Vibrations of Embedded Foundations—A Comparative Study

by

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Introduction

TORSIONAL vibrations are excited in foundations of structures and machinery when there exist eccentric forces in a horizontal plane. Foundations of radar and communication antennae and certain reciprocating machinery are some examples. Torsional vibrations can be excited in foundations of structures during earthquakes. These foundations are essentially embeded. Hence it is of great practical importance to develop a rational theoretical approach to predict the torsional response and design these embedded foundations.

Although the effect of embedment has been noted by many investigators, significant efforts towards the theoretical solution of this problem have been advanced only within the last few years. Several analytical methods, including the finite element method, have been used to study the problem of embedment. Baranov (1967), Tajimi (1969), Novak and Beredugo (1971, 1972), Beredugo and Novak (1972), Krishnaswamy (1972), Novak and Sachs (1973), Thau and Umek (1974), Beredugo (1976), Ramiah et al (1977) and Sankaran et al (1978) have used various analytical techniques to predict the dynamic response of embedded foundations. Kaldjian (1969, 1971), Kuhlemeyer (1969), Lysmer and Kuhlemeyer (1969), Krizek et al (1972), Waas and Lysmer (1972), Kuhlemeyer (1973), Johnson et al (1975) and Luco (1976) have used numerical methods including finite element method for the solution of this problem. The major portion of the work has dealt with vertical vibrations while the least amount of work has been on torsional vibrations. Novak and Sachs (1973) and Sankaran et al (1978) have presented analytical solutions to the embedded machine foundation response when subjected to steady-state torsional vibrations.

In this paper results of the experimental investigations on full scale model embedded foundations subjected to torsional mode of vibration have been presented and discussed. The results have been analysed making use of the analytical solutions. The theoretical aspects of both the models are briefly discussed herein.

Model Proposed by Novak and Sachs

The equation of torsional oscillation θ of an embedded cylindrical

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footing about the vertical axis of symmetry is,

$$I_\theta \ddot{\theta} = T(t) - R_\theta(t) - N_\theta(t) \qquad ...(1)$$

in which I_θ = mass moment of inertia of footing about vertical axis, $T(t)$ = moment of excitation, $R_\theta\ (t)$ = torsional reaction of soil at footing base and $N_\theta\ (t)$ = torsional reaction of layer adjacent to footing vertical sides.

The base reaction may be written in terms of the displacement function f_{θ_1} and f_{θ_2} as,

$$R_\theta(t) = -G\,r_0^3 \frac{1}{f_{\theta_1} + i f_{\theta_2}} \theta(t)$$

$$= G\,r_0^3 \,(C_{\theta_1} + i\ C_{\theta_2})\ \theta(t) \qquad ...(2)$$

in which G = shear modulus of the base soil, r_0 = radius of the footing, G_{θ_1} = stiffness parameter of base soil and G_{θ_2} = damping parameter of base soil.

$$C_{\theta_1} = -\frac{f_{\theta_1}}{f_{\theta 1}^2 + f_{\theta 2}^2} \qquad ...(3)$$

$$C_{\theta_2} \qquad \frac{f_{\theta 2}}{f_{\theta 1}^2 + f_{\theta 2}^2} \qquad ...(4)$$

The side reaction, $N_\theta(t) = G_s\ r_0^2\ H\,(S_{\theta 1} + iS_{\theta 2})\ \theta\ (t) \qquad ...(5)$

in which G_s = shear modulus of the soil on the sides, H = depth of embedmet, $S_{\theta 1}$ = side layer stiffness parameter and $S_{\theta 2}$ = side layer damping parameter. Substitution of Equations 2 and 5 into Equation 1 provides the differential equation of motion,

$$I_\theta\ \ddot{\theta}\,(t) + G\,r_0^3\ [C_{\theta 1} + \frac{G_s}{G}\frac{H}{r_0}\ S_{\theta 1} + i\,(C_{\theta 2} + \frac{G_s}{G}\frac{H}{r_0}\ S_{\theta 2}\ )]\ \theta(t)$$

$$= T(t) \qquad ...(6)$$

With the notation of the frequency dependent stiffness constant

$$K_\theta = G\,r_0^3\ (C_{\theta 1} + \frac{G_s}{G}\frac{H}{r_0}\ S_{\theta 1}\ ) \qquad ...(7)$$

and the frequency dependent damping constant,

$$C_\theta = \frac{G\,r_0^3}{\omega}\ (C_{\theta 2} + \frac{G_s}{G}\ \frac{H}{r_0}\ S_{\theta 2}) \qquad ...(8)$$

the real amplitude of steady-state torsional vibration

$$A_\theta = \frac{T_0}{\sqrt{[(K_\theta - I_\theta \omega^2)^2 + (C_\theta \omega)^2}}$$

$$= \frac{T_0}{K_\theta} \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4D_\theta^2\left(\frac{\omega}{\omega_\theta}\right)^2}} \quad ...(9)$$

in which the damping ratio, $D_\theta = \dfrac{C_\theta}{2 I_\theta \omega_n}$ ...(10)

and the undamped natural frequency, $\omega_n = \sqrt{K_\theta \; I_\theta}$ ...(11)

The excitation moment amplitude $T_0 = m_0 \, e l \omega^2$ ...(12)

in which m_0 = total unbalanced mass, e = eccentricity, l = distance of the masses from the axis of the footing and ω = circular frequency of excitation. The dimensionless rotational amplitude

$$= \frac{I_\theta \; A_\theta}{m_0 \; e \; l} \quad ...(13)$$

The computations are very much simplified by considering stiffness factors $C_{\theta 1}$ and $S_{\theta 1}$ as constant (frequency independent) and damping factors $C_{\theta 2}$ and $S_{\theta 2}$ proportional to the dimensionless frequency a_0 ·

$$a_0 = \omega \, r_0 \sqrt{\rho / G} \quad ...(14)$$

in which ρ is the mass density of the base soil. The model is discussed in detail by Novak and Sachs (1973). A comparison of the theoretically predicted and experimentally observed values of resonant amplitudes by Novak and Sachs is presented in Figure 1. Similar values of resonant frequencies are compared in Figure 2

Model Proposed by Sankaran et al

The authors have developed a single degree of freedom analogue model, the parameters of which are expressed in terms of the results of the half-space theory and an additional parameter to account for slip damping in embedded foundation. This parameter takes into account, the relevant physical characteristics of the interface between the foundation walls and the surrounding soil as well as the foundation base and the soil beneath.

The differential equation of torsional vibrations of an embedded footing is,

$$I_\theta \ddot{\theta} + C_\theta \dot{\theta} + K_\theta \theta \pm M_{F\theta} = T(t) \quad ...(15)$$

in which C_θ = viscous damping coefficient in torsion, K_θ = torsional spring stiffness and $M_{F\theta}$ Coulomb frictional moment.

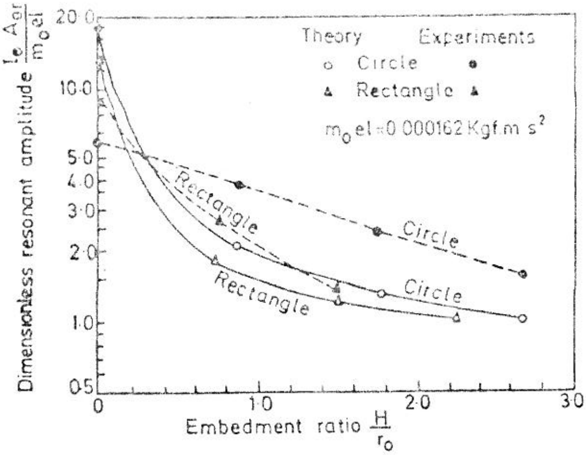

FIGURE 1 Comparison of theoretical and experimental resonant amplitudes of rotation (Novak and Sachs, 1973).

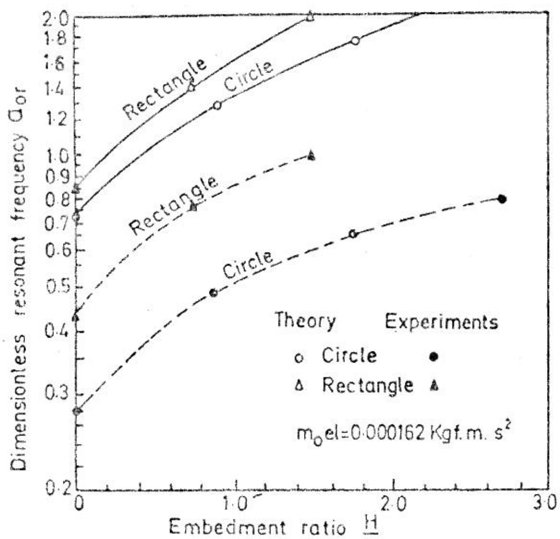

FIGURE 2 Comparison of theoretical and experimental resonant frequencies of vibration (Novak and Sachs, 1973).

The value of K_θ is given by the equation,

$$K_\theta = \frac{16}{3} G r_o^3 \qquad ...(16)$$

C_θ is obtained from the expression,

$$D_\theta = \frac{C_\theta}{2\sqrt{K_\theta I_\theta}} = \frac{0.5}{1+2 B_\theta} \qquad ...(17)$$

in which $$B_\theta = \text{inertia ratio} = \frac{I_\theta}{\rho\, r_o^5} \qquad \ldots(18)$$

The exciting torque $T(t)$ is given by,

$$T(t) = T_o \sin \omega t = m_o\, e\, l\, \omega^2 \sin \omega t \qquad \ldots(19)$$

The natural frequency, ω_n is obtained from Equation 11.

The above differential equation has been solved following the procedure reported by Den Hartog (1931) with suitable modifications.

Defining a non-dimensional parameter $\frac{M_{F\theta}}{m_o\, e\, l\, \omega_n^2}$ as Coulomb frictional moment factor, the steady-state solution of Equation 15 for a rotating mass type of exciting torque is obtained as,

$$\frac{I_\theta\, A_\theta}{m_o\, e\, l} = \left(\frac{\omega}{\omega_n}\right)^2 \left[\left(-X\beta^2 \frac{M_{F\theta}}{m_o e l^2 \omega_n^2}\right) + \sqrt{\frac{1}{n^2} - Y^2 \beta^4 \left(\frac{M_{F\theta}}{m_o e l \omega_n^2}\right)^2}\,\right] \qquad \ldots(20)$$

in which $$\beta = \frac{\omega_n}{\omega} \qquad \ldots(21)$$

$$X = \frac{\sinh \beta\pi D_\theta - \sqrt{\dfrac{D_\theta^2}{1-D_\theta^2}}\, \sin \beta\pi \sqrt{1-D_\theta^2}}{\cosh \beta\pi D_\theta + \cos \beta\pi \sqrt{1-D_\theta^2}} \qquad \ldots(22)$$

$$Y = \frac{\beta}{\sqrt{1-D_\theta^2}} \quad \frac{\sin \beta\pi \sqrt{1-D_\theta^2}}{\cosh \beta\pi D_\theta + \cos \beta\pi \sqrt{1-D_\theta^2}} \qquad \ldots(23)$$

$$n = \sqrt{\left(\frac{\beta^2}{\beta^2-1}\right)^2 + \left(\frac{2D_\theta}{\beta}\right)^2} \qquad \ldots(24)$$

Equation 20 is the complete mathematical solution of the steady-state torsional vibrations of embedded footings. For a convenience of field engineers the closed-form solutions are obtained in the form of non-dimensional graphs. The above results are reploted in a form suitable for ready use for purposes of design and analysis. Figure 3 illustrates the variation of resonant amplitude of rotation with the Coulomb frictional moment factor for various values of damping factor. Figure 4 shows the variation of resonant frequency with frictional moment factor for various values of the damping factor.

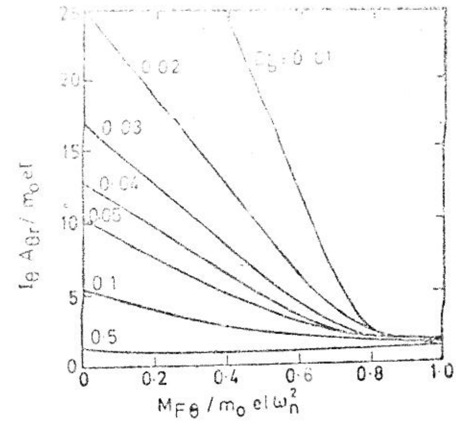

FIGURE 3 Decrease of resonant amplitude with Coulomb friction moment factor (Sankaran et al, 1978)

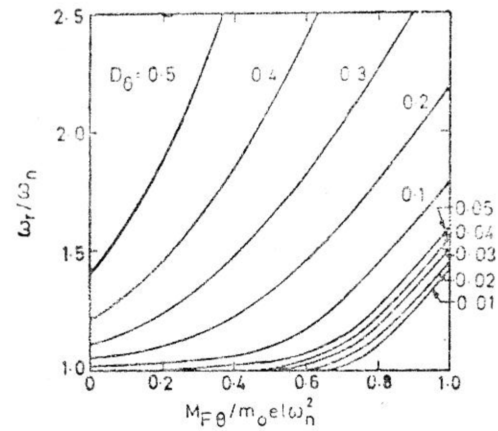

FIGURE 4 Increase of resonant frequency with Coulomb moment factor (Sankaran et al, 1978)

Thus, if the shear modulus of the soil, G, inertia ratio of the footing, B_θ, and the frictional moment, $M_{F\theta}$ are known, the dynamic response of embedded footings subjected to torsional vibration can be predicted. Simple equations for evaluating the frictional moment have been presented by Sankaran et al (1978).

Comparing Novak and Sachs' model and the model by Sankaran et al the following factors come to light.

1. The lumped parameter model of Novak and Sachs (1973) has been developed from the results of elastic half-space theory and does not take into account slip damping. The lumped parameter model developed by Sankaran et al (1978) is also based on the results of the elastic half-space theory ; the model however, takes into consideration slip damping.

2. The model proposed by Novak and Sachs is for a circular footing. For other shapes, the equivalent radius is normally calculated by equating the moment of inertia of the base areas of the two footings about the axis of rotation. In the model by Sankaran et al (1978) an effort has been made to take into account the shape effect.

Experimental Investigations

A series of field vibratory tests was carried out at I.I.T., Madras, with the aim of collecting valuable data with regard to the torsional vibrations of embedded footings. The soil at the site was silty sand with some clay binder. The bulk density, angle of internal friction and cohesion of the soil were found to be 1970 kg/m^3, 27° and 2400 kg/m^2 respectively. Eight precast reinforced concrete footings featuring circular, square and rectangular shapes were used in these investigations. The dimensions of the footing are given in Table 1.

The experimental investigations were made in an excavated pit, the depth of which was kept equal to the height of the footing tested. The sides of the footing were backfilled with clean dry river sand, the bulk density of which was maintained uniform at 1600 kg/m^3. Uniform contact of backfill sand with the foundation block, uniform shear modulus and uniform density of the backfilled sand were ensured by running steady-state field vibratory tests at a frequency approximately equal to the resonant frequency of the foundation-soil system for a sufficient length of time as to eliminate the effect of vibrations on the material properties themselves. The effectiveness of the above procedure was confirmed by an examination of test data which were reproducible. Tests were performed for various depths of embedment by backfilling. A Lazan type mechanical oscillator was used to produce pure torsional vibration. Care was taken to see that the centres of gravity of the footing, vibrator and other attachments were located along the same vertical line so as to minimise any tendency for rocking vibration. The vibrator was run by a motor through a flexible shaft. Electrodynamic vibration pick-ups were used in conjunction with the amplitude measuring aparatus. Electrical speed indicating techeometer was used to indicate the spot speeds of the revolving shafts. The footing were handled by a 5-ton capacity mobile crane. A typical response curve is given in Figure 5.

Analysis of the Test Data

The authors have recognised that the shear modulus, G, of the soil varies with depth and have developed a procedure to evaluate the same. (Sankaran et al, 1979).. In this analysis the value of G is determined by the above procedure.

The field test data obtained by the authors have been analysed by using the two mathematical models discussed in this paper. The value predicted by Novak and Sach's model were computed with the help of I.B.M. 373/155. They are presented in columns 3 and 6 of Table 1. The prediction of the resonant amplitudes and frequencies by Sankaran et al model was made by using Figures 3 and 4. These values are listed in columns 4 and 7 of Table 1. The observed data are tabulated in columns 5 and 8 of Table 1. The predicted values of resonant frequency and amplitude for a typical footing are compared in Figures 6 and 7.

TABLE 1

Comparison between Predicted and Observed Values

Footing	Embed-ment ratio H/r_o	Resonant amplitude in 10^{-4} radians			Resonant frequency in rad/sec		
		Novak & Sach's model	Authors' model	Observed	Novak & Sach's model	Authors' model	Observed
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Base 1	0.000	19.65	17.97	10.05	163	181	192
0.50×0.50	0.584	6.43	16.75	8.80	239	181	198
×0.50 (m)	1.168	4.50	13.98	7.45	295	183	207
	1.752	3.67	11.26	5.90	352	185	220
Base 2	0.000	11.51	11.62	7.80	195	218	220
0.60×0.60	0.438	4.44	10.95	6.65	264	218	236
×0.45 (m)	0.876	3.15	9.63	5.45	327	218	245
	1.314	2.58	7.69	4.20	390	220	254
Base 3	0.000	6.70	5.20	3.75	214	240	236
0.70×0.70	0.417	2.61	5.01	3.30	295	240	242
×0.50 (m)	0.834	1.87	3.91	2.75	371	246	251
	1.251	1.54	2.50	1.55	434	256	264
Base 4	0.000	2.71	1.23	1.15	232	296	257
0.90×0.90	0.290	1.28	1.12	1.00	302	331	273
×0.45 (m)							
Base 5	0.000	9.28	6.46	4.75	207	235	233
0.677 dia×	0.886	2.72	5.24	3.42	333	240	242
1.20 (m)	1.772	1.91	2.50	1.75	434	254	258
Base 6	0.000	8.98	6.02	4.05	214	237	239
0.60×0.60	0.876	2.64	4.84	3.38	333	242	245
×1.20 (m)	1.752	1.86	2.00	1.70	434	261	261
Base 7	0.000	8.97	5.99	4.05	214	238	242
0.6545×0.55	0.872	2.61	4.82	3.27	333	243	251
×1.20 (m)	1.163	2.26	3.66	2.60	371	243	258
	1.744	1.84	1.99	1.55	434	262	267
Base 8	0.000	8.63	5.57	4.00	214	241	245
0.72×0.50	0.862	2.53	4.52	3.30	339	244	251
×1.20 (m)	1.149	2.19	3.27	2.50	371	249	261
	1.724	1.79	1.93	1.40	440	268	273

$m_o el = 0.00098$ kgf.m.s[1]

It is seen that the shape of the footing has negligible influence on the resonant frequency of vibration while it has significant effect on the resonant amplitude of rotation. For the same base area, the circular footing shows larger amplitude compared to the square and rectangular footings. Also as the length to breadth ratio of the rectangular footing increases, the amplitude decreases. This trend conforms with the theoretical behaviour of the mathematical model proposed by Sankaran et el. The

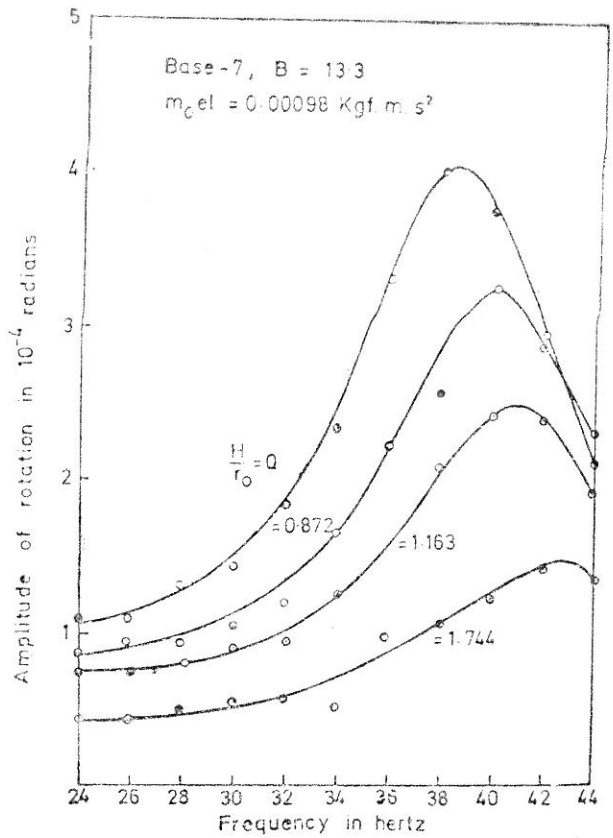

FIGURE 5 Typical measured response curve for torsional vibration (Sankaran et al, 1978)

model proposed by Novak and Sachs does not take into account the shape of the footings. This is perhaps one of the factors, that brings in a closer agreement between the authors' model and the experimental results.

From a close examination of Table 1 and Figures 6 and 7, it is seen that the model proposed by Sankaran et al shows better overall agreement with regard to the prediction of resonant frequencies of vibration and the resonant amplitudes of rotation for the footings tested. It must be recognised that Novak and Sachs' model does not take into account slip damping.

Summary

The model proposed by Sankaran et al (1978) to predict the torsional vibration response of an embedded machine foundation soil system and the corresponding model by Novak and Sachs (1973) are compared with the results of the experimental investigations carried out at I.I.T., Madras, where the soil at the test site is silty sand. The comparison shows that the model proposed by Sankaran et al (1978) provides a better prediction with respect to both resonant amplitudes and resonant frequencies.

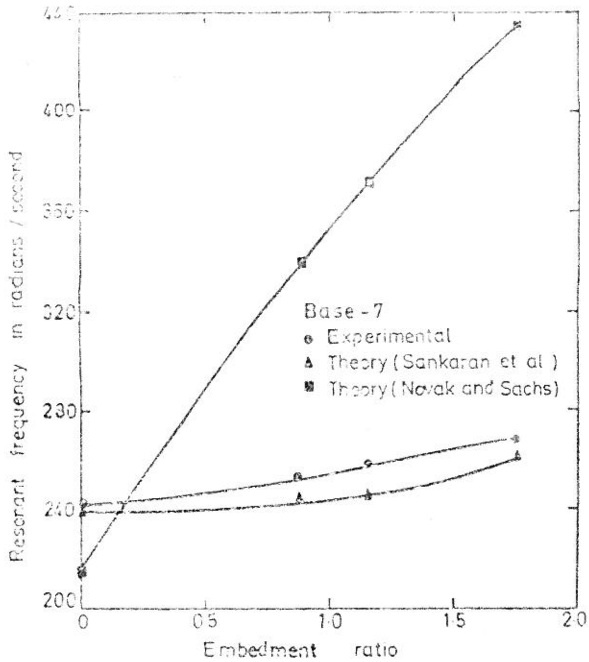

FIGURE 6 Comparison of measured resonant frequencies with theory

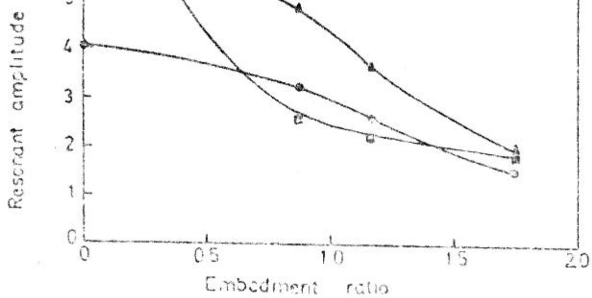

FIGURE 7 Comparison of measured resonant amplitudes with theory

References

BARANOV, V.A., (1967), 'On the calculation of Excited vibrations of an Embedded Fundation', (In Russian), *Voprosy Dynamiki: Prochnocti*, No. 14, Polytechnical Institute of Riga, pp. 195-209.

BEREDUGO, Y.O., (1971), 'Vibrations of Embedded Symmetric Footings', *Ph.D. Dissertation*, University of Western Ontario, London, Canada.

BEREDUGO, Y.O., (1976), 'Model Analysis of Coupled Motion of Horizontally excited Embedded Footings', Int. Journal, *Earthquake Engineering and Structural Dynamics*, Vol. 4, pp. 403-410.

BEREDUGO, Y.O., and NOVAK, M., (1972), 'Coupled Horizontal and Rocking Vibration of Embedded Footings', *Canadian Geotechnical Journal*, Vol. 9, No. 4, pp. 477-497.

DEN HARTOG, J.P., (1931), 'Forced Vibrations with combined Coulomb and Viscous Friction', *Transactions, ASME* APM 53-9, Vol. 53, pp. 107-125.

JOHNSON, G.R., CHRISTIANO, P., and EPSTEIN, H.I., (1975), 'Stiffness Coefficients for Embedded Footings', *Journal Geotechnical Engineering Division, ASCE*, Vol. 8, pp. 789-800.

KALDJIAN, J.M., (1969), Discussion of 'Design Procedures for Dynamically loaded Foundations', by R.V. Whitman and F.E. Rechart, Jr., Journal, Soil Mechanics and Foundations Division, ASCE, Vol. 95, No. SM 1, pp. 364-366.

KALDJIAN, J.M., (1971), 'Torsional Stiffness of Embedded Footings', *Journal, Soil Mechanics and Foundations Division, ASCE*, Vol. 97, No. SM 7, pp. 969-980.

KRISHNASWAMY, N.R., (1972), 'Study of the Effect of Embedment and Related Factors on the Dynamics Behaviour of Footings', *Ph.D. Dissertation*, Indian Institute of Technology, Kanpur.

KRIZEK, R.J., GUPTA, D.C., and PARMELEE, R.A., (1972), 'Coupled Sliding and Rocking of Embedded Foundations', *Journal Soil Mechanics and Foundations Division, ASCE*, Vol. 98, No. SM 12, pp. 1347-1358.

KUHLEMEYER, R.L., (1965), 'Vertical Vibrations of Foolings Embedded in Layered Media,' *Ph.D. Dissertation*, University of California, Berkeley.

LUCO, J.E., (1976), 'Torsion of a Rigid Cylinder Embedded in an Elastic Half Space', *Journal of Applied Mechanics*, Sep. pp. 419-423.

LYSMER, J., and KUHLEMEYER, R.L. (1969), 'Finite Dynamic Mode for Infinite Media', *Journal Engineering Mechanics Division, ASCE*, Vol. 95 No. EM 4, pp. 859-877.

NOVAK, M., and BEREDUGO, Y.O., (1971), 'The Effect of Embedment on Footing Vibrations', *Proceedings, 1st Canadian Conference of Earthquake Engineering Research*, Vancouver, B.C. pp. 1-14.

NOVAK, M., and BEREDUGO, Y.O., (1972), 'Vertical Vibration of Embedded Footings', *Journal, Soil Mechanics and Foundations Division, ASCE*, Vol. 98, No. SM 12, pp. 1291-1310.

NOVAK, M., and SACHS, K., (1973), 'Torsional and coupled vibrations of Embedded Footings', *Int. Journal of Earth quake Engineering and Structural Dynamics*, Vol. 2, No. 1. pp. 11-33.

RAMIAH, B.K., CHICKANAGAPPA, E.S., and RAMAMURTHY, T.N., (1977), 'Vertical Vibrations of Embedded Footings', *PROCEEDINGS IX International Conference on Soil Mechanics and Foundation Engineering*, Tokyo, pp. 343-346.

SANKARAN, K.S., KRISHNASWAMY, N.R., and BHASKARAN NAIR, P.G., (1978), 'Rotational Vibrations of Embedded Foundations'. *Proceedings, VI Symposium on Earthquake Engineering*, University of Roorkee, pp. 219-224.

SANKARAN, K.S., KRISHNASWAMY, N.R., and BHASKARAN NAIR, P.G., (1979), 'In-situ Determination of Dynamic Soil Properties', Accepted for Publication in the *Proceedings of the International Symposium on In-situ Testing of Soils and Rocks and Performance of Structures*, University of Roorkee, Dec. 1979.

TAJIMI. H., (1969), 'Dynamic Analysis of a Structure Embedded in an Elastic Stratum', *Proceedings, 4th World Conference, Earthquake Engineering*, Sontiago, pp. 53-69.

THAU, S.A., and UMEK, A., (1974), Coupled Rocking and Translating Vibrations of a Buried Foundation', *Journal of Applied Mechanics*, ASME, Vol. 41, No. 3, pp. 697-702.

WAAS, G., and LYSMER, J., (1972), 'Vibrations of Footings Embedded in Layered Media', *Symposium on Application of the Finite Element Method in Geotechnical Engineering*, U.S. Army Engineer Waterways Experiment Station Vicksburg, Mississippi, pp. 581-604.