Flow Under a Weir with a Sloping Floor and a Downstream Sheet Pile

by

B.K. Mohanty*

G.C. Mishra**

Introduction

A stepped weir with a downstream cutoff may be substituted by a weir with a sloping step to achieve economy. Also according to Lane (vide Harr, 1962) weirs with sloping step when the slope is greater than 45° ensure better contact between the bottom of the masonry with the foundation material. Malhotra (vide Khosla, 1954) has analysed the flow under a weir with sloping floor resting on a porous medium of infinite depth. In the present investigation flow under weir with sloping floor having a downstream vertical sheet pile has been analysed by using the Schwarz-Christoffel transformation. Uplift pressure and exit gradient are presented at salient points of the hydraulic structure.

Analysis

Figure 1(*a*) shows the physical flow domain in Z plane. The weir rests on a homogeneous isotropic porous medium of infinite extent. According to the Schwarz-Christoffel transformation the conformal mapping of the flow domain described by the polygon ABC DEF onto lower half of the t plane is given by

$$Z = \int_0^t \frac{M\,(t-\beta)}{t^{\alpha}\,(1-t)^{1/2}\,(\gamma-t)^{1/2-\alpha}}\,dt + N \qquad ...(1)$$

the vertices $A, B, C, D, E, F,$ being mapped onto $-\infty, 0, \gamma, \beta\ 1, \infty$ respectively. $\alpha\pi$ is the inclination of the sloping step with horizon. In equation (1) M and N are constants to be evaluated. For the point B, $t=0$ and $Z=0$. Therefore, $N=0$. The other three unknowns M, γ and β are found making use of the condition at points C, D, E. Equation (1) can be written as

$$Z=M\int_0^t t^{1-\alpha}\,(\gamma-t)^{\alpha-1/2}\,(1-t)^{-1/2}\,dt-M\beta\int_0^t t^{-\alpha}(\gamma-t)^{\alpha-1/2}\,(1-t)^{-1/2}\,dt \qquad ...(2)$$

Expanding the term $(1-t)^{-1/2}$ in equation (2) according to Binomial theorem i.e.

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*Assistant Design Engineer, MECON, Rourkela Site Officer, Rourkela, India.

**Reader, School of Hydrology, University of Roorkee, India.

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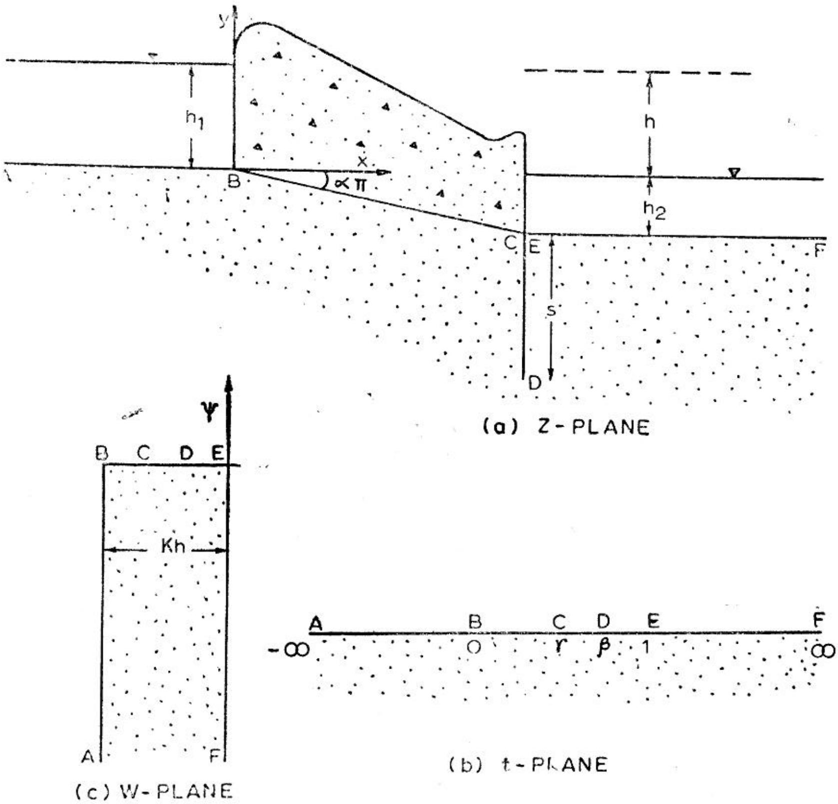

FIGURE 1. Steps of conformal mapping

$$(1-t)^{-1/2} = \sum_{n=0}^{\infty} A_n \, t^n$$

where $A_o=1$, $A_1=\frac{1}{2}$, and $A_n= \dfrac{\frac{1}{2}\,(\frac{1}{2}+1)\ \ (\frac{1}{2}+n-1)}{n!}$

and integrating, equation 2 is reduced to

$$Z=M \sum_{n=0}^{\infty} A_n \, \gamma^{n+3/2} \, B_{t/\gamma} \, (n+2-\alpha,\ 0.5+\alpha)$$

$$-M\,\beta \sum_{n=0}^{\infty} A_n \, \gamma^{n+1/2} \, B_{t/\gamma} \, (n+1-\alpha,\ 0.5+\alpha) \qquad \text{...(3)}$$

where $B_{t/\gamma}$ ( ) = incomplete beta function.

Equation (3) governs the relationship between Z and t for $0 \leqslant t \leqslant \gamma$. For point C, $t = \gamma$ and $Z = Z_c$. Therefore from equation (3)

$$Z_c=M \sum_{n=0}^{\infty} A_n \, \gamma^{n+3/2} \, B\,(n+2-\alpha,\ 0.5+\alpha)$$

$$-M\,\beta \sum_{n=0}^{\infty} A_n \, \gamma^{n+1/2} \, B\,(n+1-\alpha,\ 0.5+\alpha) \qquad \text{...(4)}$$

where B ( ) = complete beta function.

The relationship between Z and t for $\gamma \leqslant t \leqslant 1$ is given by

$$Z = M \int_{\gamma}^{t} t^{1-\alpha} (\gamma - t)^{\alpha - 1/2} (1-t)^{-1/2} \, dt$$

$$- M\beta \int_{\gamma}^{t} t^{-\alpha} (\gamma - t)^{\alpha - 1/2} (1-t)^{-1/2} \, dt + Z_c \qquad \text{...(5)}$$

In order to evaluate the integral, the term $t^{1-\alpha}$ and $t^{-\alpha}$ appearing in the integrand in equation (5) are replaced by $[1-(1-t)]^{1-\alpha}$ and $[1-(1-t)]^{-\alpha}$ respectively and by making use of Binomial theorem these terms are expressed in power series of $(1-t)$. Thus

$$Z = M \int_{\gamma}^{t} \left[\sum_{n=0}^{\infty} (\gamma - t)^{\alpha - 1/2} (1-t)^{-1/2} C_n (1-t)^n \right] dt$$

$$- M\beta \int_{\gamma}^{t} \left[\sum_{n=0}^{\infty} (\gamma - t)^{\alpha - 1/2} (1-t)^{-1/2} D_n (1-t)^n \right] dt + Z_c \qquad \text{...(6)}$$

where $C_o = 1$, $C_1 = -(1-\alpha)$, $C_2 = \dfrac{-(1-\alpha)\alpha}{2!}$

$$C_n = -\frac{(1-\alpha)(\alpha)(1+\alpha)\ldots(\alpha+n-2)}{n!}$$

$D_o = 1$, $D_1 = \alpha$ and

$$D_n = \frac{\alpha(1+\alpha)\ldots(n-1+\alpha)}{n!}$$

Integration and simplification lead to

$$Z = M(-1)^{\alpha - 1/2} \sum_{n=0}^{\infty} (1-\gamma)^{\alpha+n} \times B_{\frac{t-\gamma}{1-\gamma}} (\alpha + \tfrac{1}{2}, n + \tfrac{1}{2}) (C_n - \beta D_n) + Z_c \qquad \text{...(7)}$$

For point D, $t = \beta$ and $Z = Z_D$. Hence from equation (7)

$$Z_D - Z_C = -i S$$

$$= M(-1)^{\alpha - 1/2} \sum_{n=0}^{\infty} (1-\gamma)^{\alpha+n} \times$$

$$B_{\frac{\beta-\gamma}{1-\gamma}} (\alpha + \tfrac{1}{2}, n + \tfrac{1}{2}) (C_n - \beta D_n) \qquad \text{...(8)}$$

where S is the length of the sheet pile. For point E, $t=1$ and $Z = Z_E = Z_C$. Therefore from equation 7

$$\sum_{n=0}^{\infty} (1-\gamma)^n B(\alpha + \tfrac{1}{2}, n + \tfrac{1}{2}) (C_n - \beta D_n) = 0 \qquad \text{...(9)}$$

From equations, 4, 8 and 9 the three unknown γ, β and M can be evaluated.

For $1 \leqslant t \leqslant \infty$ relationship between Z and t is given by $Z = \int_1^t M\,(t-\beta)$ $t^{-\alpha}\,(1-t)^{-1/2}\,(\gamma-t)^{\alpha-1/2}\,dt + Z_E$

Integration by parts leads to

$$Z = -2M\,(1-t)^{1/2}\,(\gamma-t)^{\alpha-1/2}\,t^{-\alpha}\,(t-\beta)$$

$$+2M\int_1^t (1-t)^{1/2}\,(\gamma-t)^{\alpha-1/2}\,t^{-\alpha}$$

$$\times\left[\frac{(0.5-\alpha)\,(t-\beta)}{\gamma-t} - (\alpha/t)(t-\beta)+1\right]dt + Z_E \qquad \ldots(10)$$

The integrals in equation 10 are to be evaluated numerically.

The complex potential W for the flow domain is shown in Figure 1(c) where $W = \phi + i\,\psi$. Here ψ is the stream function and ϕ is the potential function defined as

$$\phi = -K\,(p/\gamma_w + y) + C \qquad \ldots(11)$$

in which

K = coefficient of permeability;

p = pressure;

y = co-ordinate;

γ_w = unit weight of water and

C = constant.

For the present case the constant C is assumed to be $K\,(h_2 - y_E)$ where

h_2 = depth of water in the downstream side and

y_E = y co-ordinate of point E

The mapping of the W plane onto lower half of t plane is given by

$$dW/dt = M'/(\sqrt{t}\,\sqrt{1-t}) \qquad \ldots(12)$$

Integrating

$$W = 2\,M'\,\sin^{-1}\sqrt{t} + N' \qquad \ldots(13)$$

For point B, $t=0$ and $W=-Kh$. Therefore $N'=-Kh$

For point E, $t=1$ and $W=0$. Hence

$$M' = Kh/\pi$$

The expression for exit gradient is given by

$$I_E = i/K\,(dW/dt\,.\,dt/dZ) \qquad \ldots(14)$$

Substituting the expression for dW/dt and dt/dZ from equation (12) and (1) respectively

$$I_E = i/K.\ M'/M.\ t^{\alpha-1/2}\ \frac{(\gamma-t)^{1/2-\alpha}}{t-\beta} \qquad ...(15)$$

Results and Discussion

Numerical results are presented for ϕ at points C and D and exit gradient at point E, in non-dimensional form for different values of α and for different length of the inclined floor. From corresponding values of ϕ, using equation (11) pressures at points C and D can be obtained easily. Figures 2(a) and 2(b) show the variation of potential at points C and D respectively with $|Z_c|/S$, $|Z_c|$ being length of the sloping floor for different values of α.

The case $\alpha=0$ corresponds to a flat bottom weir with down stream vertical sheet pile. For $\alpha=0$, the results obtained from the present analysis

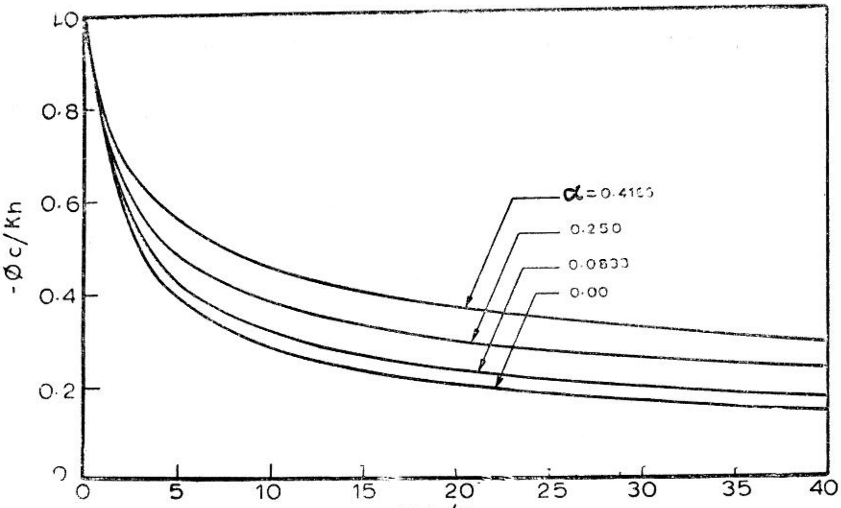

FIGURE 2 (a). Variation of ϕ_C/Kh with $|Z_c|/S$

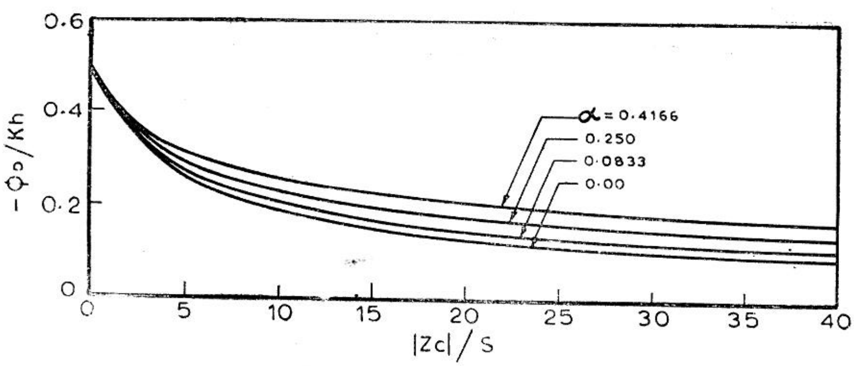

FIGURE 2 (b). Variation of ϕ_D/Kh with $|Z_c|/S$

compare with result given by Khosla *et al* (1954) for flat bottom weir with a vertical sheet pile at the down stream end. Figure 3 shows the variation of exit gradient at point *E*, with length of the sloping floor for values of $\alpha=0.4166$, 0.25, 0.0833 and 0. Exit gradient is presented only for point *E* since the exit gradient is maximum at this point. When $\alpha=0$, the results compare with the results presented by Khosla for a flat bottom weir with a downstream vertical sheet pile.

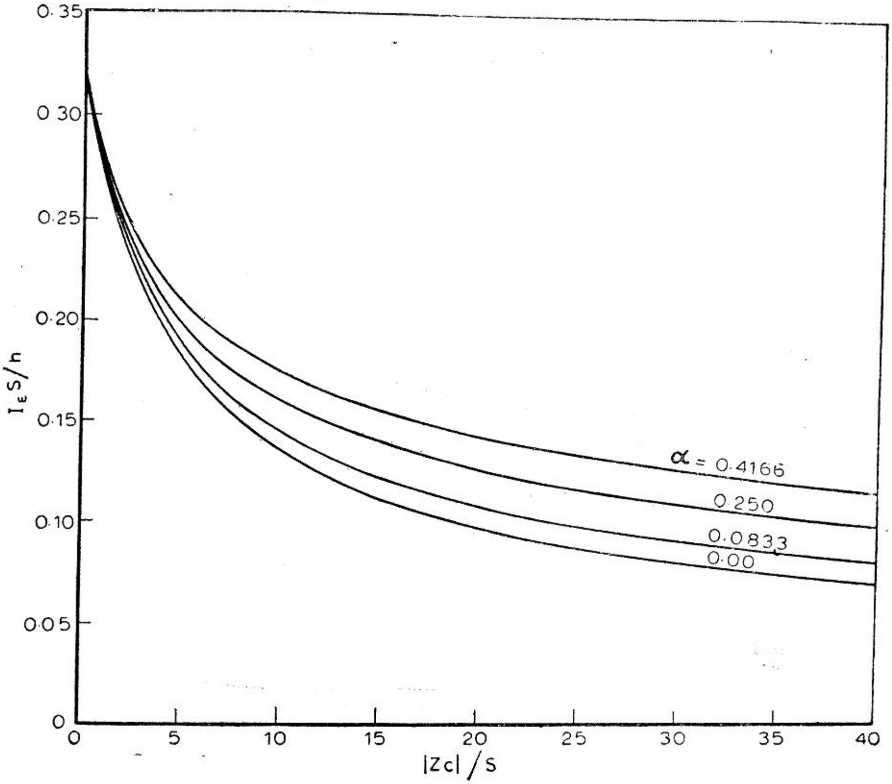

FIGURE 3. Variation of exit gradient with length of floor

The results presented here are useful for evaluation of pressure and exit gradient required for design of weir with sloping floor having a downstream vertical sheet pile.

Conclusions

Using the Schwarz-Christoffel transformation, analytical solution is presented for determining uplift pressure and exit gradient for a weir having sloping floor and a vertical sheet pile. The results are useful in design of weirs with sloping floor on permeable foundation of infinite depth.

References

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