Nonlinear Programming in Automated Slope Stability Analysis

by

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Introduction

In most of the limit equilibrium methods of slope stability analysis a number of trial surfaces are analysed and the critical surface is obtained by some iterative procedure. With the advent of high speed digital computers attempts (Little and Price 1958, Horn 1960) for automated embankment analysis have been made. In this communication the usefulness of nonlinear programming technique has been demonstrated for automatic determination of critical slip circle and the corresponding minimum factor of safety. As with the use of any type of analytic or numerical techniques within the context of problem solving, the focus for discussion falls not only on the various techniques available for the analysis but also on the art of how such mathematical procedures are applied. No one procedure or series of procedure will be the panacea that solves all problems to the last detail. Optimization is a useful tool for design and analysis, but its successful application depends to a great extent on how it is used. Since the efficiency of the optimization techniques is problem oriented, the application of these methods to new problems needs critical evaluation.

The object of the paper is to present the usefulness of such techniques and to provide a comparative study of the different minimizing schemes, when applied to slope stability analysis. The problem is one of nonlinear programming with strict inequality constraints.

Analysis

Derivation of the Objective Function

Using the well-known ordinary method of slices and neglecting the imbalance of lateral forces on an individual slice, the factor of safeiy for a non-submerged finite slope in a homogeneous and isotropic $c-\phi$ soil with a circular critical potential slip surface (Figure 1) is

$$F = \frac{c\,\Sigma b_i \sec \alpha_i + \tan \phi\, \Sigma(\gamma b_i h_i \cos \alpha_i - u_i b_i \sec \alpha_i)}{\gamma\, \Sigma\, b_i h_i \sin \alpha_i} \qquad \dots (1)$$

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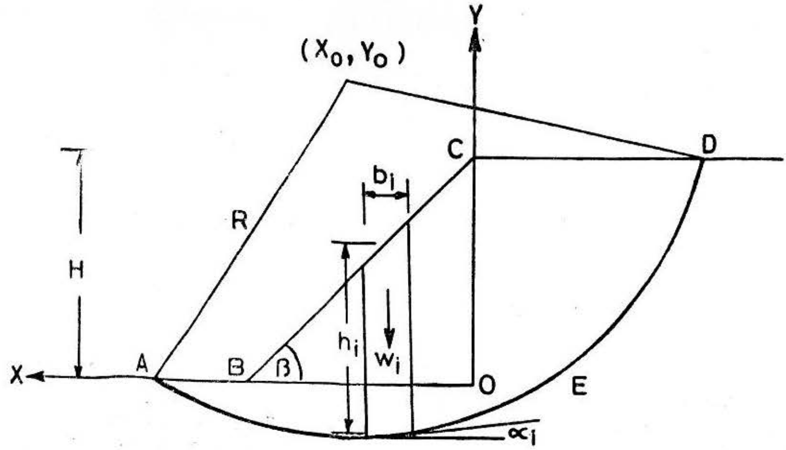

FIGURE 1 Typical slope with a trial arc and a slice

In reference to the figure and the slice equation, c = effective cohesion, ϕ = effective angle of friction, b_i = width of the i th slice, α_i, u_i = angle of inclination of the slip surface and the pore water pressure at the base of the i th slice respectively, w_i = weight of the individual slice, γ = unit weight of the soil.

The factor of safety of a slope in purely cohesive soil or under undrained condition ($\phi = 0$) can be obtained as follows:

$$F = \frac{c}{\gamma H} N \qquad \ldots (2)$$

where

$$N = \frac{\sum b_i^* \sec \alpha_i}{\sum b_i^* h_i^* \sin \alpha_i} \qquad \ldots (3)$$

and $b_i^* = b_i / H$. $h_i^* = h_i / H$, H is the height of the slope.

For a given slope b_i^* h_i^* and α_i are functions of the coordinates of the centre and the radius of the slip circle. Depending on the types of failure, *viz*, base failure, toe failure and slope failure, b_i^* h_i^* and α_i are expressed in terms of the x and y coordinates of the centre (x_0, y_0) and the radius (R) of the slip circle. For reasons of space and brevity these are not presented herein. The factor of safety, as expressed by Equations 1 and 2, is a nonlinear function of the design variables.

Design Variables and the Objective Function

There are many possible circular arcs through a cross section. The location of the critical, or the most dangerous, arc is determined by minimizing the factor of safety. The design variables are the co-ordinates of the centre and the radius of the slip circle, and the objective function is the factor of safety F. The design variables are collected in D vector as follows :

$$D^T = \left(x_0^* \quad y_0^* \quad R^* \right)$$

where $$x_0^* = x_0/H, \quad y_0^* = y_0/H, \quad R^* = R/H \qquad \text{... (4)}$$

F, the factor of safety is expressed as a function of the design vector

$$F = f(D)$$

Design Restrictions

In the absence of any underlying hard stratum the only constraint is that at least a portion of the slip circle must pass through the slope. This condition will be satisfied if the radius of the slip circle is greater than the perpendicular distance of the centre of the slip circle from the sloping surface and the bottom intersection point must lie to the left of y axis (Figure 1). These conditions are expressed in mathematical form as follows :

$$g_1(D) = \left[\left(H/x_n\right) x_0^* + y_0^* - 1\right] / \left[\left(H/x_n\right)^2 + 1\right]^{0.5} - R^* < 0 \qquad \text{... (6a)}$$

$$g_2(D) = -x_{f_b} < 0 \qquad \text{... (6b)}$$

x_{f_b} is the x co-ordinate of the lower intesection point of the slip surface with the slope and is a function of the design variables. x_n is the x co-ordinate of the toe.

Mathematical Programming Problem

Finding the minimum value of the factor of safety (Equations 1 and 2) subject to the constraints as expressed by Equation 6, is formulated as a mathematical programming problem which is stated as follows :

Find the design D_m such that,

$F = f(D_m)$ is minimum,

subject to $g_j(D_m) < 0 \; ..... \; j = 1, 2$ ... (7)

In the present analysis the strict inequality constraints are satisfied indirectly and unconstrained minimization of the objective function is carried out. The initial starting point is chosen to be in the feasible zone. Due to the long step nature of some of the algorithms, the path of optimization could become diverted into region where the constraints are violated and the function is undefined. Such an eventuality is guarded by assigning a very large value for F if the step is beyond the admissible zone, and the minimization algorithm is left to correct the situation on its own. In general, it is more efficient to provide direct logic to accommodate this situation than penalty formulation using tolerance for the constraints so that the strict inequalities are satisfied.

The Unconstrained Minimization Techniques

The unconstrained minimization techniques that are used alongwith quadratic fit for linear minimization are as follows :

1. Powell's method : Conjugate direction.
2. Fletcher and Reeve's method : Conjugate gradients.
3. Davidon-Fletcher-Powell method : Variable metric.

These methods are described in all standard text books on mathematical optimization (Fox 1971).

In the gradient methods, the gradients are evaluated by finite difference method using central difference scheme. Theoretically all the three methods have quadratic convergence.

Convergence Criterion

When the changes in the function values and the design variables in between two cycles are less than 0.001 the minimization is stopped.

Results and Discussion

IBM 7044 computer is used for the computational work.

The stability numbers as expressed by Equation 3, are obtained for slopes in homogeneous and isotropic purely cohesive soil. Instead of preassigning the total numbers of slices into which the area *ABCD* (Figure 1) is divided, it is advantageous to assign a reasonable slice width which is arrived at by dividing the distance *OB* by a number N_2. The total number of slices is then automatically fixed depending on the design vector (D) as can be seen from Table 1. It has been found that for flat slopes, N_2, strongly effects the value of stability number (Table 1*a*). For steep slopes this trend was not observed (Table 1*b*). For 10° slope (Table 1*a*) values of

TABLE 1

Influence of N_2 on the stablility number obtained by using Powell's method of minimization

(*a*)

Slope angle = 10°, Initial $x_0^* = 2.13$, $y_0^* = 2.50$ and $R^* = 4.32$, $\phi = 0$.

N_2	Final design vector D_m			Total number of slices	N
	x_0^*	y_0^*	R^*		
10	2.74	1.39	4.45	16	6.30**
15	2.76	1.70	4.33	22	6.50**
20	2.80	8.56	20.63	135	5.56
25	2.84	7.94	19.10	156	5.57
30	2.83	7.51	18.02	176	5.57
40	2.84	1.83	6.60	58	6.60+

**Values do not converge to the reported minimum.

(*b*)

Slope angle = 70°, Initial $x_0^* = 0.615$, $y_0^* = 1.54$ and $R^* = 1.54$, $\phi = 0$.

N_2	Final design vector D_m			Total number of slices	N
	x_0^*	y_0^*	R^*		
5	0.514	1.54	1.54	18	4.81
10	0.512	1.54	1.54	36	4.83
15	0.512	1.54	1.56	54	4.83
20	0.510	1.54	1.54	72	4.83

N_2 greater or equal to 20 lead to a stability number which is nearly the same as the reported value (Lo 1965). When N_2 was increased to 40. the obtained minimum stability number, 6.6, is much higher than Taylor's (1927) value, 5.53. It could be due to the fact that the direction of minimization is such that the function value decreases very slowly and, as such, premature termination has occurred.

To investigate this aspect, the starting point was altered keeping N_2 same as 40. It now led to a stability number comparable to the exact value. Similar trend is observed for other values of N_2 (Table 2). The

TABLE 2

Influence of starting point on stability number obtained by using Pewell's method of minimization, Slope angle = 10°, ϕ = 0

N_2	starting point D_o			final point D_m			N
	x_0^*	y_0^*	R^*	x_0^*	y_0^*	R^*	
20	1.26	3.08	5.37	2.72	4.54	6.83	6.32**
	2.13	2.50	4.33	2.80	8.56	20.64	5.56
	2.83	9.00	21.00	2.83	9.54	22.90	5.55
40	2.13	2.50	4.32	2.83	1.83	4.33	6.60**
	2.84	7.94	20.64	2.84	8.57	20.64	5.56

** Values do not converge to the reported minimum.

above behaviour brings out also the importance of starting point in an optimization scheme. However, only very flat slopes in homogeneous purely cohesive soil exhibit this dependence on the initial starting point. For reasons of space and brevity only the detailed results obtained by using Powell's algorithm are reported herein.

As an initial guess, the position of the centre of the most dangerous rupture surface through the toe of the slope is found out using the direction angles, compiled by Jumikis (1965) from Fellenius data. This as a guideline was found to be inadequate for flat slopes ($\beta < 50^\circ$) in homogeneous soils but for steep slopes it gives a good starting point. From experience, for flat slopes the initial starting point D_0 is assumed as follows to get a quick and good results:

$$D_0 = (x_n^* \,/\, 2, 9.0,\ 21.00).$$

A comparison of the different algorithms (Table 3) showed that the Davidon-Fletcher-Powell method is the most efficient when applied to this problem. It can be observed that though different algorithms led to different final design point, D_m (Table 3), the minimum stability number is either identical or very close to the exact value. This may be due to the fact that the objective function is almost flat in that zone and, as such, insensitive to the variation of the design vector (D).

As a typical example of the application of the optimization technique for a slope stability analysis in a c—ϕ soil, the following problem has been solved and is compared with Jumikis's (1965) graphical solution. *Example*: A slope (1:2), the height of which is $H = 13.716$ metres is to be made in a c—ϕ soil the unit weight of which is $= 1762.031$ kg/m^3, the angle of internal friction, $\phi = 7^\circ$, and the cohesive strength is found to be $c = 5858.4$ kg/m^2. Compute the factor of safety against rupture of slope.

Slope angle corresponding to 1:2 slope is taken as 26.6°. The initial starting point is chosen as follows:

$$D_0^T = (x_0^* \quad y_0^* \quad R^*) = (34.8950 \quad 69.4337 \quad 88.6664)$$

The starting point function value $F_s = 2.05947$. After optimization the final design vector is obtained as:

$$D_m^T = (49.9013 \quad 67.6726 \quad 92.7463)$$

and the optimized factor of safety F_s is 1.9501. The minimum factor of safety obtained by Jumikis is 2.06. The two values are close enough to show the suitability of the mathematical programming techniques for automated slope stability analysis.

If at any stage of optimization the step length is such that the constraints are violated, the function N_c is assigned an arbitrary high value of 10^3 and the minimization algorithm is left to correct itself on its own. In the present study it was observed that during the process of optimization such possible oscillations of the design between feasible and infeasible zones occur occasionally and, as such, its influence on the optimization scheme might be presumed to be insignificant. Hence the study of the probable influence of the value of the multiplier ($10^3/N$) on the convergence of the optimization scheme were not undertaken.

The present approach has also been successfully applied to slope stability analysis in nonhomogeneous and anisotropic soils (Basudhar 1976) which are not reported here in.

TABLE 3

Comparison of different unconstrained minimization techniques applied to slope stability analysis, $\phi = 0$

Slope angle in degrees	N_2	Unconstrained minimization method	Starting point D_o			Final point D_m			Time in seconds	N
			x_0^*	y_0^*	R^*	x_0^*	y_0^*	R^*		
10	20	POWELL	2.83	9.00	21.00	2.83	9.53	22.90	63	5.52
		FLRV				2.84	8·66	21.40	29	5.56
		DFP				2.83	9.00	21.00	14	5.56
30	7	POWELL	0.86	9.00	21.00	0.88	9.00	21.00	69	5.52
		FLRV				0.86	8.99	21.68	13	5.52
		DFP				0.86	8.99	21.00	9	5.52

POWELL — Powell's method
FLRV — Fletcher-Reeves method
DFP — Davidon-Fletcher-Powell's method

Conclusion

The study brings out clearly that a blind application of sophisticated algorithms may not produce the desired accuracy especially for flat slopes wherein the results are strongly dependent on the proper choice of the width of the slice and on the initial starting point. However, with a little care all the three algorithms can be applied to find the critical slip circle and obtain the corresponding minimum factor of safety. Daviden-Fletcher-Powell method gives the solution in the shortest time.

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