Plane Strain Consolidation Under Shear Loads

by

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Introduction

Present methods of estimating magnitudes and rates of settlement employ the conventional one-dimensional Terzaghi (1923) and pseudo three-dimensional Rendulic (1936) theories of consolidation. Biot (1941) proposed a comprehensive three-dimensional theory of primary consolidation based on poro-elastic approach. However, not many problems of foundation engineering interest have been solved based on Biot's theory.

Foundations may be subjected to horizontal (shear) loads due to friction between footing and the supporting soil or by way of horizontal component of applied inclined loads on footings. The problem of a circular footing subjected to tangential loads was solved by Schiffman and Fungaroli (1965). The problem of planestrain consolidation of a semi-infinite medium subjected to surface tangential loads uniformly distributed over a strip is solved by Babu Shanker et. al. (1973) using Biot's theory. The solutions for the problem of line shear load are obtained in this paper as particular cases of the corresponding solutions of the strip shear problem mentioned above. A detailed discussion on the type of settlements and special porepressure effects under line shear loads is presented.

Governing equations and method of solution

The governing equations of Biot's theory for a saturated soil for plane strain consolidation are :

$$\nabla^2 u_x - (2n'-1)\frac{\partial e_v}{\partial x} - \frac{1}{G'}\frac{\partial u}{\partial x} = 0$$

$$\nabla^2 u_z - (2n'-1)\frac{\partial e_v}{\partial z} - \frac{1}{G'}\frac{\partial u}{\partial z} = 0$$

$$c_v \triangle^2 e_v = \frac{\partial e_v}{\partial t} \qquad \ldots(1)$$

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where u_x, u_z are the displacements in x and z directions, u is the excess pore water pressure, e_v is the volumetric (compressive) strain, c_v is the coefficient of consolidation, t is the time, G' is the effective shear modulus and $n' = \frac{1-\nu'}{1-2\nu'}$, where ν' is the effective Poisson's ratio of the soil (i.e., skeleton), and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$

McNamee and Gibson (1960) defined two displacement functions E and S which are related to the original variables by :

$$u_x = -\frac{\partial E}{\partial x} + z\frac{\partial S}{\partial x}$$

$$u_z = -\frac{\partial E}{\partial z} - S + z\frac{\partial S}{\partial z}$$

$$e_v = \nabla^2 E$$

$$u = 2G'\left[\frac{\partial S}{\partial z} - n'\nabla^2 E\right] \qquad \ldots(2)$$

Thus governing Equations (1) get transformed to

$$c_v \nabla^4 E = \nabla^2\left(\frac{\partial E}{\partial t}\right)$$

$$\nabla^2 S = 0 \qquad \ldots(3)$$

The total stresses (which act at the boundaries) may be related to E and S by

$$\frac{\sigma_{zz}}{2G'} = -\frac{\partial^2 E}{\partial x^2} - z\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}$$

$$\frac{\sigma_{xz}}{2G'} = \frac{\partial^2 E}{\partial x \partial z} - z\frac{\partial^2 S}{\partial x \partial z} \qquad \ldots(4)$$

The solution of Equations (3) in non-dimensional form making use of Laplace Transforms and Fourier Sine Transforms is obtained by Babu Shanker, et. al. (1973) for the problem of plane strain shear where a uniformly distributed unidirectional surface shear load of intensity q acts on a strip of width $2b$. Solutions for line shear load problem (with a surface intensity of q_1) can be obtained as particular cases of the above solutions (after dimensionalising) on using

$$q_1 = \underset{b \to 0}{\text{Limit}}\,(2bq) \qquad \ldots(5)$$

Settlements and porepressures so obtained are given below for both the surface drainage boundary conditions. The boundary value problem under discussion is illustrated in Figure 1.

Solutions for pervious drainage boundary

(*a*) *Settlements*

Settlement or the surface vertical displacement s (i.e., u_z at $z = 0$) is obtained as

$$2G's = \frac{q_1}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha}\left[\frac{1}{2n'-1} - n' e^{-\alpha^2 c_v t} I_6\right] d\alpha \qquad \text{...(6)}$$

where,

$$I_6 = \frac{2}{\pi n'^2}\int_0^\infty e^{-\alpha^2 c_v t \beta^2} \frac{\beta^2}{(\beta^2+1)\left\{\beta^2 + \left(\frac{n'-1}{n'}\right)^2\right\}} d\beta.$$

Immediate settlement s_o is obtained by letting $t \to 0$ in Equation (6) which reduces to zero. Thus immediate settlements are absent under shear loads and hence time-dependent consolidation settlements ($s_c = s - s_o$) are identical to total settlements. Ultimate settlement s_u is obtained by letting $t \to \infty$ in Equation (6) which then reduces to

$$s_u = \frac{q_1}{4G'(2n'-1)} = \frac{q_1(1+\nu')(1-2\nu')}{2E'} \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x = 0 \qquad \text{...(7)}$$

Presence of sine term in Equation (6) suggests that settlement is zero directly under the shear load ($x = 0$) and that the soil mass in the direction of shear load ($x > 0$) settles whereas soil mass on the other side ($x < 0$) heaves up.

For $n' = 1$ (or $\nu' = 0.0$) Equation (6) reduces to

$$2G's = \frac{q_1}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} \, erf(\alpha\sqrt{c_v t})\, dx \qquad \text{...(8)}$$

where, $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2} dx$ is the Error function.

For $n' \to \infty$ ($\nu' \to 0.5$), the settlements are zero. The time-settlement relations obtained for point A (of Figure 1) are shown in Figure 2. It is seen from Figure 2 that larger the n' value (or lesser the ν' value), lesser is the settlement. This is an expected trend since n'-value (or Poisson's ratio) reflects the incompressibility of the material. Progress of settlement (at a given surface point) can best be judged by the degree of consolidation

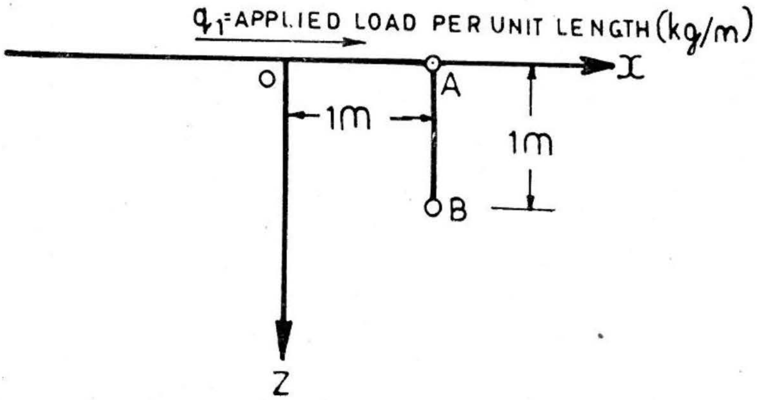

FIGURE 1. Boundary value problem under study

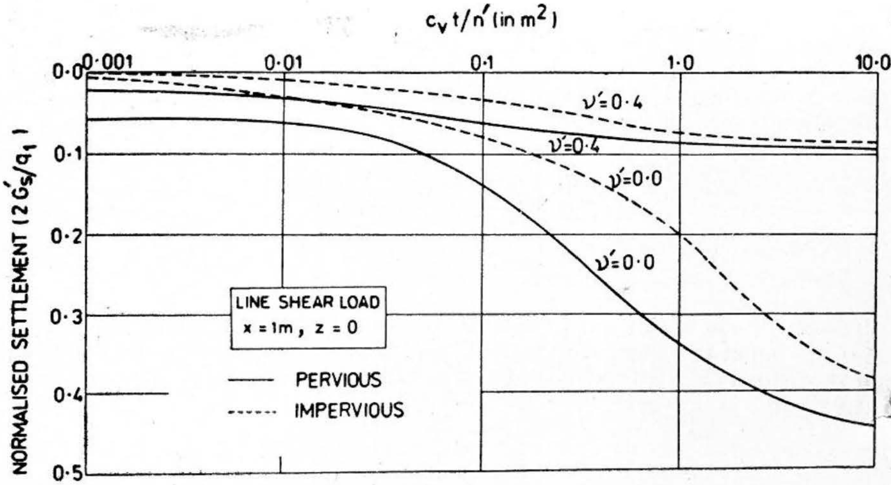

FIGURE 2. Time-settlement relations

settlement (U_s) defined as

$$U_s = \frac{S_{ct}}{S_{cu}} = \frac{\text{consolidation settlement at any time}}{\text{ultimate consolidation settlement}}$$

$$= \frac{S}{S_u}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha x}{\alpha} \, erf\,(\alpha\sqrt{c_v t})\, d\alpha \qquad \ldots(9)$$

for $n' = 1$

The rates of settlement as indicated by U_s (for point A of Figure 1) are shown in Figure 3. Larger the n' value, faster is the progress of settlement.

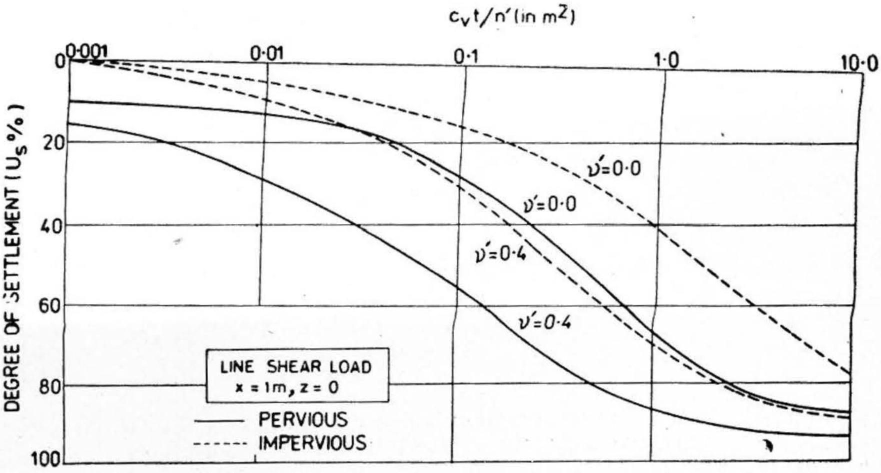

FIGURE 3. Degree of settlement

This is explained by the fact that magnitude of ultimate settlement depends upon the compressibility of the material and hence on the amount of water to be drained. The less the amount of water to be drained, the shorter is the time taken for drainage and hence settlement.

(*b*) *Porepressures*

The porepressure is given by

$$u = \frac{q_1 n'}{\pi} \int_0^\infty \sin \alpha x . e^{-\alpha^2 c_v t} \left[I_4 - \frac{1}{\alpha^2} \frac{d^2 I_4}{dz^2} - e^{-\alpha z} I_6 \right] d\alpha \qquad ...(10)$$

where,

$$I_4 = \frac{2}{\pi n'^2} \int_0^\infty e^{-\alpha^2 c_v t \, \beta^2} \frac{\beta^2 \cos \beta \alpha z + \{n'(\beta^2+1)-1\} \beta \sin \beta \alpha z}{(\beta^2+1)^2 \left\{ \beta^2 + \left(\frac{n'-1}{n'} \right)^2 \right\}} d\beta$$

The initial porepressure u_o is given by letting $t \to 0$ in the above equation which then reduces to

$$u_o = \frac{q_1}{\pi} \int_0^\infty \sin \alpha x \, e^{-\alpha z} \, d\alpha = \frac{q_1}{\pi} \frac{x}{x^2+z^2} \qquad ...(11)$$

Ultimate porepressure is obviously zero at the end of consolidation as $t \to \infty$. Porepressure directly under the centre of the load (at $x = 0$) is zero and it increases beyond that point sinusoidally. Positive porepressures generate in the direction of load application ($x > 0$) and negative pressures in the other direction ($x < 0$). This porepressure distributional pattern is consistent with the settlement pattern discussed earlier.

For $n' = 1$, Equation (10) reduces to

$$u_{n'-1} = \frac{q_1}{\pi}\int_0^\infty \sin \alpha x \, e^{-\alpha z}\left[erfc\left(\alpha\sqrt{c_v t} - \frac{z}{2\sqrt{c_v t}}\right) - erfc(\alpha\sqrt{c_v t})\right] d\alpha \quad ...(12)$$

similarly for $n' \to \infty$, u reduces to

$$u_{n' \to \infty} = \frac{q_1}{\pi}\int_0^\infty \sin \alpha x . \left[e^{-\alpha z}\, erfc\left(\alpha\sqrt{c_v t} - \frac{z}{2\sqrt{c_v t}}\right) - e^{\alpha z}\, erfc\left(\alpha\sqrt{c_v t} + \frac{z}{2\sqrt{c_v t}}\right)\right] d\alpha \quad ...(13)$$

where $erfc(x) = 1 - erf(x)$ = complimentary error function. Equations (12) and (13) can further be reduced in terms of Dawson's Integrals etc. (Babu Shanker, 1974). Figure 4 shows the time-porepressure relations for point B (of Figure 1). The most striking result of Figure 4, is the fact that for $n' = 1$, the porepressure increases with time beyond its initial value (u_o) before it decreases or dissipates. This peculiar porepressure effect is known as "Mandel-Cryer effect" (Mandel, 1950 and Cryer, 1963) and is absent for $n' \to \infty$. It has been shown that for $n' \to \infty$ ($\nu' = 0.5$), the true consolidation theory reduces to pseudo (diffusion) theory (Gibson and Lumb, 1953). And predictions from pseudo theories for porepressure do not show up Mandel-Cryer effect. The possible reasons for manifestation of this porepressure effect are investigated in the following paragraphs.

(c) *Mandel-Cryer Effect*

Schiffman et. al. (1969) investigated the reasons for Mandel-Cryer effect under normal loads. One of the reasons given by them is the fact that if at a given (isolated) point, the total volumetric stress $\left(\sigma_v = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}\right)$

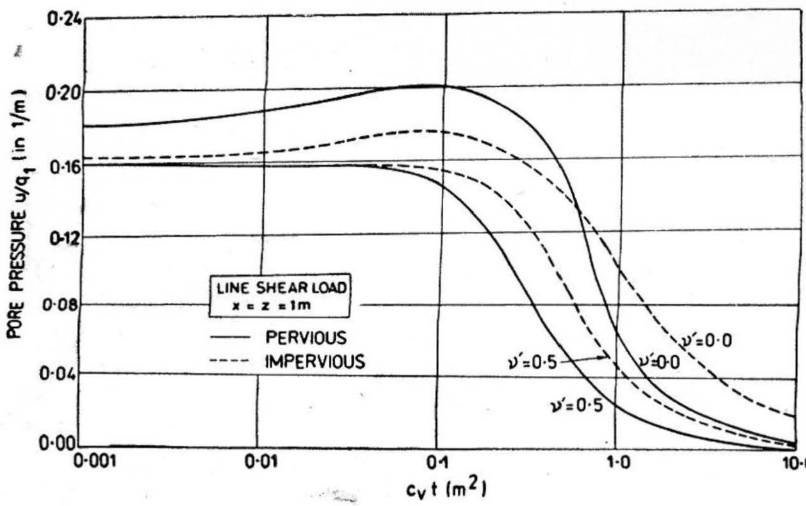

FIGURE 4. Time-porepressure relations

increases with time with no change in effective volumetric stress ($\sigma_v' = \sigma_v - u$) then porepressure should increase locally at such a point. For $n' = 1$, the total volumetric stress (σ_v) is obtained as :

$$\sigma_v = \frac{q_i}{3\pi}\int_0^\infty \sin \alpha x \, e^{-\alpha z}\left[2\text{-}3\, erf\,c(\alpha\sqrt{c_v t})+2\, erf\,c\left(\alpha\sqrt{c_v t}-\frac{z}{2\sqrt{c_v t}}\right)\right] d\alpha \quad \ldots(14)$$

Figure 5 shows the variations of σ_v, σ_v' and u with time. It is observed that during initial period the effective volumetric stress σ_v' essentially remains constant, *whereas the corresponding total stress* σ_v *increases during the* period and this results in temporary porepressure build up. It is seen that the peak points of σ_v and u curves in Figure 5, correspond to almost the same time.

For $n' = 1$, the volumetric strain is given by

$$e_v = \frac{q_1}{2\pi G'}\int_0^\infty \sin \alpha x \, e^{-\alpha z}\left[2-erfc\left(\alpha\sqrt{c_v t}-\frac{z}{2\sqrt{c_v t}}\right)\right] dx \quad \ldots(15)$$

At time $t = 0$, $e_v = 0$ (which in fact is the initial condition used in the solution of basic governing equations). The mobilisation of e_v with time t for point B (of Figure 1) is shown in Figure 6. It is seen that not only at $t = 0$ but also for a long time even after the application of the load, there is no volumetric strain at the point under study. This fact can be attributed

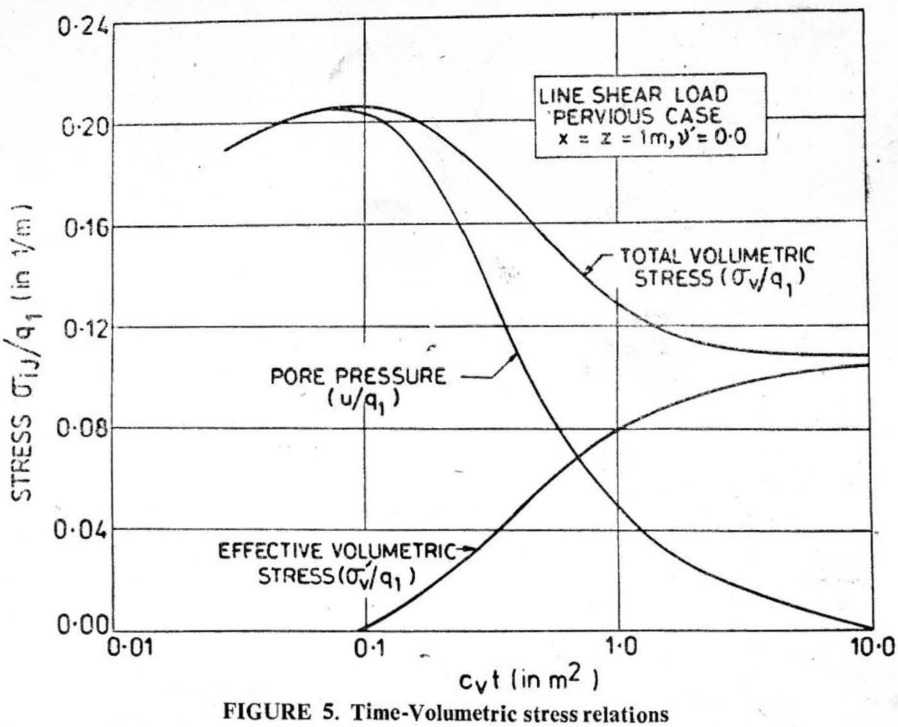

FIGURE 5. Time-Volumetric stress relations

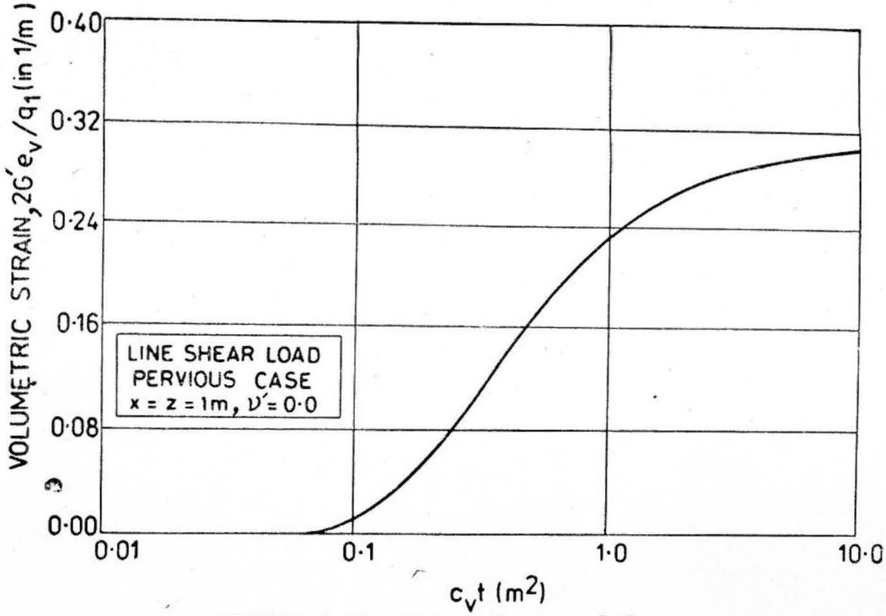

FIGURE 6. Time-Volumetric stress relations

to large lateral strains mobilised during this period, which while allowing vertical strains do not result in volumetric strains. These large lateral strains are in fact responsible for Mandel-Cryer effect. It is seen that e_v curve of Figure 6 is similar to that of σ_v' curve of Figure 5, as it should be.

Consolidation can be looked upon as a process where the elastic properties of soil change. For example, the initial (undrained) value of Poisson's ratio is 0.5 and it decreases to its effective (drained) value ν' at the end of consolidation. Assuming $\nu' = 0.0$ for a soil, the Poisson's ratio with respective to total stresses (ν) can be obtained as (Schiffman, et. al., 1969) :

$$\nu = \frac{u}{3\sigma_v - u} \qquad \ldots(16)$$

Thus from known variations of u and σ_v with time from Figure 5, one can obtain time variation of ν (see Figure 7). The value of ν in this case varies from an initial value of 0.5 to a final value of zero. It is interesting to note that the time corresponding to a change in the initial slope of the curves ν, e_v and σ_v' in Figures 7, 6 and 5 respectively is the same.

In one-dimensional and pseudo three-dimensional theories it is tacitly assumed that the rate of settlement is same as the average rate of porepressure dissipation. However, these two rates need not be the same for a three-dimensional or plane strain situation (Davis and Poulos, 1972). This fact is investigated below.

Unlike surface settlement (which represents the averaging effect of vertical strains throughout the consolidating layer), porepressure is a local variable (and so is volumetric strain). For a given x, let an average porepressure u_{av} be defined as :

$$u_{av} = \int_0^\infty u \, dz \qquad \ldots(17)$$

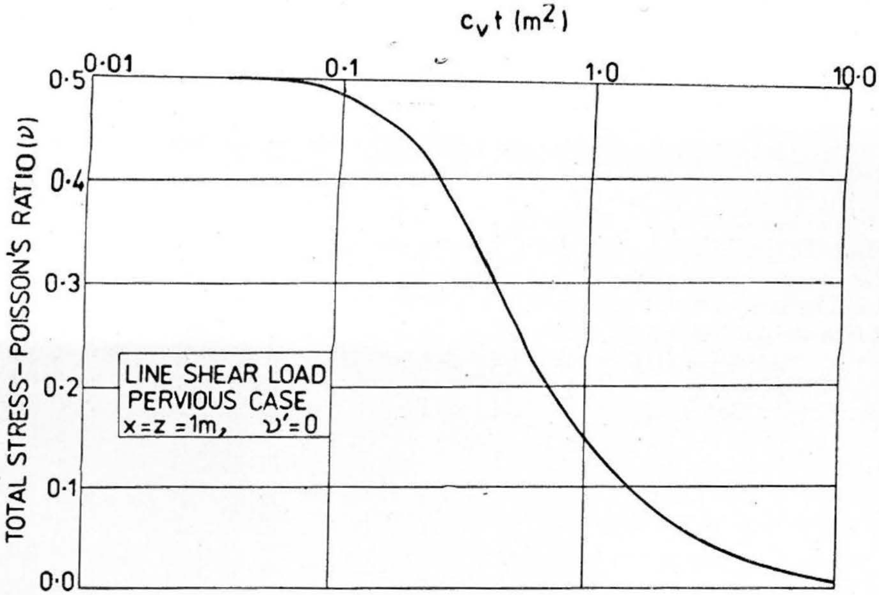

FIGURE 7. Time-total stress Poisson's ratio relation

Substitution of u from Equation (12) into above equation leads to

$$u_{av} = \frac{q_1}{\pi} \int_0^{\infty} \frac{\sin \alpha x}{\alpha} e^{-\alpha^2 c_v t} \, d\alpha \qquad \ldots(18)$$

Degree of porepressure dissipation is defined as

$$U_p = \frac{(u_{av})_{t=0} - (u_{av})_t}{(u_{av})_{t=0}} \qquad \ldots(19)$$

which from Equation (18) reduces to

$$U_p = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha x}{\alpha} (1 - e^{-\alpha^2 c_v t}) \, d\alpha$$

$$= erf\, c(x/2\sqrt{c_v t}) \qquad \ldots(20)$$

Similarly the average volumetric strain e_{av} is defined as

$$e_{av} = \int_0^{\infty} e_v \, dz$$

and Degree of volume change U_v is defined as

$$U_v = (e_{av})_t / (e_{av})_{t \to \infty} \qquad \ldots(21)$$

which from Equation (15) results in

$$U_v = \frac{2}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} \cdot \left[\frac{(1-e^{-\alpha^2 c_v t})+erf(\alpha\sqrt{c_v t})}{2}\right] d\alpha$$

$$= \frac{U_p+U_s}{2} \qquad \ldots(22)$$

Thus the three degrees 'U_s' defined above are different for $n' = 1$ whereas they are assumed to be identical in one-dimensional consolidation and hence are known by one name "the degree of consolidation". From Equation (22) it is seen that the degree of volume change is average of the degrees of settlement and porepressure dissipation. These degrees for point B (of Figure 1) are shown in Figure 8 for comparison. It is interesting to note that degree of settlement is always more than the degree of porepressure dissipation i.e., $U_s > U_v > U_p$. This incidentally explains the Mandel-Cryer effect, since the time lag between rate of progress of settlement and average rate of porepressure dissipation must result in the manifestation of increase in porepressure at some local points.

Solutions for impervious boundary

The different variables of interest for this case are summarised below :

$$s = \frac{q_1}{2G'\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha}\left[\frac{1}{2n'-1} - n' e^{-\alpha^2 c_v t} I_3 - \frac{2n\sqrt{\delta}\, e^{-(1-\delta)\,\alpha^2 c_v t}}{(n'+1)^2-(2-n'\delta)^2}\right] d\alpha$$

$$u = \frac{q_1 n'}{\pi}\int_0^\infty \sin \alpha x\, e^{-\alpha^2 c_v t}\left[I_1 - \frac{1}{\alpha^2}\frac{d^2 I_1}{dz^2} - e^{-\alpha z} I_3 + \frac{2e^{\delta\alpha^2 c_v t}\{e^{-\alpha z\sqrt{\delta}} - \sqrt{\delta}\, e^{-\alpha z}\}}{(n'+1)^2-(2-n'\delta)^2}\right] d\alpha$$

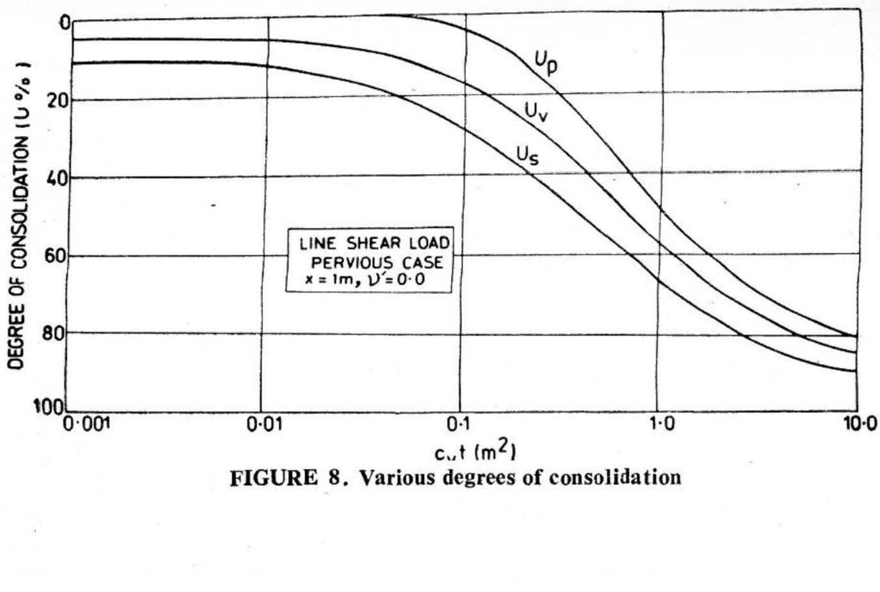

FIGURE 8. Various degrees of consolidation

$$= \frac{q_1 x}{\pi(x^2+z^2)}\left[1-e^{-(x^2+z^2)/4c_v t}\right]$$

where,

$$I_1 = \frac{2}{\pi n'^2}\int_0^\infty e^{-\alpha^2/\beta^2 c_v}\left[\frac{\{1+n'(\beta^2+1)\}\,\beta^2\cos\alpha\beta z-\beta\sin\alpha\beta z}{(\beta^2+1)^2\,(\beta^2+\delta)\left(\beta^2+\frac{1}{n'^2\delta}\right)}\right]d\beta$$

$$I_3 = \frac{2}{\pi n'^2}\int_0^\infty e^{-\alpha^2/\beta^2 c_v}\frac{\beta^2}{(\beta^2+1)\,(\beta^2+\delta)\left(\beta^2+\frac{1}{n'^2\delta}\right)}\,d\beta$$

and $\delta = \frac{(n'+2)-\sqrt{(n'^2+4n')}}{2n'}$

It is seen from these expressions that immediate settlement (s_o), ultimate settlement (s_u) and initial porepressure (u_o) are all independent of surface drainage condition. Settlement and porepressure distributional patterns are also similar to those of pervious drainage boundary case. Mandel-Cryer effect is seen in porepressure (for $n' = 1.0$) but the intensity of this effect is not felt as severely as in the other case. Time-settlement and time-porepressure relations for this case are shown along with the other case also in Figures 2, 3 and 4.

Conclusions

Some of the conclusions drawn from the present analysis are as follows :

(1) Immediate settlements are absent under shear loads.

(2) Settlements directly under the centre of the unidirectional shear loads are absent. Positive (downward) settlements occur in the direction of load while negative settlements (heaving) occur in the opposite direction. Settlements are totally absent for a medium with $\nu' = 0.5$.

(3) Pattern of porepressure distribution is consistent with the settlement pattern.

(4) Biot's theoretical predictions show up Mandel-Cryer effect (for $\nu' < 0.5$). Increasing total stresses under constant effective stresses is one of the reasons for this effect. The degree of settlement is always more than the degree of porepressure dissipation and degree of volume change is the average of the above two degrees. These differential rates are primarily responsible for Mandel-Cryer effect in soils.

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Notation

The following symbols are used in this paper :

b = half width of strip

c_v = coefficient of consolidation

E = displacement function

E' = effective value of Young's modulus

e_v = volumetric strain

e_{av} = average volumetric strain

$erf(x)$ = error function

$erfc(x)$ = complementary error function

G' = effective shear modulus

n' = auxiliary elastic constant

I_1, I_3, I_4, I_6 = Integral expressions

q = intensity of load per unit area

q_1 = intensity of load per unit length

S = displacement function

s = total (surface) settlement
s_c, s_{ct} = consolidation settlement
s_{cu} = ultimate consolidation settlement
s_o = immediate settlement
s_u = ultimate (total) settlement
t = time
U_p = degree of porepressure dissipation
U_s = degree of settlement
U_v = degree of volumetric strain mobilisation
u = excess porepressure
u_{av} = average excess porepressure
u_x = displacement in x-direction (horizontal)
u_z = displacement in z-direction (vertical)
x = space coordinate (horizontal)
z = space coordinate (vertical)
$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}$ = Parameters of integration (variables)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

δ = a function of n'
ν = total (stress) Poisson's ratio
ν' = effective (stress) Poisson's ratio
σ_v = total volumetric stress = $\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$
σ_v' = effective volumetric stress
σ_{xz} = shear stress
σ_{xx}, σ_{yy}, σ_{zz} = normal stresses