Simple Analytical Solutions to Terzaghi Consolidation Theory

by

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Terzaghi consolidation theory

The solutions of the one dimensional consolidation equation (viz.)

$$C_v \frac{\partial^2 u}{\partial Z^2} = \frac{\partial u}{\partial t} \qquad \text{...(1)}$$

are given by

$$u = \frac{4}{\pi} \sigma^1 \sum_{N=0}^{N=\infty} \frac{1}{(2N+1)} \cdot \text{Sin}\left[\frac{(2N+1)\pi Z}{2H}\right] \cdot \text{exp.} -\left[\frac{(2N+1)^2 \pi^2 T_v}{4}\right] \qquad \text{...(2)}$$

and

$$U = 1 - \frac{8}{\pi^2} \sum_{N=0}^{N=\infty} \frac{1}{(2N+1)^2} \cdot \text{exp.} -\left[\frac{(2N+1)^2 \pi^2 T_v}{4}\right] \qquad \text{...(3)}$$

where u = pore water pressure, U = average degree of consolidation and T_v = time factor.

Since these equations are cumbersome for calculation purposes, Terzaghi (1943) gave the following two emperical expressions to represent Equation (3) besides presenting the exact values in graphical form.

$$T_v = \frac{\pi}{4}\left(\frac{U}{100}\right)^2 \qquad \text{...(4)}$$

for values of U per cent between 0 and 52.6 and

$$T_v = 1.781 - 0.933 \log_{10} (100 - U \text{ per cent}) \qquad \text{...(5)}$$

for values of U per cent between 52.6 and 100.

However, no emperical expressions were proposed by Terzaghi (1943) for calculating the pore water pressure in the soil layer.

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* *The paper is open for discussions till the end of February, 1977.*

Assumptions and the governing equation

A convenient non-dimensional form of Equation (1) could be obtained by writing $u' = \frac{u}{\sigma'}$, $Z_o = \frac{Z}{2H_o}$ and $T_v = \frac{C_v t}{H_o^2}$.

Thus

$$\frac{\partial}{\partial Z_o}\left(\frac{\partial u'}{\partial Z_o}\right) = 4\frac{\partial u'}{\partial T_v}. \quad \ldots(6)$$

The initial and boundary conditions for u' are shown in Figures 1(a) and 1(b). Terzaghi (1943) pointed out that the isochrones for u' have approximately the shape of parabolas. The consolidation process can be conveniently divided into the following two stages. During the first stage of consolidation, the apex of the isochrones 'c' and 'e' advance towards 'd' (Figure 1a). At the end of the first stage, points 'c' and 'e' meet at point 'd' and the isochrone moves away from the line bf. This is the second stage of consolidation.

Solution during the first stage of consolidation

During the first stage of consolidation, the isochrones are assumed to be represented by parabolas of the form

$$u_a' = \frac{z_o(2z_1 - z_o)}{z_1^2} \qquad \text{for } z_o < z_1, \quad z_1 < \tfrac{1}{2} \quad \ldots(7a)$$

and

$$u_a' = 1 \qquad \text{for } z_1 < z_o < \tfrac{1}{2} \quad \ldots(7b)$$

where z_1 is an unknown function of time factor and has to be determined. If Equation 7(a) is substituted Into Equation (6), a residual R is obtained depending upon the error involved. Thus

$$R = \frac{\partial}{\partial Z_o}\left[\frac{\partial u_a'}{\partial Z_o}\right] - 4\frac{\partial u_a'}{\partial T_v} \quad \ldots(8)$$

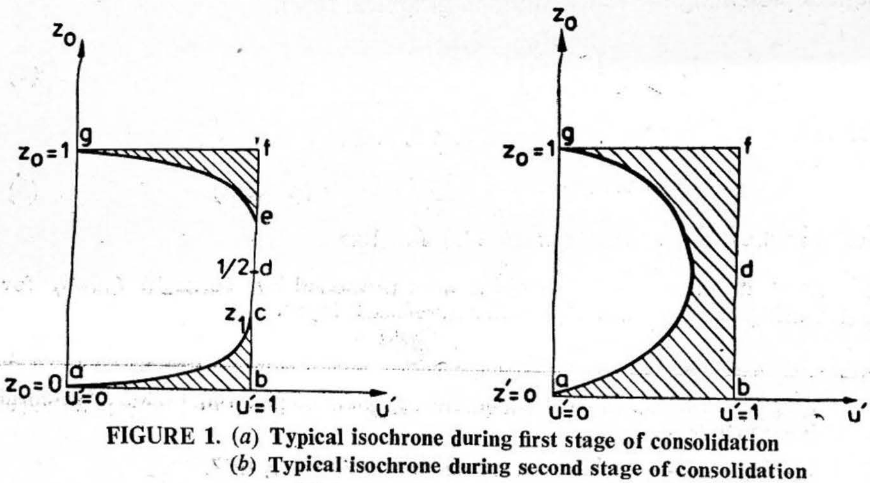

FIGURE 1. (a) **Typical isochrone during first stage of consolidation**
(b) **Typical isochrone during second stage of consolidation**

Taking a weighting function of z_o and applying Galerkin's weighted residual method,

$$\int_0^{Z_1} R.z_o.dz_o = 0 \qquad \ldots(9)$$

i.e.

$$\int_0^{Z_o}\left[\frac{\partial}{\partial z_o}\left(\frac{\partial u_a'}{\partial z_o}\right)-4\,\frac{\partial u_a'}{\partial T_v}\right].z_o\,.\,dz_o = 0 \qquad \ldots(10)$$

Substitution of the values of $\frac{\partial u_a'}{\partial z_o}$ and $\frac{\partial u_a'}{\partial T_v}$ from equation 7(a) and subsequent integration of Equation (10) yeilds

$$z_1 dz_1 = 3/2\ dT_v \qquad \ldots(11)$$

which on integration between the limits 0 to z_1 and 0 to T_v gives

$$z_1 = \sqrt{3T_v} \qquad \ldots(12)$$

The average degree of consolidation U is given by the shaded area of Figure 1(a). Thus

$$U = 1-2\int_0^{z_1} \frac{z_o(2z_1-z_o)}{z_1^2}.\ dz_o-(1-2z_1)$$

$$= \frac{2}{3}\,z_1 \qquad \ldots(13)$$

Substitution of Equation (13) into Equation (12) gives,

$$U = \sqrt{4/3\,T_v}$$

This equation gives a simple relationship between U and T_v during the first stage of consolidation. It is also observed that the first stage of consolidation is over where $U = 1/3$ or $T_v = 0.0833$.

Solution during the second stage of consolidation

During the second stage of consolidation, the isochrones for u' can be expressed by parabolas of the form

$$u_a' = z_o(1-z_o)\ F \qquad \ldots(15)$$

where F is a function of time.

If Equation (15) is substituted into Equation (6), the residual R is given by

$$R = \frac{\partial}{\partial z_o}\left[\frac{\partial u_a'}{\partial z_o}\right]-4\,\frac{\partial u_a'}{\partial T_v} \qquad \ldots(16)$$

Application of Galerkin's weighted residual method with a weighting function of $z_o(1-z_o)$ gives,

$$\int_0^1 \left[\frac{\partial}{\partial z_o}\left(\frac{\partial u_a'}{\partial z_o}\right) - 4\,\frac{\partial u_a'}{\partial T_v}\right] . z_o(1-z_o)\, dz_o = 0 \qquad \text{...(17)}$$

As before substitution of $\frac{\partial u_a'}{\partial z_o}$ and $\frac{\partial u_a'}{T \partial_v}$ values from Equation (15) into Equation (17) and subsequent integration yeilds

$$dT_v = \tfrac{2}{5} . dF/F \qquad \text{...(18)}$$

Integration of Equation (18) between the limits 0 to T_v gives

$$T_v = \tfrac{2}{5} . \log_e (F/F_o) \qquad \text{...(19)}$$

where F_o is the initial value of F and is obtained as follows : when $T_v = 0$, equation (15) gives

$$1 - z_o(1-z_o)\, F_o = 0. \qquad \text{...(20)}$$

Applying Galerkin's method with a weighting function of $z_o(1-z_o)$ to the above equation one gets

$$\int_0^1 [1 - z_o(1-z_o) F_o]\, z_o(1-z_o)\, dz_o = 0 \qquad \text{...(21)}$$

whence $F_o = 5$. Thus Equation (19) can be written as

$$F = 5 \exp(-2.5 T_v) \qquad \text{...(22)}$$

The average degree of consolidation U is given as before by the shaded area of Figure 1(b). Thus

$$U = 1 - \int_0^1 z_o(1-z_o)\, F\, dz_o \qquad \text{...(23)}$$

which gives $\quad U = (1 - \tfrac{1}{6} F)$.

Thus Equation (22) can be written as

$$U = 1 - \tfrac{5}{6} \exp. (-2.5\, T_v) \qquad \text{...(24)}$$

This equation provides a simple relationship between U and T_v during the second stage of consolidation. The values of U for various values of T_v are calculated using Equations (14) and (24) and are presented in Table I along with those calculated from Terzaghi's rigorous solution given in Equation (3). It is observed that the values of average degree of consolidation calculated with the present analysis are within 2 to 3 per cent of the exact values.

TABLE I

Time Factor T_v	Average degree of consolidation U		Mid-plane pore water pressure ratio u_o'	
	Terzaghi's theory Equation (3)	Present Analysis : Equation (14) for T_v values upto 0.083 and Equation (24) for T_v values beyond 0.083	Terzaghi's theory Equation (2)	Present Analysis : Equation (25)
0.004	0.0735	0.0731	1.0000	1.0000
0.008	0.1038	0.1033	1.0000	1.0000
0.012	0.1248	0.1265	1.0000	1.0000
0.020	0.1598	0.1633	1.0000	1.0000
0.028	0.1889	0.1932	1.0000	1.0000
0.036	0.2141	0.2191	0.9996	1.0000
0.048	0.2464	0.2530	0.9974	1.0000
0.060	0.2764	0.2829	0.9922	1.0000
0.072	0.3028	0.3098	0.9828	1.0000
0.083	0.3233	0.3328	0.9716	1.0000
0.100	0.3562	0.3512	0.9493	0.9731
0.150	0.4370	0.4275	0.8642	0.8588
0.200	0.5041	0.4945	0.7723	0.7582
0.250	0.5622	0.5541	0.6842	0.6688
0.300	0.6132	0.6063	0.6068	0.5905
0.350	0.6582	0.6546	0.5355	0.5205
0.400	0.6973	0.6934	0.4745	0.4599
0.500	0.7640	0.7615	0.3708	0.3532
0.600	0.8156	0.8141	0.2897	0.2793
0.700	0.8559	0.8554	0.2264	0.2170
0.800	0.8874	0.8872	0.1769	0.1691
0.900	0.9119	0.9121	0.1382	0.1319
1.000	0.9313	0.9316	0.1080	0.1026
2.000	0.9942	0.9944	0.0092	0.0084

Calculation of pore water pressure in the soil layer

For calculating the pore water pressure during consolidation at any depth in the soil layer, Equations (7*a*) and (12) are used during the first stage of consolidation and Equations (15) and (22) are used during the second stage of consolidation. As an example, if it is required to calculate the dissipation of pore water pressure at the mid-plane with time in a consolidating soil layer with double drainage, a valve of $Z_o = 0.5$ is substituted into Equation (15) which gives $u_a' = 0.25\ F$. Thus from Equation (22),

$$u_o' \text{ (mid-plane)} = 0.25 \exp(-2.5\ T_v) \qquad \text{...(25)}$$

Using the above simple equation, the pore water pressure ratios u_o' at the mid-plane are calculated for different values of T_v and are presented in Table I along with those calculated from Terzaghi's rigorous solution given in Equation (2). In this case also, the pore water pressure ratios calculated from the present analysis are within 2 to 3 per cent of the exact values. By substituting the relevant values for Z_o (viz.), $Z_o = 0.1$, 0.2, 0.3 etc....in Equation (15) and following the above procedure, one could calculate the pore water pressure at any other depth in the soil layer. This procedure is much simpler than using Equation (2) for calculating pore water pressure dissipation with time.

Conclusions

Simple analytical solutions are developed for the one dimensional consolidation theory using Galerkin's method. The final solutions developed are of closed-form, enabling one to calculate the average degree of consolidation and the pore water pressure at any depth during the process of consolidation.

References

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