Analysis

The generalized method of lower bound limit analysis as developed by Lysmer (1970) and subsequently modified by Basudhar (1976) to incorporate the nonlinear no yield condition Constraints directly in the analysis is used for the problem formulation. For the sake of completeness the method is presented herein in brief and the readers are referred to the original work of Lysmer (1970) and Basudhar et al (1979b) for details.

The first step in the analysis of a typical problem, such as, the bearing capacity problem shown in Figure 1, is the discretisation of the soil mass under consideration into a mesh of finite number of triangular elements. All nodal points, elements, ane element sides are then numbered in some arbitrary order. The geometry of a typical element, n, the six external stresses and the body forces acting on the element are shown in Figure 2. Only the stresses at the nodes are considered since the stresses are assumed to vary linearly within each element. In addition, one internal stress σ^n is defined as the normal stress at node i acting on a plane parallel to the side jk. The normal stresses on each element are combined to form a vector $\{\sigma\}^n$ defined as

$$\{\sigma\}^{n^T} = \{\sigma^n \; \sigma_{ik} \; \sigma_{ij} \; \sigma_{ji} \; \sigma_{jk} \; \sigma_{kj} \; \sigma_{ki}\} \qquad ...(1)$$

The external shear stresses are combined into a vector $\{\tau\}^n$, where

$$\{\tau\}^{n^T} = \{\tau_{ik} \; \tau_{ij} \; \tau_{ji} \; \tau_{jk} \; \tau_{kj} \; \tau_{ki}\} \qquad ...(2)$$

The internal stresses in each element are expressed as

$$\{s\}^T = \overline{\{s_i \; s_j \; s_k\}} \text{ with } \{s_i\}^T = \{\sigma_{z,\,i} \; \sigma_{x,\,i} \; \tau_{zx,\,i}\} \text{ etc.} \qquad ...(3)$$

$\overline{\{s_i\}}$ are the internal stresses at node i. From the equilibrium of infinitismal elements at node i, j and k the following relations between the internal and external stresses can be established:

$$\{\sigma\}^n = [S]\,\{s\} \qquad ...(4)$$

$$\{\tau\}^n = [T]\,\{s\} \qquad ...(5)$$

The matrices $[S]$ and $[T]$ consist of geometric properties of each element.

Following Lysmer (1970) the internal equilibrium can be maintained by assuming a linear stress field and the external shear stress vector $\{\tau\}^n$ and the internal stress vector $\{s\}$ can be expressed in terms of the element normal stress vector $\{\sigma\}^n$ as follows:

$$\{\tau\}^n = [T]\,[B]\,\{\sigma\}^n + [T]\,\{h\} \qquad ...(6)$$

$$\{s\} = [B]\,\{\sigma\}^n + \{h\} \qquad ...(7)$$

in which

$$[B] = [G]\,([S]\,[G])^{-1} \qquad ...(8a)$$

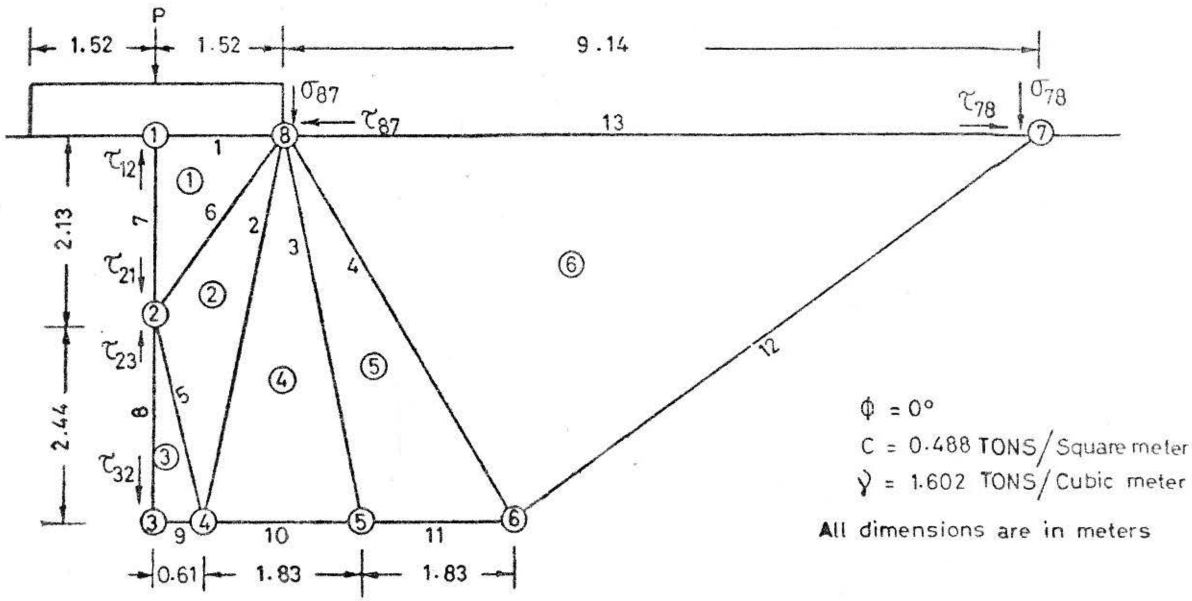

FIGURE 1 Mesh for Beariug Capacity Problem

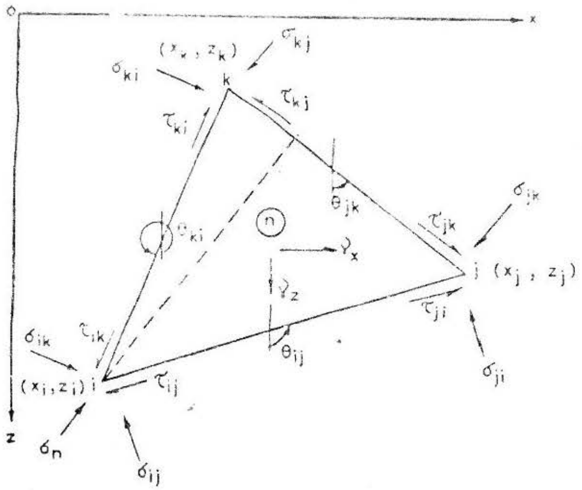

FIGURE 2 Definition Sketch Geometry Nodal Normal and Shear Stresses and body forces

$$\{h\} = \{g\} - [B]\,[S]\,\{g\} \qquad \ldots(8b)$$

The matrix $[G]$ consists of the node co-ordinates.

The vector $\{g\}$ is as follows:

$$\{g\}^T = \{\gamma_z\, z_i,\ \gamma_x\, x_i,\ 0,\ \gamma_z\, z_j,\ \gamma_x\, x_j,\ 0,\ \gamma_z\, z_k,\ \gamma_x\, x_k,\ 0\} \qquad (8c)$$

The elements of all $\{\sigma\}^n$ vectors for all the elements are collected into a general $\{\sigma\}$ vector as per the rule enunciated by Lysmer (1970). The stresses are the principal unknowns. The interface and the boundary conditions are expressed in terms of these values. A system consisting of p elements connected at q nodal points will have $(3p+2q-2)$ stress variables.

The stresses acting on the interface between two typical elements m and n are shown in Figure 3. The continuity of normal and shear stresses across any interface requires

$$\overset{m}{\sigma}_{ij} = \overset{n}{\sigma}_{ij} \text{ and } \overset{m}{\tau}_{ij} = \overset{n}{\tau}_{ij} \qquad \ldots(9)$$

for all corresponding values of i, j, m and n. These conditions yields a set of linear equality constraints in terms of the principal unknowns. The boundary stresses on the external faces of the system may be expressed either in the form

$$\tau_{ij} \leqslant \mu\, \sigma_{ij} \qquad \ldots(10)$$

or

$$\sigma_{ij} = \eta \text{ and } \tau_{ij} = \zeta \sigma_{ij} \qquad \ldots(11)$$

where μ, η ane ζ are known constants.

Equations (9), (10) and (11) can be transformed into the form

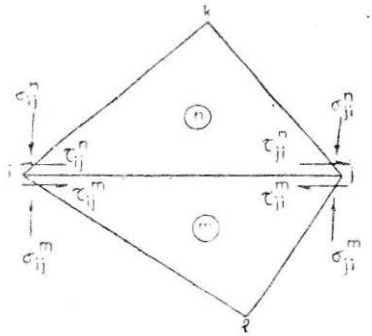

FIGURE 3 Continuity of Nodel Stresses

$$\sum_{j=1}^{3p+2q-2} a_{ij}\,\sigma_j = b_i \quad \text{and/or} \quad \sum_{j=1}^{3p+2q-2} a_{ij}\,\sigma_j \leqslant b_i \qquad ...(12)$$

At any node i, the stresses should not violate the Mohr Coulomb failure condition, i.e.

$$(\sigma_{z,i}-\sigma_{x,i})^2+(2\tau_{zx,i})^2 \leqslant [(\sigma_{z,i}+\sigma_{x,i})\,\text{Sin}\,\phi+2c\,\text{Cos}\,\phi]^2 \qquad ...(13)$$

The friction angle ϕ and c are assumed to be constant within each element. Equation (13) can easily be expressed in terms of the principal unknown as follows:

$$\sigma_{z,i} = Z_i\,\{s\},\ \sigma_{x,i} = X_i\,\{s\} \text{ and } \tau_{zx,i} = T_i\,\{s\} \qquad ...(14)$$

Where

$$Z_i = (1, 0, 0, 0, 0, 0, 0, 0, 0) \qquad ...(15a)$$

$$X_i = (0, 1, 0, 0, 0, 0, 0, 0, 0) \qquad ...(15b)$$

$$T_i = (0, 0, 1, 0, 0, 0, 0, 0, 0) \qquad ...(15c)$$

Similarly for nodes j and k

$$Z_j = (0, 0, 0, 1, 0, 0, 0, 0, 0) \qquad ...(16a)$$

$$X_j = (0, 0, 0, 0, 1, 0, 0, 0, 0) \qquad ...(16b)$$

$$T_j = (0, 0, 0, 0, 0, 1, 0, 0, 0) \qquad ...(16c)$$

and

$$Z_k = (0, 0, 0, 0, 0, 0, 1, 0, 0) \qquad ...(17a)$$

$$X_k = (0, 0, 0, 0, 0, 0, 0, 1, 0) \qquad ...(17b)$$

$$T_k = (0, 0, 0, 0, 0, 0, 0, 0, 1) \qquad ...(17c)$$

Substitution of the values of $\sigma_{z,i}$, $\sigma_{x,i}$ and $\tau_{zx,i}$ from Equation 14 in Equation 13 yields

$$(A_i\,\{s\})^2+4(T_i\,\{s\})^2-(B_i\,\{s\}\,\text{Sin}\,\phi+2c\,\text{Cos}\,\phi)^2 \leqslant 0 \qquad ...(18)$$

Where

$$A_i = Z_i - X_i \qquad ...(19a)$$

$$B_i = Z_i + X_i \qquad ...(19b)$$

Substituting 7 in Equation 18 one gets

$$[A_i([B]\{\sigma\}^n-\{h\})]^2+4[T_i([B]\{\sigma\}^n+\{h\})]^2$$
$$-[B_i([B]\{\sigma\}^n+\{h\}) \text{ Sin } \phi+2c \text{ Cos } \phi]^2 \leqslant 0 \qquad ...(20)$$

Similar relations can be obtained for the nodes j and k. The elements of $\{\sigma\}^n$ vector can easily be picked up from the general stress vector $\{\sigma\}$ by following the scheme enunciated by Lysmer (1970).

Since in general infinitely many stress fields will satisfy the aforementioned conditions of static admissibility, the problem is therefore to isolate the stress field which optimizes the bearing capacity. In such problems the stress quality which is desired to be minimized is a linear combination of the principal unknowns σ_j, as follows:

$$\text{optimize} \quad \Sigma\, a_j\, \sigma_j \qquad ...(21)$$

The design restrictions are the interface equilibrium conditions and the external boundary conditions, Equation 12, and the no yield criterion Equation 20. As the soil can not take tension, the following constraints are also introduced.

$$-\sigma_j \leqslant 0 \qquad ...(22)$$

The inequality constraints are designated as

$$g_j \leqslant 0 \qquad ...(23)$$

The equality constraints of Equation 12 are written in matrix notation as follows.

$$[A]\{\sigma\} = \{b\} \qquad ...(24)$$

In some of the elements of vector are specified at the boundary the following relation can be arrived at by eliminating the corresponding columns of $[A]$ matrix. Then

$$[A^*]\{\sigma^*\} = \{b^*\} \qquad ...(25)$$

By expressing some design variables, in terms of the remaining variables the equality constraints (Equation 25) are implicitly satisfied. Such a technique helps in reducing the complexity of the problem by elimination of the equality constraints and reducing the dimensionality of the problem.

The following calculations are performed for the general rectangular matrix $[A^*]$.

(*i*) The rank and the linearly dependent rows and columns if there be any, of the matrix, is determined.

(*ii*) A submatrix of maximal rank is expressed as product of triangular factors.

(*iii*) The non-basic rows are expressed in terms of the basic ones.

(*iv*) The basic variables are expressed in terms of the free variables.

The rank of the general rectangular matrix $[A^*]$ is determined using the standard Gaussian elimination technique with complete pivoting. The values of the free variables contained in $\{D\}$ may be chosen arbitrarily. In the present study the standary library subroutine MFGR developed by IBM has been used to perform the reduction of the design variables as mentioned.

After elimination of all the linear equality constraints the problem contains only inequality constraints of the form presented in Equations 12, 20 and 22.

Finding the minimum value of the objective function subjected to the inequality constraints as described is formulated as a non-linear programming problem which is stated as follows.

Find D_m such that,

$$\sum_{j} a_j \sigma_j = F(D_m) \text{ is minimum} \qquad \ldots(26a)$$

Subject to $$g_j(D_m) \leqslant 0 \qquad \ldots(26b)$$

In many complex physical systems, it is extremely difficult to obtain an initial feasible design vector, and, as such the interior penalty function method can not be used. In such cases the problem has to be solved either by using the exterior penalty function method or obtaining an initial feasible design vector following a procedure as suggested by Fox (1971) and using the interior penalty function method. Even when the interior penalty function method is used, during the progress of the unconstrained minimization the path may be diverted into infeasible regions. In such cases the function is set to an arbitrary high value and the minimization procedure is left to correct the situation on its own. Sometimes this approach presents numerical difficulties. As such, in the present study an extended penalty function method enunciated by Kavlie (1971) has been used. This readily accepts infeasible design points and needs no special treatment. In the penalty function method the constraints are blended into a composite function $\phi(D, r_k)$ and a sequential unconstrained minimization of this function is carried out. In some cases the penalty function approach is the most efficient means of solving a problem. However, in a number of cases it is preferred because of its simplicity rather than its efficiency. The problem is stated as follows:

$$\min_{D} \phi(D, r_k) = F(D) + r_k \sum_{j=1}^{M} G[g_j(D)] \qquad \ldots(27)$$

where M is the total number of inequality constraints. The function $G[g_j(D)]$ is chosen as suggested by Kavlie (1971).

$$G[g_j(D)] = \begin{cases} 1/g_j(D) & g_j \leqslant 0 \\ [2\,\epsilon - g_j(D)/\epsilon^2];\ g_j(D) > \epsilon \end{cases} \qquad \ldots(28)$$

where $\epsilon = -r_k/\delta_t$ and δ_t is a constant that defines the transition between the two types of penalty term. In this approach infeasible starting points are acceptable to the minimization algorithms. Unconstrained minimization is carried out using Powell's algorithm along with quadratic fit (Fox, 1971) for linear minimization.

Results and Discussions

To show the effectiveness of the present approach one of the examples presented by Lysmer (1970) is considered and solved.

The physical system consists of a smooth strip footing on a homogeneous purely cohesive ($c = 488.2$ kg/m^2) soil having a unit weight (γ_z) of 1601.8463 kg/m^3.

Figure 1 shows the mesh used for the calculation of the bearing capacity.

The boundary conditions are :

$$\tau_{18} = \tau_{81} = \tau_{87} = \tau_{78} = \sigma_{87} = \sigma_{78} = \tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = 0 \quad \ldots (29)$$

The expression to be optimized is $(\sigma_{18} + \sigma_{81})$.

In this six-element problem there are 32 elements in the general vector. Two normal stresses are known from boundary condition, Rank analysis is performed for the matrix $[A^*]$ obtained from the coefficient matrix $[A]$. $[A^*]$ is a 18×30 matrix. The rank is observed to be 17. Hence there will be 13 number of free design variables. The rest of the variables are expressed in terms of these free variables.

The optimal solution is obtained by nonlinear programming and the complete solution is presented in Table 1. The Terzaghi bearing capacity factor N_c is

$$N_c = 0.5\,(4.977 + 5.016) = 4.996.$$

The value obtained by Lysmer (1970) is 5.03 and the exact value is 5.14. From the comparison of the complete stress field (Table 1) it may be concluded that the present approach gives fairely accurate results.

The problem is solved with two different starting points and ϵ values (Table 2). For both the cases δ_f is taken as 10. To show the influence of the starting point the objective function plotted against the total number of function evaluation and presented in Figure 4. It may be observed that for set 1 when ϕ function is obtimized there is an initial decrease *in* function value. As the number of function evaluations increases, there is a sharp rise in the objective functiōn value upto a certain limit after which the increase in the function value is gradual. It may be observed that for set 1 when the total number of function evaluation is more than 1100, the increase in the objective function value is not at all appreciable and the solution does not converge to the Lysmers value. For the set 2 the design point is near optimum and the progress of the solution is very slow but converges to the Lysmer's solution (1970).

In Table 3 the optimum stress field along with the values of the equality and inequality constraints are present. It may be observed from the table that the order of the magnitude of the equality constraints are so small that they may be considered to be satisfied. It is also seen that all the inequality constraints are satisfied.

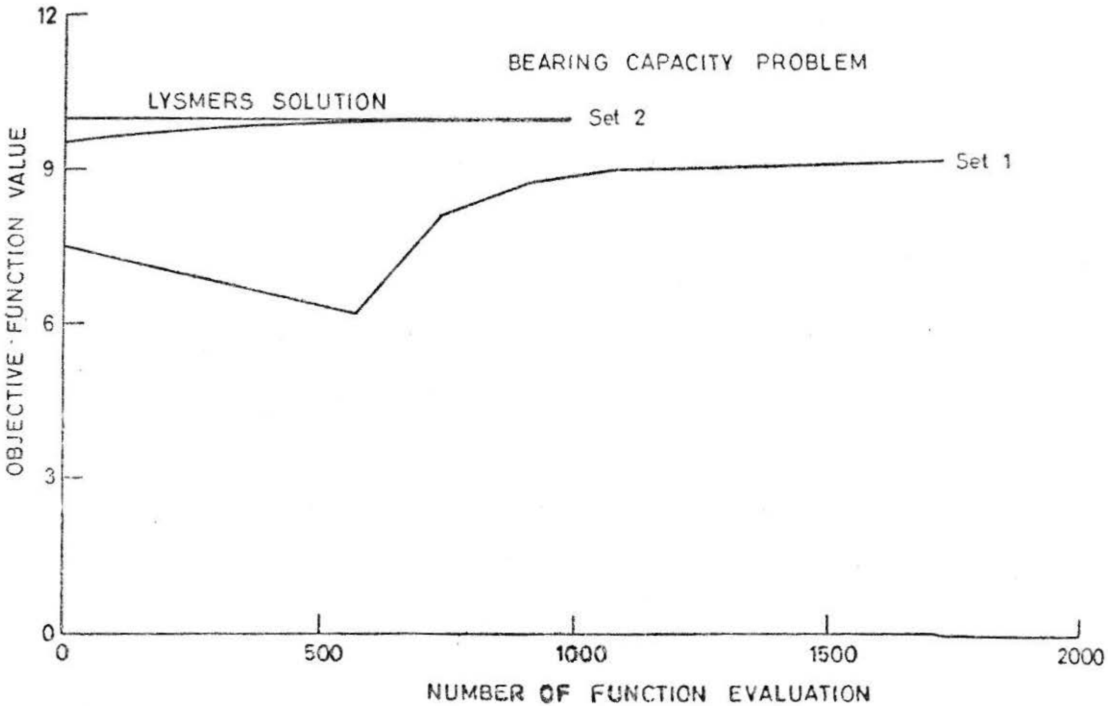

FIGURE 4 Path of objective function with Number of function evaluation

TABLE 1

Comparison of the present solution with Lysmer's solution for a bearing capacity problem

Element No.	Nodal Point	Bearing Capacity Problem: $c = 488.2$ kg/m^2, $\gamma_z = 1601.8463$ kg/m^3					
		σ_z/c		σ_x/c		τ_{xz}/c	
		Lysmer	Present Analysis	Lysmer	Present Analysis	Lysmer	Present Analysis
1	1	5.030	4.977	3.030	3.016	0.0	0.0
	2	12.000	11.977	10.000	10.000	0.0	0.0
	8	5.030	5.016	3.030	3.013	0.0	0.0
2	2	11.200	11.173	9.600	9.591	0.620	0.574
	4	19.200	18.924	17.600	17.520	0.610	0.705
	8	4.170	4.160	2.590	2.580	0.614	0.611
3	3	15.500	16.315	17.500	17.379	0.0	0.0
	4	17.900	16.665	17.500	17.379	0.289	0.141
	2	8.680	8.878	9.440	9.447	0.0	0.0
4	4	17.300	17.481	17.500	17.463	0.991	0.993
	5	17.300	17.129	17.500	17.463	0.993	0.980
	8	2.280	2.271	2.520	2.504	0.993	0.989
5	5	16.300	16.343	17.500	17.432	0.791	0.823
	6	16.400	16.231	17.500	17.419	0.832	0.798
	8	1.340	1.339	2.480	2.466	0.805	0.802
6	6	15.000	15.000	17.000	16.975	0.0	0.059
	7	0.0	0.0	2.000	1.866	0.0	0.0
	8	0.0	0.0	2.000	1.985	0.0	0.0

TABLE 2

Initial design point and ϵ value
Bearing capacity problem

Set 1

Design vector (D)

2.5000	3.1100	9.1400	8.9300	15.7000	16.300	
17.0000	16.2300	16.1300	15.7400	15.4100	16.3000	4.5200

$\epsilon = -0.1.$

Set 2

Design vector (D)

4.9365	2.9700	9.9670	9.3898	16.4500	15.8447	
16.5507	17.4677	17.2127	16,3844	16.1870	17.0896	4.7005

$\epsilon = -10^{-5}$

TABLE 3

Bearing capacity problem

Final optimal design point

(D) Vector

4.9769	3.0164	10.0000	9.4472	16.4002	16.3153
16.6655	17.4813	17.1293	16.3437	16.2313	17.0810
5.0161					

(σ) Vector

3.6787	9.8723	17.3165	17.0810	16.4177	15.0000
4.9769	5.0161	17.8455	2.8756	17.0736	2.1147
16.4002	1.4597	9.4137	17.2709	10.6683	3.6920
3.0164	10.006	9.4472	17.3791	16.3153	16.6655
17.4813	17.1293	16.3437	16.2313	15.7235	0.6306
0.0	0.0				

Equality constraints

0.566E−05	0.536E−06	0.700E−06	−0.357E−06	−0.301E−05	0.262E−05
−0.241E−05	−0.161E−05	−0.625E−06	−0.103E−05	0.596E−07	0.268E−06
−0.357E−06	0.953E−06	−0.596E−07	−0.506E−06	0.204E−05	0.125E−05

Inequality constraints

−0.1565E00	−0.9441E−01	−0.1350E−02	−0.1775E00	0.3991E−01
−0.6797E−02	−0.2868E01	−0.3411E01	−0.3676E01	−0.4911E−01
−0.4517E−01	−0.315 E−01	−0.1057E00	−0.4215E−01	−0.1489E00
−0.8332E−01	−0.5156E00	−0.5868E−01	−0.3678E01	−0.9872E01
−0.1732E02	−0.1708E02	−0.1942E02	−0.1500E02	−0.4977E01
−0.5016E01	−0.1784E02	−0.2876E01	−0.1707E02	−0.2115E01
−0.1640E02	−0.1459E01	−0.9414E01	−0.1727E02	−0.1067E02
−0.3692E01	−0.3016E01	−0.1000E02	−0.9447E01	−0.1738E02
−0.2631E02	−0.1666E02	−0.1748E02	−0.1713E02	−0.1634E02
−0.1623E02	−0.1572E02	−0.6306E00		

Optimal value of the objeective function = 9.9928.

Conclusions

The usefuluess of the penalty function technique in conjunction with finite elements for solving bearing capacity problems is demonstrated with remarkkable success. The non-linear no-yield conditions are incorporated directly in the analysis. By expressing some of the unknown variables in terms of the design variables, the linear equality constraints are implicitly satisfied. Such a technique helps in the reduction of the number of design variables and elimination of equality constraints, saving much of the computational effort. Some times premature termination may occur. As such, the problem should be solved with different starting point to have an idea about the global optimum.

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Notations

a_j = Coefficient to σ_j in linear function to be optimized.

a_{ij} = Coefficient to σ_j in linear constraint number i.

$[A]$ = Coefficient matrix of the linear equality constraints.

b_i, $\{b\}$ = Coefficients.

$[B]$ = 9×7 matrix, geometrical property of the element.

c = Cohesion for nth element.

$\{D\}$ = Design Vector.

D_m = Optimum design vector.

$F(D)$ = Objective function.

$[G]$ = 9×7 matrix, geometrical praperty of the element.

$\{g\}$ = 9 component vector, related to body forces in nth element.

g_j = Inequality constraints.

$\{h\}$ = 9 component vector related to body forces in nth element.

i = Subscript referring to nodal point i or running index.

j = Subscript referring to nodal point j or running index.

k = Subscript referring to nodal point k.

m = Element number.

n = Subscript or superscript referring to element n.

N = Total number of design variables.

γ_k = Response factor

$[S]$ = 7×9 matrix, geometrical property of nth element.

$\overline{s_i}$ = 3—component stress vector which defines stress state at corner i of nth element.

$\{s\}$ = 9—component stress vector which defines internal stress in nth element.

$[T]$ = 6×9 matrix. geometrical property of nth element.

x_i = x Coordinate to nodal point i.

z_i = z Coordinate to nodal point j.

γ_z, γ_x = body force per unit volume in z direction and x direction, respectively.

δ_t = Parameter showing the transition between the two types of penalty terms.

ϵ = Coefficient.

η = Coefficient.

θ_{ij} = Slope of element side connecting nodal points i and j.

$\sigma_{x,i}$ = Normal stress on vertical plane through nodal point i.

$\sigma_{z,i}$ = Normal stress on horizontal plane through nodal point i.

σ^n = Normal stress on vertical plane through nodal point i.

$\overset{n}{\sigma}_{ij}$ = Normal stress at point i of element n on side connecting nodal point i and j.

$\{\sigma\}$ = Stress vector which defines the complete stress state.

$\{\sigma\}^n$ = 7—component stress vector which defines external normal stresses on nth element.

$\tau_{zx,i}$ = Shear stress on horizontal plane nodal point i.

τ_{ij}^{n} = Shear stresss at point i of element n on side connecting nodal points i and j.

$\{\tau\}^n$ = 6—component stress vector which defines the external shear stresses on nth element.

ϕ = Angle of internal friction.