Steady State Solution for Non-Penetrating well with Nonlinear Flow

by

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Introduction

For many decades, Darcy's simple linear flow law connecting velocity and gradient served a unique role in the field of flow through porous media. While using Darcy's linearity for various field problems it is always cautioned about its applicability in the high velocity zone where inertial forces dominate over viscous forces and thus causing a less than proportional increase of velocity with gradient resulting non-linear form of velocity. gradient response. The deviation from Darcian linear response is expected when Reynold's number of flow exceeds unity (Taylor, 1948). Various investigators have proposed various forms of velocity-gradient relationships which are mainly based on extensive experimental results. A summary of the available relations is given in Table I.

Of late various field situations have been recognised where it is felt that for accurate predictions a suitable nonlinear velocity-gradient response will have to be employed. Typical examples of such situations are:

(*i*) Flow through coarse grained soils and rockfill banks and dams.

(*ii*) Flow in coarse grained acquifer under high drawdown, especially in the area adjacent to pumping well.

(*iii*) Flow in filters.

In recent past, several attempts have been made to analytically solve some of the steady and unsteady state field problems incorporating one or the other form of nonlinear velocity-gradient relationship shown in Table-I. Amongst the steady state problems, Slepicka (1961) studied the discharge from wells placed in confined and unconfined acquifer, Volker (1969) studied the nonlinear seepage in isotropic and anisotropie porous media, Valsangkar, Subramanya and Rao (1973) studied the nonlinear seepage in trenches. Exitsing solutions for unsteady state problems with nonlinear flow response number very few. Hansbo (1960) and Schimidt and Westmann (1975) gave solutions for one dimensional unsteady flow through compressible porous media incorporating velocity gradient response of the type $v=Mi^n$. Basak (1975) solved the same problem with different initial condition. Basak (1975) also presented an analytical solution for unsteady flow in semi-infinite embankments with $v=Mi^n$.

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TABLE—I

Available nonlinear flow equations at high gradients

Sr. No.	Equation	Original propser	Explanation of terms	Comments
1.	$v=Mi^n$ $n<1$	Izbash (1931)	M=Proportionality constant having a unit of velocity	Purely experimental relation
2.	$v=(Bi)^{\frac{1}{2}}$	Eseande (1953)	B=Constant	(*i*) Purely empirical (*ii*) B is found to vary both. 80 to 290 cm^2/sec^2 for particle dia. of 1 inch
3.	$v=32.9m^{\frac{1}{2}}\ i^{0.54}$	Wilkinson (1956)	m=Hydraulic radius	(*i*) semi-empirical (*ii*) Experiments were conducted on particle size of 3/4 inch to 3 inch
4.	$v=\alpha\ \left(\frac{\mu}{\sigma}\right)^{f-1}\ (ki)^f$	Slepicka (1961)	α, f, k=Constants μ=viscosity σ=surface tension	(*i*) Semi-empirical (*ii*) The equation was derived from dimensional analysis
5.	$i=av+bv^2$	Forchhiemer	a and b are constants with units of reciprocal and square of reciprocal of velocity respectively	(*i*) Empirical (*ii*) Theoretical basis was found later by Ahmed (1969)
6.	$i=av+bv^2+cv^3$	Forchhiemer (1901)	a, b, c=constants	Empirical

Out of the various types of non-linear velocity gradient response available as shown in Table-I, the two most widely used are that due to Izbash (1931) and Forchhimer (1901). While Izbash's equation is purely empirical, Forchhiemer's equation, though initially proposed on the basis of experimental results alone, is found to have theoretical justification also (Ahmed, 1967).

In this paper, a steady state analytical solution for the case of non penetrating well in a semi-infinite medium incorporating Izbash's flow law is presented. The solution is compared with the available Darcian solution (Harr, 1962). The effect of nonlinearity in the flow response on the discharge characteristics and piezometric pressure distribution in relation to corresponding linear case is brought out.

Analysis

The non-penetrating well is a special case of partially penetrating well when it just penetrates the top surface of the acquifer and displays spherical flow. Figure 1 is the definition sketch of the problem under consideration where,

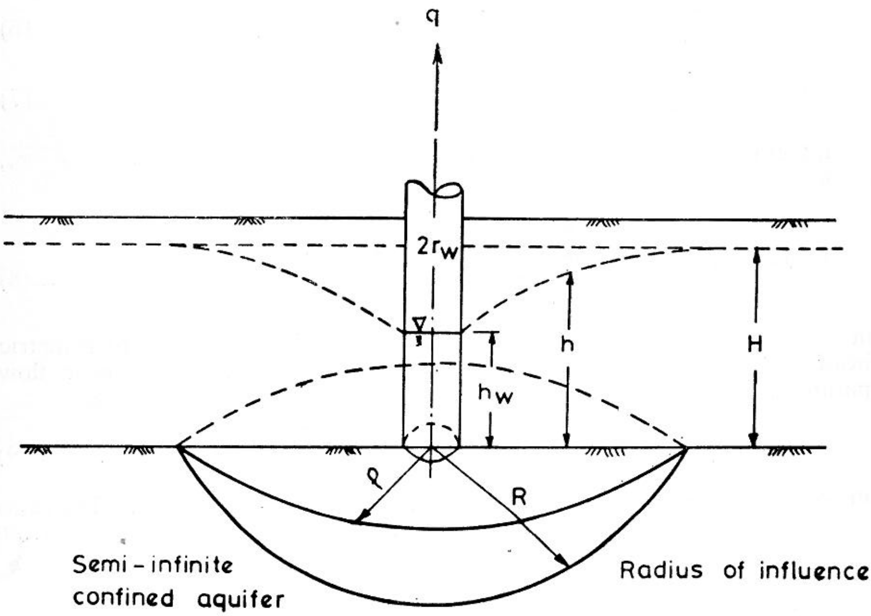

FIGURE 1. Definition sketch for non-penetrating well

r_w = Radius of well

R = Influence radius

ρ = Any radius

h_w = Head at $\rho = r_w$

H = Piezometric head at $\rho = R$

h = Piezometric head at any ρ

q = Tubewell discharge

If v =velocity at any distance then

$$v = \frac{q}{4\pi\rho^2} \qquad \text{...(1)}$$

As mentioned earlier the velocity-gradient response due to Izbash is

$$v = M_i^n \qquad \text{...(2)}$$

Combining Equation (1) and (2) and assuming Dupuit's assumption to be valid,

$$\frac{q}{4\pi\rho^2} = M\left(\frac{dh}{d\rho}\right)^n \qquad \text{...(3)}$$

or
$$\frac{H}{R}\frac{dy}{dx} = \frac{q_*^{1/n}}{x^{2/n}} \qquad \text{...(4)}$$

where,

$$x = \frac{\rho}{R} \qquad \text{...(5)}$$

$$y = \frac{h}{H} \qquad \text{...(6)}$$

$$q_* = \frac{q}{4\pi MR^2} \qquad \text{...(7)}$$

Integration of the Equation (4) with the boundary condition $\rho = r_w$, $h = h_w$ results

$$y - y_o = \frac{n}{2-n}\left(\frac{q_*^{1/n}}{H/R}\right)\left[x_o^{-\left(\frac{2-n}{n}\right)} - x^{-\left(\frac{2-n}{n}\right)}\right] \qquad \text{...(8)}$$

in which $x_o = r_w/R$ and $y_o = h_w/H$. This equation gives the piezometric head distribution as a function of H/R, q_*, x_o, y_o and the nonlinear flow parameter n. The corresponding relation for Darcian flow ($n=1$) is

$$y_D - y_o = \frac{q_*}{H/R}\,(x_o^{-1} - x^{-1}) \qquad \text{...(9)}$$

in which y_D is the piezometric head for Darcian linear flow. The ratio between piezometric head for nonlinear and linear flow is obtained from Equations (9) and (8) as

$$\frac{y - y_o}{y_D - y_o} = \frac{n}{2-n}\,\frac{q_*^{\frac{1}{n}-1}}{x_o^{\frac{2}{n}-2}}\left[\frac{1 - (x/x_o)^{-\frac{2-n}{n}}}{1 - (x/x_o)^{-1}}\right] \qquad \text{...(10)}$$

Note that

$$\underset{\frac{x}{x_o} \to 1}{Lt}\ \frac{y - y_o}{y_D - y_o} = \frac{q_*^{\frac{1}{n}-1}}{x_o^{\frac{2}{n}-2}}$$

Equation (10) is plotted for various values of nonlinear parameter 'n' and is shown in Figure 2.

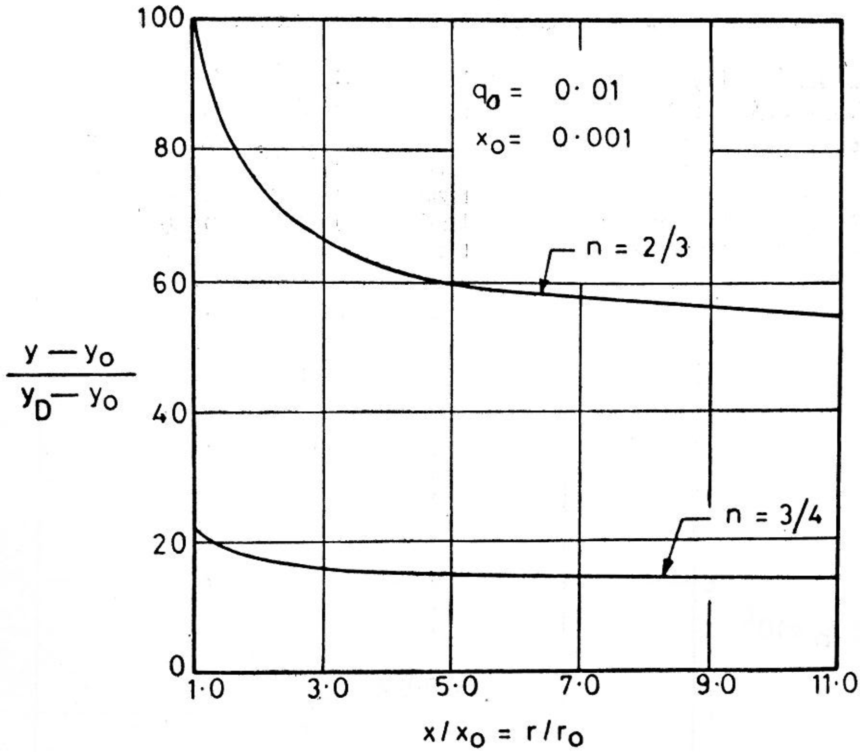

FIGURE 2. The Variation of the ratio of piezometric head distribution for non-linear and linear flow

Inserting second boundary condition $h=H$ at $\rho=R$, i.e. $y=1$ at $x=1$ in Equation (8), one gets

$$1-y_o = \frac{n}{2-n}\,\frac{q_*^{1/n}}{H/R}\left(x_o^{-\frac{2-n}{n}} - 1\right) \qquad \text{...(11)}$$

This can further be written as

$$q_* = \left[\frac{2-n}{n}\left(\frac{r_w}{R}\right)^{\frac{2-n}{n}}\right]^{n}\left(\frac{H-h_w}{R}\right)^{n} \qquad \text{...(12)}$$

which gives the discharge q_* as a function of well drawdown $(H-h_w)$, radius r_w, influence radius R and the nonlinear parameter n. Equations (12) can also be expressed as

$$q_* = T_n\ I_{av}^{n} \qquad \text{...(13)}$$

Where, $I_{av} = \frac{H-h_w}{R}$ = Average hydraulic gradient ...(14)

and $T_n = \left(\frac{2-n}{n}\right)^n \left(\frac{r_w}{R}\right)^{2-n}$ ...(15)

= may be called as 'Average transmission coefficient'.

Expression (13) is of the similar form of the governing nonlinear flow equation ($v=M_i^n$) itself. For linear flow $n=1$, and Equations (11), (12) or (13) reduces to the well known Darcian discharge equations for the problem. The average transmission coefficient T_n as given by Equation (15) is plotted against nonlinear parameter 'n' for various values of r_w/R and is shown in Figure 3. The error involved in the discharge estimation by assuming the flow to be linear is plotted against the average gradient in Figure 4.

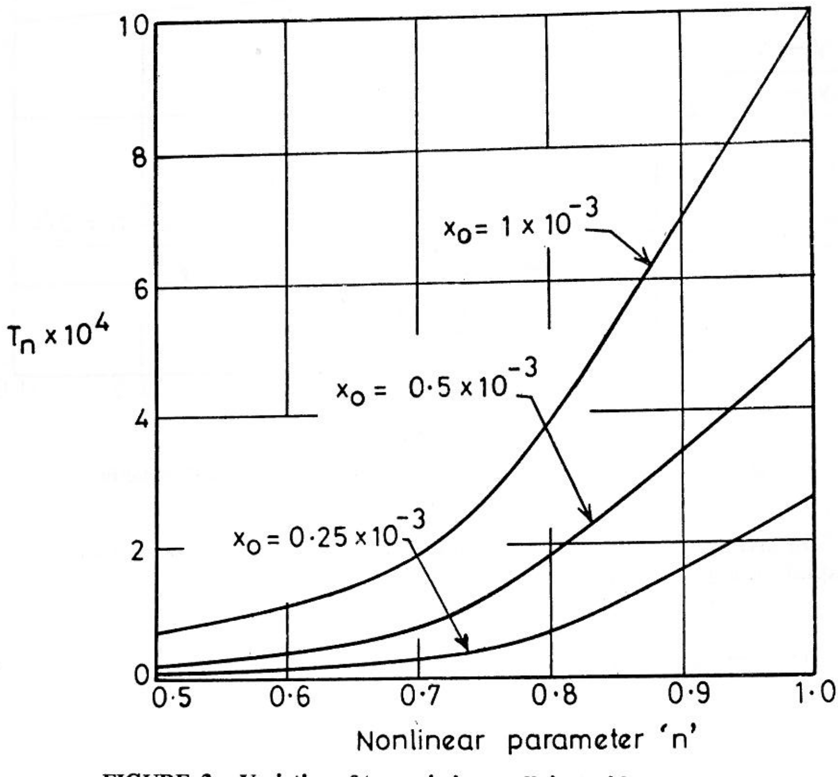

FIGURE 3. Variation of transmission coefficient with n ann X_o

Result and Discussions

The piezometric head distribution as given by Equation (8) indicates that to maintain in same discharge, head at any distance from the well is required to be more in case of nonlinear flow than that of linear case or in otherway, for same drawdown at any distance Darcian linear flow will give more discharge than that due to nonlinear flow (n being less than unity). Equation (10) and it's plot in Figure 2, shows that the error involved in piezometric head variation in assuming the flow to be linear is considerable. A nonlinear flow characterised by $n=2/3$, will have approximately 60 times more

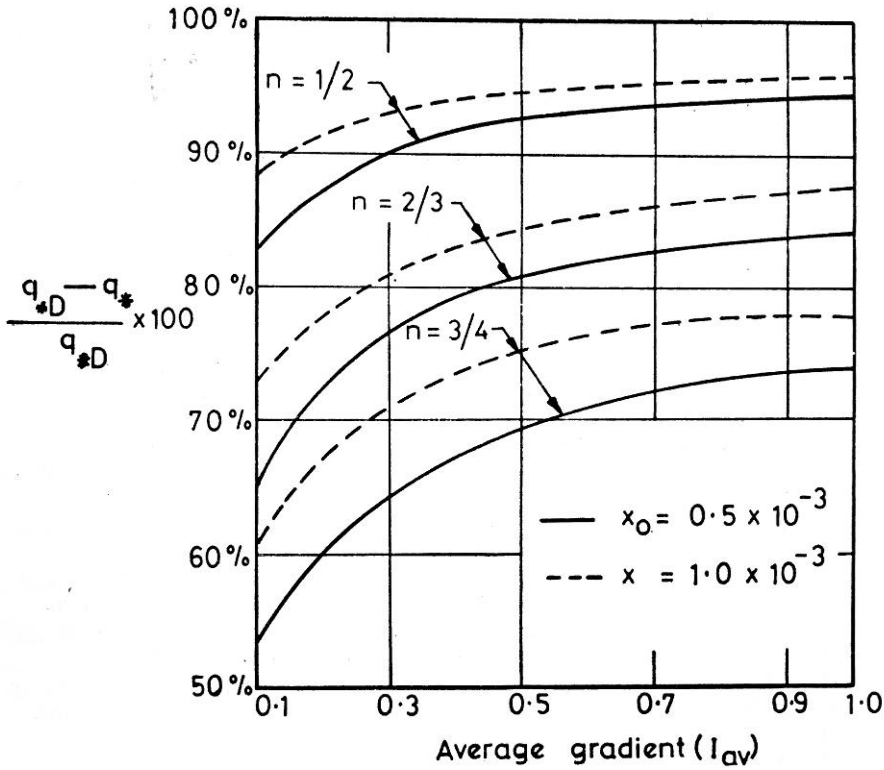

FIGURE 4. Error involved in discharge estimation in assuming the flow

piezometric head based on Darcian linear flow at a distance of $5r_w$ from the well centre. The error is more near the well and increases with the increase in the nonlinear parameter n. This indicates that safe well spacing for designing group of well will be adversely affected by the assumption of Darcian linear flow response.

In practical problems, the value of average gradient (I_{av}) and the nonlinear parameter n will be always les than one (usually, the range varies between 0.5 to 0.75) and hence it is evident from Equations (13) and (15) the discharge q_* will be always less than the predicted discharge q_{*D} based on Darcian linear law. The average Transmission Coefficient T_n drastically reduces as the parameter n deviates more and more from the assumed value of unity in the linear flow. This is clearly seen in Figure 3. The error involved in prediction of discharge by assuming the flow to be linear as depicted in Figure 4 is seen to be considerable and is a function of well radius (r_w/R), average gradient (I_{av}) and the nonlinear parameter n. A weakly nonlinear flow with $n=3/4$ will yield only 25% of the predicted discharge based on Darcian linear flow at an average gradient (I_{av}) of 0.6 for $r_w/R=1\times10^{-3}$. An increase in the nonlinear parameter n, average gradient (I_{av}) and decrease in well radius (r_w/R) will cause an increase in the error.

Conclusions

A steady state nonpenetrating well flow problem is being analysed incorporating the most widely used nonlinear velocity-gradient response of the type $v=Mi^n$, given by Izbash. Essentially, the primary objective of the investigation has been to bring out the necessity of considering the nonlinear behaviour at high Reynold's numbers, as the results can be at considerable variance with results based on Darcian linear law. The representative numerical results are believed to be useful in estimating the magnitude of error that may result in neglecting the nonlinear behaviour in the case of nonpenetrating well flow problem.

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