Short Communications

Rankine's Earth Pressure Theory for Inclined Backfills

by

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Introduction

Rankine's and Coulomb's are the two classical theories of earth pressure often used in the design of earth retaining structures. Rankine's theory uses a stress-approach and hence predicts the earth pressure distribution, the magnitude and point of action of the resultant earth pressure on the wall. On the other hand, Coulomb's theory uses a force-approach and as such predicts only the magnitude of the resultant earth pressure. Though both these classical theories are based on certain simplifying assumptions, Coulomb's theory is known to grossly overestimate the passive earth pressures (Terzaghi, 1943)

Most of the text books on soil Mechanics (Alam Singh, 1975) and Foundation Engineering (Bowles, 1979) give the expressions of active and passive earth pressures resulting from the two classical theories. Such expressions due to Rankine are available for the case of inclined backfills of sand without the consideration of surcharge loading and effect of cohesion. However, Rankine's expression for horizontal backfills are available for C-ϕ soils including the effect of uniform surcharge loading.

General expressions are derived in this paper for the coefficient of earth pressure (k), intensity of earth pressure (p), magnitude and point of action of resultant earth pressure (p) based on Rankine's theory both for active and passive cases for inclined backfills in C-ϕ soils subjected to a uniform surcharge loading at the ground surface. Though these expressions look to be lengthy, they are easily solved using ordinary pocket calculators as is illustrated by a numerical example.

Analysis

The problem under discussion is illustrated in Figure 1. AB in Figure 1 indicates the back of the retaining wall which is assumed to be smooth and vertical. AD represents the sloping backfill (with a surcharge angle of (i) which has a unit weight of γ and has the two shear parameters C and ϕ. A uniform vertical surcharge loading of q (per unit of horizontal area) acts along the surface AD. The problem is treated as two dimensional (plane strain case). Earth pressure is assumed to act parallel to the sloping ground surface (AD).

If a soil element bound by two vertical planes (separated by a unit distance) and two inclined planes parallel to AD is considered at a depth of Z below the top of the retaining wall just adjacent to it, then the two resultant stresses acting on the boundaries of the soil element are σ and p. These two form a conjugate stress system, each having an angle of

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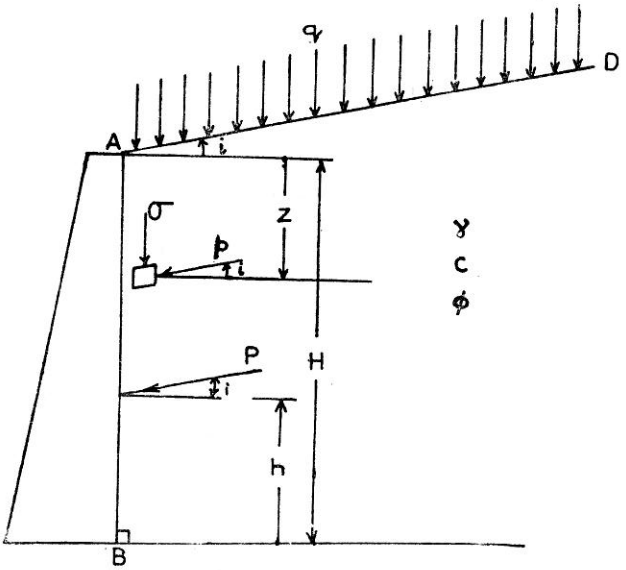

FIGURE 1 Earth retaining wall problem

obliquity of i σ is estimated from the overburden pressure of soil (γz) and the surcharge q as

$$\sigma = (\gamma z + q) \cos i \qquad \text{...(1)}$$

p is the intensity of earth pressure at that depth Z. The coefficient of earth pressure (k) is defined as

$$k = p/\sigma \qquad \text{...(2)}$$

The magnitude of resultant earth pressure (P) is obtained as

$$p = \int_0^H p \, dz \qquad \text{...(3)}$$

Where H is the height of the retaining wall. P acts at a height of h above the base, where h is given by

$$h = \frac{\int_0^H p\,(H-Z)\, dz}{p}$$

$$= H - \frac{\int_0^H p\, z\, dz}{p} \qquad \text{...(4)}$$

The suffixes a and p will be used for the quantities k, p, P and h to refer their values corresponding to active and passive conditions respectively. Expressions for k_a and k_p will be derived first which when made use of in the above equations yield expressions for p, P and h.

Derivation of k_a and k_p

The conjugate stresses p and σ are represented on the Mohr diagram shown in Figure 2. Shear strength (s) of the backfill soils is represented by the lime *FGJ* in Figure 2 with an inclination of ϕ with respect to the normal stress (σ_n) axis and with an intercept of C on the shear stress (τ) or shear strength (s) axis. Since both p and σ are the resultant stresses (on vertical plane and inclined plane parallel to the sloping ground surface respectively) with an obliquity of i, they will be represented along the line ODME passing through the origin (O) and inclined to σ_n axis at an angle of (i). For the active case $\sigma > p$ and hence σ and P are represented as lengths OE and OD, where EJD is the Mohr's stress circle passing through the known point E and tangent to the known line FG (the point of tangency is denoted as J and the centre of the circle as N). For the passive case $p > \sigma$ and p and σ are represented as the lengths *OE* and *OD* respectively. Let NM be perpendicular from N to the line ODE. Thus

$$\frac{\text{OD}}{\text{OE}} = \frac{p_a}{\sigma} = k_a = \frac{\sigma}{p_p} = \frac{1}{k_p} \qquad \ldots(5)$$

But

$$\frac{\text{OD}}{\text{OE}} = \frac{\text{OM}-\text{MD}}{\text{OM}+\text{ME}} = \frac{\text{OM}-\text{ME}}{\text{OM}+\text{ME}}$$

Also

$$\text{OM} = \text{ON}\cos i$$

$$\text{MN} = \text{NE}\sin\theta = \text{ON}\sin i \;\therefore\; \text{NE} = \text{ON}\frac{\sin i}{\sin\theta}$$

$$\text{ME} = \text{NE}\cos\theta = \text{ON}\sin i\cot\theta$$

$$\therefore \quad \frac{\text{OD}}{\text{OE}} = \frac{\text{Cos}\, i-\sin i\cot\theta}{\text{Cos}\, i+\sin i\cot\theta} \qquad \ldots(6)$$

$$= \frac{\text{Sin}\,(\theta-i)}{\text{Sin}\,(\theta+i)} \qquad \ldots(6a)$$

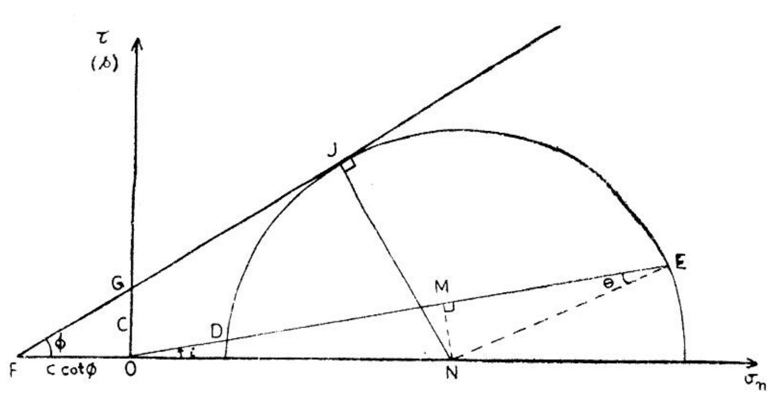

FIGURE 2 Mohr's circle of stress

From NJF,
$$\frac{NJ}{NF} = \frac{NE}{ON+OF} = \sin\theta$$
$$OF = C\cot\theta$$
$$\therefore \quad \frac{NE}{ON} = \sin\phi + \frac{c\,\mathrm{Cos}\,\phi}{ON}$$
$$= \frac{\sin i}{\sin\theta} \text{ from sine rule for } \triangle\ EON$$

Also
$$\frac{ON}{OE} = \frac{\mathrm{Sin}\,\theta}{\mathrm{Sin}\,(i+\theta)} \textit{ from sine rule for } \triangle\ EON$$
$$\therefore \quad ON = OE\,\frac{\mathrm{Sin}\,\theta}{\mathrm{Sin}\,(\theta+i)}$$
$$= OD\,\frac{\sin\theta}{\sin(\theta-i)} \text{ from Equation (6a)}$$
$$\frac{\sin i}{\sin\theta} = \sin\phi + \frac{C\cos\phi\sin(i+\theta)}{OE\sin\theta}$$
$$= \sin\phi + \frac{(C\cos\phi)}{OE}(\cos i + \sin i\cot\theta) \qquad ...(7)$$
$$= \sin\phi + \frac{(C\cos\phi)}{OD}(\cos i - \sin i\cot\theta) \qquad ...(7a)$$

From Equation (5) and (6) one gets
$$\frac{k_p-1}{k_p+1} = \frac{1-ka}{1+ka} = \tan i\cot\theta \qquad ...(8)$$

Elimination of θ from Equations (7) and (8) yield the values of k_a and k_p with
$$\lambda = \frac{C\cos\phi}{\sigma} \qquad ...(10)$$

and $\sigma = OE$ for active case and $\sigma = OD$ for passive case
Thus
$$k_a = -1 + \frac{2\cos i}{\cos^2\phi}\left[(\cos i + \lambda\sin\phi) - \sqrt{(\cos^2 i - \cos^2\phi) + \lambda(\lambda + 2\cos i\sin\phi)}\right] \qquad ...(11)$$

which for $i = 0$ (horizontal backfills) reduces to
$$k_a = -\frac{1+2(1-\lambda)}{1+\sin\phi} = \frac{1}{N_\phi} - \frac{2\lambda}{1+\sin\phi}$$

where N_ϕ is the flow value $= \dfrac{1+\sin\phi}{1-\sin\phi}$

k_a for $C = 0$ (cohesionless backfills) reduces to

$$k_a = -1 + \frac{2\cos i}{\cos^2\phi}\left[\cos i - \sqrt{(\cos^2 i - \cos^2\phi)}\right]$$

$$= \frac{\cos i - \sqrt{\cos^2 i - \cos^2\phi}}{\cos i + \sqrt{\cos^2 i - \cos^2\phi}}$$

Similarly

$$k_p = -1 + \frac{2\cos i}{\cos^2\phi}$$

$$[(\cos i + \lambda\sin\phi) + \sqrt{(\cos^2 i - \cos^2\phi) + \lambda(\lambda + 2\cos i\sin\phi)} \quad \text{...(11a)}$$

which for $i = 0$ reduces to

$$k_p = N_\phi + \frac{2\lambda}{1 - \sin\phi}$$

and for $C = 0$ reduces to

$$k_p = \frac{\cos i + \sqrt{\cos^2 i - \cos^2\phi}}{\cos i - \sqrt{\cos^2 i - \cos^2\phi}}$$

Earth Pressure

Once expressions for k are obtained, their substitution in Equation (2), (3) and (4) yield corresponding expression of p, P and h. Structurally all such expressions are more or less similar for both the active and passive cases and hence they are recorded below with the understanding that in all the following equations upper signs are to be used for active conditions and lower signs for passive conditions.

$$P = a + bz \mp d\sqrt{e + fz + gz^2} \quad \text{...(12)}$$

where

$$a = \frac{\cos i}{\cos^2\phi}\left[C\sin 2\phi + q(2\cos^2 i - \cos^2\phi)\right]$$

$$b = \frac{\gamma\cos i}{\cos^2\phi}\left(2\cos i - \cos^2\phi)\right) \quad \text{...(13)}$$

$$d = \frac{2\cos i}{\cos^2\phi}$$

$$e = C^2\cos^2\phi + q\cos^2 i\,\{C\sin 2\phi + q(\cos^2 i - \cos^2\phi)\}$$

$$f = 2\gamma\cos^2 i\,[C\sin\phi\cos\phi + q(\cos^2 i - \cos^2\phi)]$$

$$g = \gamma^2\cos^2 i\,(\cos^2 i - \cos^2\phi)$$

For $i = 0$, Equation (12), reduces to the following familiar expressions

$$p_a = \frac{\gamma z}{N_\phi} + \frac{q}{N_\phi} - \frac{2C}{\sqrt{N_\phi}}$$

$$p_p = \gamma z N_\phi + qN_\phi + C\sqrt{N_\phi}$$

Equation (12) also reduce to the following familiar expressions for $C = 0$.

$$p = (\gamma z + q) \cos i \frac{\cos i \pm \sqrt{\cos^2 i - \cos^2 \phi}}{\cos i \pm \sqrt{\cos^2 i - \cos^2 \phi}}$$

The depth (z_c) at which p_a becomes zero is given by

$$z_c = (2c/\gamma)\sqrt{N_\phi} - q/\gamma \qquad \text{...(14)}$$

$$p = aH + \frac{bH^2}{2} \mp d\, I_H \qquad \text{...(15)}$$

where

$$I_H = \frac{1}{4g}\Bigg[\Big\{(2gH+f)\sqrt{e+fH+gH^2} - f\sqrt{e}\Big\} + \frac{4eg-f^2}{2\sqrt{g}}\Big\{ ln(2\sqrt{g}\sqrt{e+fH+gH^2} + 2gH+f) - ln\,(2\sqrt{ge}+f)\Big\}\Bigg] \qquad \text{...(15a)}$$

For $i = 0$

$$p_a = \frac{\gamma H^2}{2N_\phi} + \frac{qH}{N_\phi} - \frac{2CH}{\sqrt{N_\phi}}$$

$$p_p = \frac{\gamma H^2}{2} N_\phi + qHN_\phi + 2CH\sqrt{N_\phi}$$

and for $C = 0$

$$P = \left(\frac{\gamma H^2}{2} + qH\right) \cos i \frac{\cos i \mp \sqrt{\cos^2 i - \cos^2 \phi}}{\cos i \pm \sqrt{\cos^2 i - \cos^2 \phi}}$$

which are the familiar expressions

$$h = H - \frac{\frac{aH^2}{2} + \frac{bH^3}{3} - \frac{d}{3g}\left\{\left(e+fH+gH^2\right)^{3/2} - e^{3/2}\right\} \pm \frac{df}{2g} I_H}{P} \qquad \text{...(16)}$$

For $i = q = c = 0$ Equation (16) reduces to $h = H/3$ which is an expected result.

Numerical Example

Assuming the data : $q = 1\ t/m^2$, $c = 0.5\ t/m^2$, $\gamma = 2\ t/m^3$ $H = 10$ m, $i = 10°$ and $\phi = 30°$, we get the following constants from Equation (13)

$a = 2.1308$ $\qquad e = 0.8206$

$b = 3.1243$ $\qquad f = 1.6925$

$c = 2.6261$ $\qquad g = 0.8526$

Equation (12) yield at a depth (z) of 2m the following pressure intensities

$$p_{az} = 1.13\ t/m^2;\ p_{p2} = 15.63\ t/m^2$$

For active case $z_c = 0.37$m from Equation (14)

The magnitudes of resultant earth pressures and their points of action are given by the following values obtained from Equations (15) and (16) repectively.

$$P_a = 32.28 \text{ t/m} \qquad h_a = 3.20\text{m}$$

$$p_p = 322.77 \text{ t/m} \qquad h_p = 3.57\text{m}$$

Note that $h_p > H/3$ because of the effects of cohesion and surcharge while $h_a < H/3$ due to the fact that effect of cohesion (leading to negative active pressure) is more pronounced than that of surcharge (leading to positive active pressure), the pressure distributions due to both these effects however being not hydrostatic.

Conclusions

Analytical expressions are derived based on Rankine's theory of earth pressure for the coefficients of earth pressure, distribution of earth pressure, magnitudes and points of action of the resultant earth pressures acting on the walls retaining $C-\phi$ soils at a slope with vertical surcharge loading both for active and passive conditions. These expressions are lengthy but can be evaluated easily with the help of ordinary pocket calculator as has been demonstrated by a numerical example.

These general epxressions reduce (as particular cases) to the more simpler and familiar Rankine's expression given in most of the text books on soil Mechanics and also foundation engineering.

References

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