# Computer Algorithm for Slip-Circle Analysis

*by*

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## Introduction

SPENCER (1967) described an alternative method of analysing stability of slope, based on slip-circular analysis developed by Bishop (1955). The analysis is carried out by assuming that the resultants of the pair of inter-slice forces are parallel and that the point of action passes through the centre of the base of the slice. The method of solution is based on trial and error; wherein the most critical slip-circle is found by systematically choosing the centre of the circle, and analysing each circle for its stability. According to Spencer's study the most convenient method of choosing the centres is to form a rectangular grid, one side of the grid being perpendicular to the slope. With the nodes formed by the grid as centre of the slip-circle, trial circles were obtained. For each trial circle, the factor of safety is found, first by satisfying force equilibrium, and then by satisfying moment equilibrium. The procedure is repeated by varying the angle of inclination of the resultant of the pair of inter-slice forces. The minimum factor of safety and the corresponding angle of inclination of the resultant of pair of inter-slice forces satisfying the force and moment equilibrium are obtained by graphical interpolation. The results thus obtained by Spencer showed a maximum variation of about one per cent with those obtained by Bishop's simplified analysis. Based on this, Spencer concluded that the moment equation had no significant influence on the value of factor of safety.

Though, Spencer's method seems more complete and rigorous than Bishop's simplified analysis, it has been found to be computationally quite slow. This is due to the fact that the whole procedure has to be carried out twice; once by satisfying the force equilibrium and next by satisfying moment equilibrium. However, by modifying the computational algorithm it is possible to arrive at a more effective computational approach which leads to rapid numerical solution, The scope of this section is to present the modification employed in the numerical technique which hastens the computational procedure. Furthermore, the method has been extended to account for the cases with varying pore-pressure coefficient within the potential sliding mass.

## Method of Solutions

The method of analysis is carried out by examining the equilibrium of

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the potential sliding mass, using infinite slices and is solely based on limiting equilibrium. Figure 1 shows the cross-section of any general slope section DABCE. The slope section can be any continuous function $y_s(x)$ represented by a general polynomial of N th degree of the form

$$y_s(x) = \sum_{k=0}^{N} A_s(k)x^k \quad \ldots(1)$$

where $y_s(x)$ defines the slope surface in terms of $y$ coordinate and $A_s(k)$ are the coefficients of the assumed polynomial.

In order to locate the most critical slip-circle it is necessary to analyse the slope section with several trial slip-circles whose centres are chosen along a well defined frame. In this method, the centres of these slip-circles are chosen by forming a rectangular grid (Spencer, 1967), wherein the position of centres are defined by the nodes of the rectangular grid (Figure 2). The rectangular grid has been formed by taking one side of it ($G_1$ $G_2$) coinciding with the pependicular bisector of AB. Where $G_1$ is the point of intersection of the line $G_1$ $G_2$ with the slope AB. $N_{XH}$ and $N_{XN}$ represent the magnitudes of the rectangular sides parallel and perpendicular to the slope AB respectively. The whole operation has been programmed, which frames the rectangular grid and carries out the analysis, for the slip-circles. Furthermore, the values of factor of safety obtained are interpolated and thus the slip-circle giving minimum factor of safety is located. Furthermore, the angle of inclination of the resultant of the pair of inter-slice forces is determined.

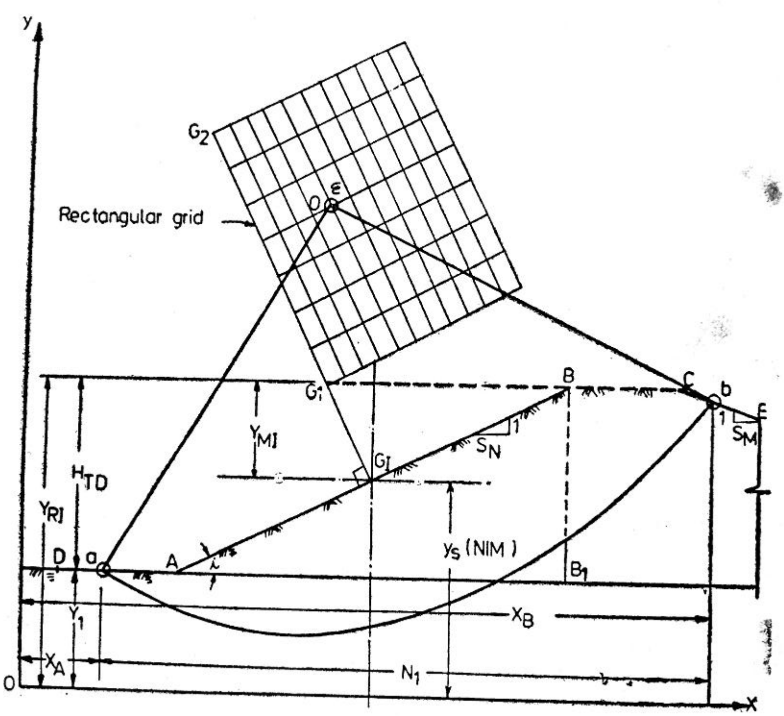


FIGURE 1 General slope section DABCE

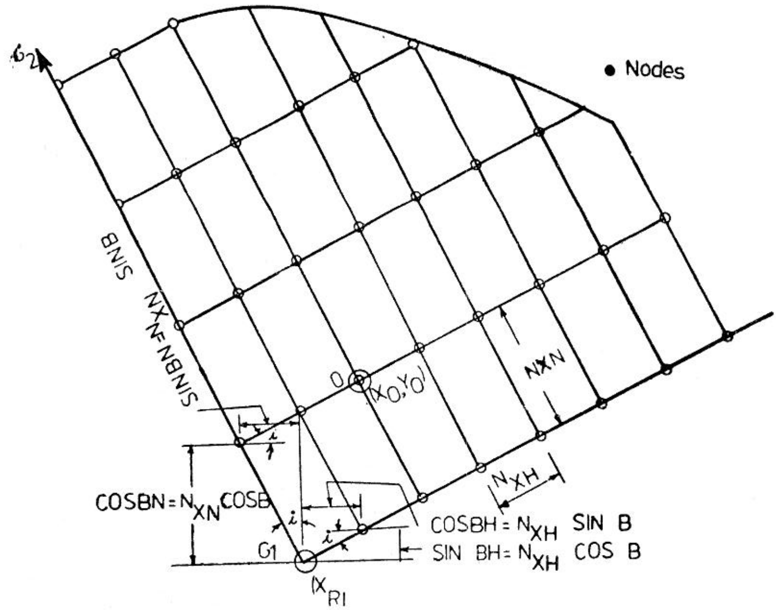


**FIGURE 2 Details of rectangular grid**

The procedure adopted for the computer solution is as follows:

(*a*) The geometrical properties, soil properties and the magnitude of the rectangular sides, $N_{XH}$ and $N_{XN}$ are fed as data.

(*b*) A general polynomial of the form defined in Equation (1) is assumed. The degree of the polynomial, $N$, is chosen before hand depending on the desired accuracy.

(*c*) With reference to X-Y axis; the rectangular grid is formed using the values of $N_{XH}$ and $N_{XN}$ as the sides of the rectangle; and the $(X_0, Y_0)$ coordinates of the nodes with reference to $X$-$Y$ plane are stored. These coordinates $(X_0, Y_0)$ are chosen as centres of trial *slip-circles for analysing* the stability of the slope section.

(*d*) The boundary of the slip-circle is chosen, and is usually based on physical aspect of the problem. It is usually taken as the toe of the slope section; however, it need not necessarily be through the toe always.

(*e*) Using the coordinates of the centres of the circle $0(X_0, Y_0)$ and radius of the circle $R$, the constant of the circle is obtained from

$$C_r = X_0^2 + Y_0^2 - R^2 \qquad \ldots(2)$$

(*f*) Next step is to locate the boundary $b$, where the assumed trial circle intersects the slope section. The algarithm developed for this is explained below.

As a best approximation, the distance $X_{INS}$ along $X$-axis of the boundary $b$, from the centre of the circle is obtained from

$$X_{INS} = \sqrt{R^2 - X_{IS}^2} \quad \text{...(3)}$$

where

$$X_{IS} = Y_0 - (Y_R + H_{TD}) \quad \text{...(4)}$$

$H_{TD}$ is the height of the dam and $Y_R$ is the $Y$-coordinate of the boundary, $a$. using the value of $X_{INS}$ the approximate $X$-coordinate, $X_{IN}$, of the boundary, $b$ is obtained as

$$X_{IN} = X_0 + X_{INS} \quad \text{...(5)}$$

Now, the $Y$-coordinate, $Y_{IN}$ of the boundary, $b$ can be obtained by using the equation of the circle

$$x^2 + y^2 + 2f_c x + 2gy + C_r = 0 \quad \text{...(6)}$$

where

$$(f_c, g) = (-X_0, -Y_0) \quad \text{...(7)}$$

and $C_r$ is given by Equation (2), and hence value of $Y_{IN}$ is obtained from

$$Y_{IN} = Y_0 + \sqrt{Y_0^2 - (X_{IN}^2 - 2X_0 X_{IN} + C_r)} \quad \text{...(8)}$$

Corresponding to the coordinate of the boundary, $b$, the $y$-coordinate of the slope section, $Y_{SP}$ is obtained from the polynomial defined by the Equation (1).

Now the problem is to find the actual point of inter-section of the slip-circle with the slope section, i.e., to find the $x$-$y$ coordinates of the boundary $b$.

It can be shown that if the $x$-coordinate, $X_{IN}$ used in Equations (1) and (8), satisfies the condition

$$Y_{IN} - Y_{SP} = 0 \quad \text{...(9)}$$

the corresponding values $X_{IN}$, $Y_{IN}$ define the coordinates of the actual boundary $b$. The solution of $X_{IN}$ is determined by using standard Newton Raphson technique. Thus by knowing the boundaries, $a$ and $b$ also the radius of the circle, the potential sliding surface is defined.

Now in general, the Y-coordinates of the slip-circle are obtained from equation

$$y(x) = y_0 + \sqrt{y_0^2 - (x^2 - 2X_0 x + C_r)} \quad \text{...(10)}$$

**Analysis of the Section**

By dividing the potential sliding mass into infinite number of slices, and starting from the boundary, $a$, the corresponding $y$-coordinates of the slope section and the potential slip surface are obtained from Equations (1) and (10) respectively. The analysis is carried out by examining the equilibrium of the potential sliding mass aABCBa of Figure 1 (Spencer, 1967). and the various forces acting over the slice are shown in Figure 3. With this modification in the analysis, the expression for the resultant of pair of inter-slice force $Q$ obtained by Spencer (1967) is modified as

$$Q = \left[ \frac{\frac{c'}{F_s} \sec\alpha + \frac{\tan\phi'}{F_s}(\gamma\bar{y} - r_u\,\gamma\bar{y}\sec\alpha) - \gamma\bar{y}\sin\alpha}{\cos(\alpha-\theta)\left[1 + \frac{\tan\phi'}{F_s}\tan(\alpha-\theta)\right]} \right] dx \quad ...(11)$$

or

$$Q = \left[ \frac{\{c'(1+y'^2) + \tan\phi'\,(1 - r_u(1+y'^2))\bar{y} - F_s\gamma y'\bar{y}\}\sec\theta}{(1+y'\tan\theta)\,F_s + (y' - \tan\theta)\tan\phi.} \right] dx \quad ...(12)$$

As explained the force and moment equations to be satisfied respectively are

$$\sum_{x=x_a}^{x_b} Q = 0 \int_{x_a}^{x_b} \left[ \frac{\{c'(1+y'^2) + (1 - r_u(1+y'^2))\bar{y} + F_s\gamma y'\bar{y}'\}}{\{1+\bar{y}'\tan\theta)F_s + (y' - \tan\theta)\tan\phi'\}} \right] dx \quad ...(13)$$

and

$$\sum_{x=x_a}^{x_b} Q\cos(\alpha-\theta) = 0 = \int_{x_a}^{x_b} \left[ \frac{\{c'(1+y'^2) + (1 - r_u(1+y'^2))\,\bar{y} + F_s\gamma y'\bar{y}'\}}{\{(1+\bar{y}'\tan\theta)\,F_s + (y' - \tan\theta)\tan\phi'\}} \times \left(\frac{1+y'\tan\phi')}{(1+y'^2)}\right)\right] dx \quad ...(14)$$

Hence, there are two equilibrium equations to be solved with two unknown variables $F_s$ and $\theta$. These variables are obtained by simultaneously solving Equations (13) and (14) and the numerical solutions are obtained by using the Newton-Raphson techniques.

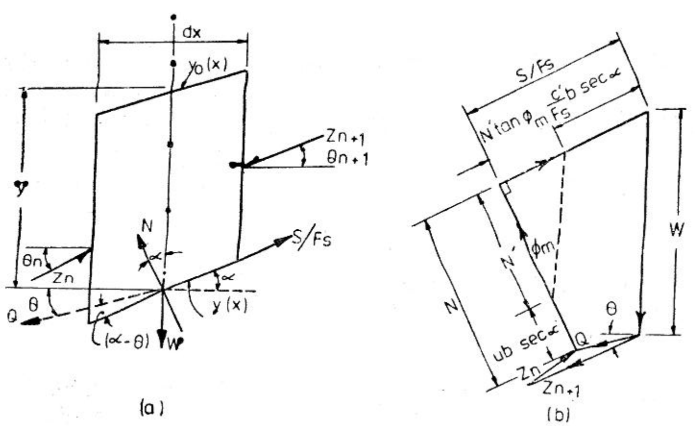


**FIGURE 3** (*a*) **Forces acting on an elemental slice**
(*b*) **Force polygon**

For cases with tension crack, the procedure remains same except that the boundary, $b$ terminates at a height z (Figure 4) given by

$$z = \frac{2c'}{F_s\gamma(1-r_u)}\sqrt{\frac{1+\sin\phi_m}{1-\sin\phi_m}} \quad ...(15)$$

Furthermore, the analysis has been extended to include any variation in the pore-pressure coefficient $r_u$ within the body of the potential sliding mass. By assuming $r_u$ as a known function, $h$ (based on field data) the force and moment equations defined by Equations (13) and (14) are modified as

$$\sum_{x=x_a}^{x_q} Q = 0 = \int_{x_a}^{x_b}\left[\frac{\{c'(1+y'^2)+(1-h(1+y'^2))\bar{y}+F_s\gamma\bar{y}'\}}{\{(1+\bar{y}'\tan\theta)\,F_s+(y'-\tan\theta)\tan\phi'\}}\right] dx \quad ...(16)$$

and

$$\sum_{x=x_a}^{x_b} Q\cos(\alpha-\theta) = 0 = \int_{x_a}^{x_b}\left[\frac{\{c'(1+y'^2)+(1-h(1+y'^2))\,\bar{y}+F_s\gamma y'\bar{y}'\}}{\{1+\bar{y}\tan\theta)\,F_s+(y'-\tan\theta)\tan\phi'\}}\right.$$

$$\left.\frac{1+y'\tan\theta}{(1+y'^2)}\right] dx \quad ...(17)$$

## Numerical Results

Using the modified computational technique developed, stability analysis was carried out for various geometrical slope sections with different combination of soil strength parameters $c'$ and $\phi'$ and also for various pore-pressure coefficient $r_u$.

Figure 5a shows a specific case of a slope section approximated by a fourth degree polynomial. The figure also shows the assumed rectangular grid frame consisting of 78 rectangles of dimension 10′×5′ (3.048 m×1.524 m)

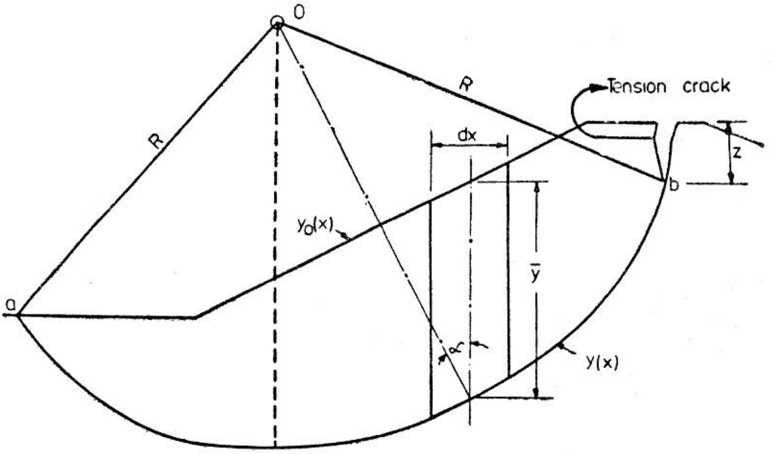


**FIGURE 4 Stability analysis of slope wlth tension crack**

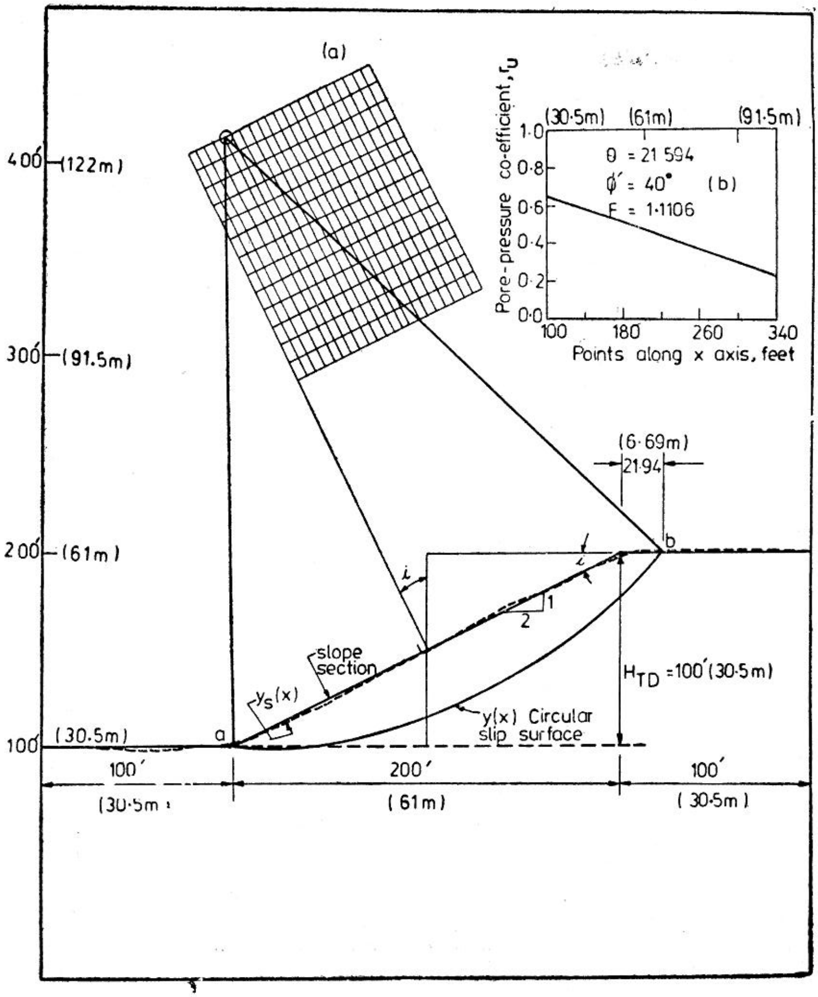


**FIGURE 5** (*a*) **Stability analysis of embankment by modified Spencer's circular slip surfaces analysis**

(*b*) **Variation of pore pressure coefficient $r_u$ along the cross section of the embankment**

with 93 node points; stability analysis was carried out for each circle and the corresponding factor of safety and the angle of slope of resultant *Q* of pair of inter-slice forces were determined. In this case, circles passing through toe of the slope section was found to give minimum factor of safety. The time taken for analysing 98 circles (satisfying both force and moment equilibrium) as well as locating the most critical slip-circle is four minutes on ICL computer 1909 series. The numerical results obtained are summarised in Figure 5*b* and Table 1.

TABLE 1

Stability analysis by slip circle

| Sr No. | $\gamma$ g/cm³ | $c'$ kg/cm² | $\phi'$ degrees | Slope $S_N$ | Factor of safety $F_s$ | $\theta$ in degrees | Remarks |
|---|---|---|---|---|---|---|---|
| 1 | 1.922 | 0.117 | 40 | 2 | 1.110 | 21.592 | Pore Pressure |
| 2 | 1.922 | 0.117 | 30 | 2 | 0.840 | 21.089 | Coefficients, $r_u$ |
| 3 | 1.922 | 0.117 | 40 | 1.5 | 0.827 | 20.101 | Varies along X-axis as in *Figure 5 b* |
| 4 | 1.922 | 0.117 | 30 | 1.5 | 0,631 | 19.871 | |

## Conclusions

Based on Spencer's (1967) work, a modified computational technique of analysing the slope section is presented. The analysis is carried out by assuming infinite slices, and is in terms of effective stress. It satisfies both force and moment equilibrium conditions. The analysis is applicable to cases with tension cracks. It has been further extended to account for any known variation of pore-pressure coefficient within the potential sliding mass. The trial slip-circle were chosen with rectangular grid nodal points as centre of the slip-circles. The analysis of each slip-circle takes about two to three seconds on ICL 1909 series computer. For each set of data 98 trial slip-circles were analysed which took about four minutes. The results obtained are comparable with existing results. The analysis presented can be conveniently extended for analysing the slope section with varying soil strength parameters $c'$ and $\phi'$ within the potential sliding mass.

## References

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NARAYAN, C.G.P., (1976), "Variational Methods in the Stability Analysis of Slopes", *Ph.D. Thesis, Indian Institute of Technology, Delhi*, India.

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## Notations

$A_{sk}$ = constant coefficients of the general polynomial where $k = 0, \ldots, N$.

$C_r$ = constant of a circle

$F_s$ = overall factor of safety

$N$ = number of items in general

$Q$ = resultant of pair of inter-slice forces

$R$ = radius of the slip-circle

$S$ = total shear strength available on the potential surface

$S_m$ = total shear strength mobilised along the potential slip surface
$w$ = total weight of potential sliding mass
$X_0$ = x-coordinate of the centre of the slip-circle
$Y_0$ = y-coordinate of the centre of the slip-circle
$Z$ = resultant inter-slice force inclined at an angle $\delta$ with respect to horizontal
$a(x_a, y_a)$ = boundary point of the slip surface defined by $(x_a, y_a)$ coordinate system
$b(x_b, y_b)$ = boundary point of the slip surface defined by $(x_b, y_b)$ coordinate system
$c'$ = cohesion intercept in terms of effective stresses
$h$ = pore-pressure coefficient function
$r_u$ = pore-pressure coefficient
$u$ = pore pressure per unit area
$x$ = $x$-coordinate of the reference axis
$y$ = function defining the potential sliding surface
$y'$ = first derivative of the function defining potential slip surface
$y_0, y_s$ = functions defining the slope section
$z$ = depth of soil strata
$dx$ = elemental width of the slice
$\infty$ = slope of the base of the elemental slice
$\gamma$ = unit density of the soil mass
$\delta$ = slope angle of the inter-slice force
$\theta$ = average slope angle of the resultaut inter-slice force
$\phi'$ = angle of shearing resistance with respect to effective stresses
$\phi'_m$ = mobilized angle of shearing resistance