Slope Stability Analysis By Variational Method

by

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Introduction

Stability analysis of slopes has attracted considerable attention of soil engineers during the past six decades due to its techno-economic importance in the construction of earth dams, road and railway embankments, levees and in the investigation of land slides. The earliest work on stability analysis of slopes was carried out by Coulomb (1773), Francis (1820), and Collin (1846), but, significant contributions in this field were largely due to the classical methods developed by Swedish engineers during the period from 1915 to 1925. Swedish slip-circle method of slices developed by Fellenius (1927, 1936) has been the most widely used conventional technique for many practical problems. Among other significant contributions in this field are works of Taylor (1948), Sokolovsky (1950), Janbu (1954), Bishop (1955), Morgenstern and Price (1965), Chugaev (1966), and Spencer (1967, 1968, 1969, 1973).

Bishop's (1955) slip circle analysis formed the basis for further research in this area. This method was rigorous in its content which satisfied both force and moment equilibrium conditions and also considered the presence of inter-slice forces. The expression for factor of safety obtained by this method is rather lengthy and involves very tedious numerical computation. To circumvent this difficulty, Bishop simplified the original expression for the factor of safety by assuming the inter-slice force to be horizontal and obtained the solution by successive approximations. The latter method came to be known in the literature as Bishop's routine method of analysis. Bishop's simplified expression satisfied only moment equilibrium condition but not force equilibrium condition. The minimum factor of safety obtained by this method was a very close approximation to the final value obtained using the rigorous method. Thus, the routine analysis presented by Bishop being the first stage of a more complete iterative process does not satisfy statical equilibrium condition and also suggests that for circular slip surfaces, the factor of safety is relatively insensitive to the distribution of the internal forces. The analysis does not seem to justify that the expression obtained for the factor of safety which does not satisfy one of the basic conditions of equilibrium yields the solution corresponding to critical

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equilibrium state. Morgenstern and Price (1965) suggested a method of analysing a stability of general slip surfaces, satisfying both force and moment equilibrium conditions, and could consider slope sections with varying shear strength parameters and pore-pressures. The analysis was based on the principles of limiting equilibrium and needed *a priori* assumptions of the shape of the potential sliding mass as well as the distribution of internal forces. This method and the circular-slip analysis suggested by Bishop gave approximately the same factor of safety which appeared to be insensitive for varying distribution of internal forces within the potential sliding mass. Spencer (1967, 1968, 1969) presented an alternative method of analysis for circular and logarithmic spiral slip surfaces, based on Bishop's earlier work. The analysis was carried out by assuming parallel inter-slice forces to pass through the centre of the inter-slice base. It also studied the influence of tension crack on factor of safety. In the recent paper (Spencer 1973) the shape of the slip surface was assumed to be of some general form and the inter-slice forces were not necessarily to be parallel. However, it was observed that a reasonably reliable value for the minimum factor of safety can be obtained by assuming the inter-slice forces to be parallel.

A detail study of earlier works on the stability analysis brings out some of the shortcomings that one encounters during the analysis of slope. None of the analyses mentioned above seem to illustrate the exact factor of safety or the critical failure surface. This may be due to, the problem being statically inderminate. The indeterminancy arises from the lack of knowledge of the stresses present in the soil mass. In order to render the problem statically determinate, assumptions have to be made regarding the internal stress distribution within the potential sliding mass for which consideration of soil properties and the forces acting over the potential sliding mass becomes imperative. The necessity of considering body forces, pore-pressures, and a wide variety of soil types in the stability analysis invalidates the application of methods in the mechanics of continua; as a result of this, limit equilibrium methods are commonly employed. These methods require an assumption to be made regarding the potential slip surface. More commonly a circular potential slip surface is assumed. The main advantage of assuming a circular slip surface is that the direction of normal force acting on the slip surface is directed towards the centre of the circle. As a result of this, the moments of these forces with respect to the centre of the slip circle do not exist. However, assumption of circular slip surface, although justified on the grounds that the analysis is made simpler, lacks physical validity. In the field observations, more non-circular slip surfaces have been observed than the circular surfaces (Cooling and Golder 1942, Hutchinson 1961, Skempton 1961, Legget 1962).

The circular slip surface analyses are generally acceptable for practical problems as an approximate solution in stability analysis. The analysis does not seem to justify that the surface obtained by this method leads to an absolute critical surface or the minimum factor of safety obtained is absolute minimum. Analysis with ill-conditioned assumptions lead to misleading results.

Review of existing analyses based on rupture considerations suggest that a more rigorous method of analysis is necessary so that it would circumvent the shortcomings attendant in the existing methods. By posing the problem more appropriately a better mathematical technique can be developed for a rigorous slope stability analysis. The rigorous method developed must be able to consider the requirements necessary to encompass more general problems, if it is to have other than restricted usage. The method should also be able to consider a wide variety of soil properties varying over a large range; it must also make an allowance for complex pore-pressure distributions; in other words, the analysis must be in terms of effective stresses.

An attempt is made in this paper to provide a technique for stability analysis of slop taking into consideration the requirements of a rigorous analysis.

The problem of analysing the stability of slope is posed as a minimization problem in the calculus of variations (Goldstein 1969), wherein, the stress distribution function has to be determined so as to minimize the factor of safety satisfying all equilibrium and boundary conditions, and also that for a slope section to be stable, the basic Coulomb-Mohr failure criterion must not be violated anywhere along the slip surface. Thus, the variational method of analysis not only offers a sound technique for solving a minimization problem of this type, but also provides physical insight into the problem itself.

The Equations of Equilibrium

General

Figure 1 (*a*) shows a section through an embankment with a general slip surface. The problem is to investigate equilibrium of the potential sliding

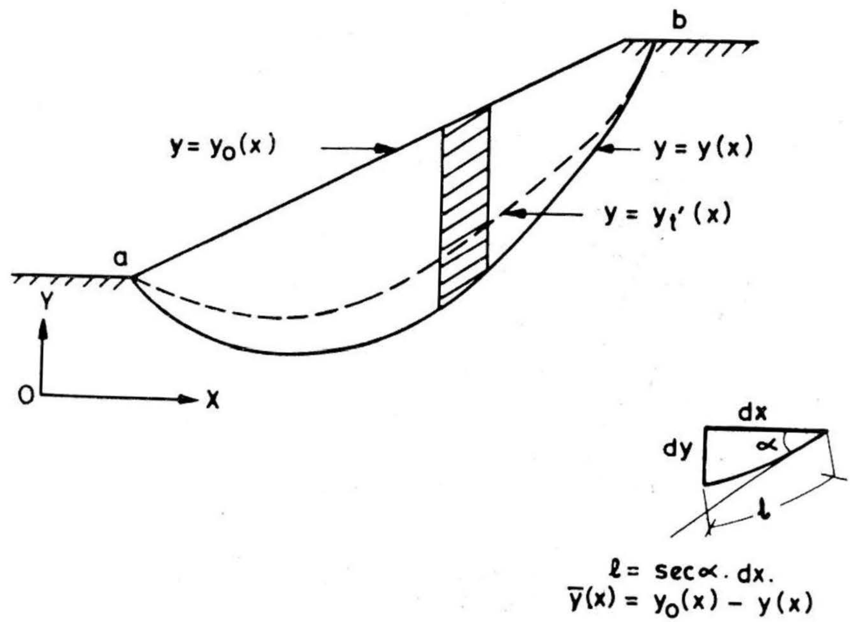

FIGURE 1(*a*). Potential sliding mass

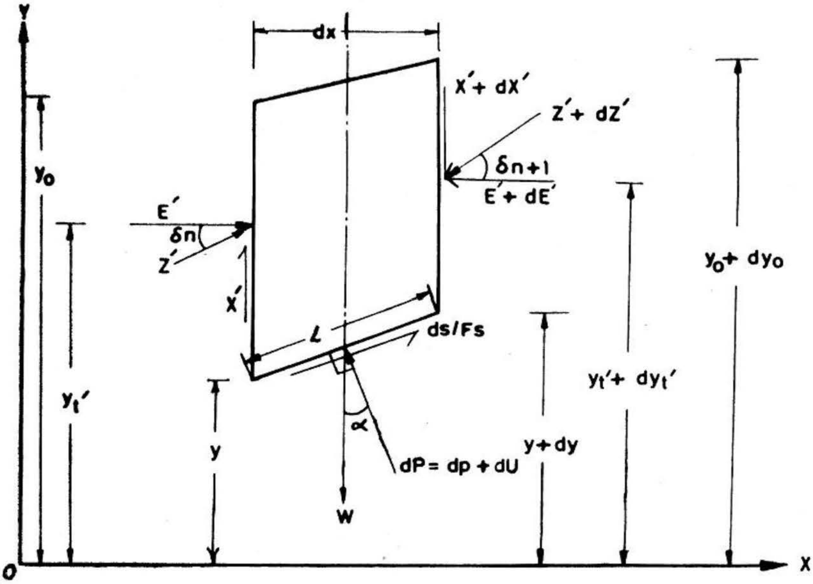

FIGURE 1(*b*). Forces acting on an infinitesimal slice

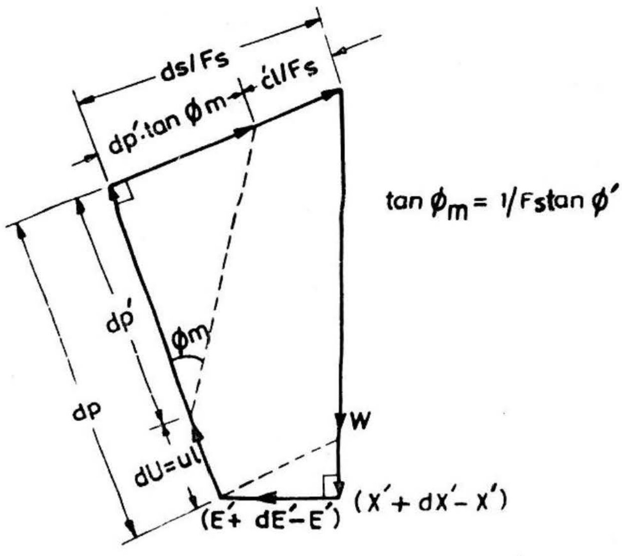

FIGURE 1(*c*). Force polygon of the forces mass of this slope section. In this figure, the equations of the general slip surface, surface of the slope, and the line of action of the thrust line are given by $y=y(x)$, $y=y_o(x)$ and $y=y_t(x)$ respectively and $y=y_t'(x)$ represents the equation of the effective thrust line. The forces acting on an infinitesimal slice of mass are shown in Figure. 1 (*b*) and the force polygon is shown in Figure 1(*c*).

Shear strength

The factor of safety with respect to shear strength has been adopted in this analysis. The factor of safety is defined as that value by which the shear strength parameters must be reduced in order to bring the potential sliding mass into a state of limiting equilibrium. Based on Coulomb-Mohr failure criterion the shear strength mobilized along the base of the slice in terms of effective stresses is expressed as :

$$dS_m = \frac{dS}{F_s} = \frac{1}{F_s} \; [c' \sec \alpha \; dx + dp' \tan \phi'] \qquad \ldots(1)$$

While estimating the shear strength mobilized and shear strength causing slide (in terms of effective stresses) along the base of the slice it should be borne in mind that this force contains a hydrostatic component which acts with equal intensity in all directions, and hence only the effective stress $(\gamma\bar{y}-u)$ should be resolved in a direction normal and tangential to the failure surface; otherwise the equation yields unreasonable results (Turnbull and Hvorslev 1967). This aspect can be better explained with the help of the specified example in consideration. Neglecting the terms involving the inter-slice force (however, inter-slice force terms are considered in the analysis), the total normal stress acting on the base of slice is given by

$$\sigma_n = (\gamma\bar{y}-u) \cos^2 \alpha + u \qquad \ldots \; (2)$$

which yields the effective stress

$$\sigma'_n = (\gamma\bar{y}-u) \cos^2\alpha \qquad \ldots \; (2a)$$

The mean pore-pressure on the base of the slice can be written :

$$u = r_u \gamma\bar{y} \qquad \ldots \; (3)$$

Where r_u is a pore-pressure co-efficient, as defined by Bishop and Morgenstern (1960).

The weight of infinitesimal slice of width dx is given by

$$dW = \gamma\bar{y}dx \qquad \ldots \; (3a)$$

Now considering the presence of inter-slice force and using Equation (2) and (3) the expression for mobilised shear strength can be written as :

$$dS_m = \frac{dS}{F_s} = \frac{1}{F_s} [c' \sec \alpha \; dx + \{\gamma\bar{y} \; (1-r_u) \cos \alpha \; dx - dE' \sin \alpha + dX' \cos \alpha\} \tan \phi'] \qquad \ldots \; (4)$$

Similarly the effective tangential component, dW_T, of the weight dW can be shown to be

$$dW_T = \gamma\bar{y} \; (1-r_u) \sin \alpha \; dx + (dE' + dX'y') \cos \alpha \qquad \ldots \; (4a)$$

Inter-slice forces

In general, the problem is statically indeterminate involving the unknown functions E', X' and y_t'. To render the problem statically determinate, an assumption is made regarding the relation between the E' and X' forces. According to Morgenstern and Price (1965) by isolating an element at the interface of the slice, for a specific geometry and slip surface, the internal forces are determined by,

$$E' = \int_{y_o}^{y} \sigma'_x (y)\, dy \qquad \dots (5)$$

and,

$$X' = \int_{y_o}^{y} \tau_{xy} (y)\, dy \qquad \dots (6)$$

we may therefore assume that,

$$X' = \tan \delta \; E' \qquad \dots (7)$$

where,

$$\tan \delta = f(x) \tan\theta \qquad \dots (7a)$$

By specifying the function $f(x)$ the problem is rendered statically determinate. Satisfying Coulomb-Mohr failure criterion along the vertical inter-slice boundary and resolving the forces shown in Figure. 2 normal and parallel to the vertical plane, an expression for the horizontal effective thrust, E', is obtained in terms of soil parameters and the average factor of safety, F_v, as :

$$E' = (b_1\bar{y} + b_4\bar{y}^2)\beta_1 \qquad \dots (8)$$

where, $$b_1 = c' \qquad \dots (8a)$$

$$b_4 = -\tfrac{1}{2}\, r_u \gamma \tan\phi' \qquad \dots (8b)$$

and, $$\beta_1 = (F_v f(x) \tan\theta - \tan\phi')^{-1} \qquad \dots (8c)$$

Using Equations (7) and (8) it can be written that :

$$X' = \tan\theta\, f(x)\; (b_1\bar{y} + b_4\bar{y}^2)\beta_1 \qquad \dots (9)$$

Similarly differentiating Equatiohs (8) and (9) it can be shown that :

$$dE' = [(b_1\bar{y} + b_4\bar{y}^2)\beta'_1 + (b_1\bar{y}' + 2b_4\bar{y}\bar{y}')\beta_1]dx \qquad \dots (10)$$

and, $$dX' = [\tan\theta\, \{\beta_2\, (b_1\bar{y} + b_4\bar{y}^2) + \beta_3\, (b_1\bar{y}' + 2b_4\bar{y}\bar{y}')\}]\, dx \qquad \dots (11)$$

where, $$\beta_2 = f'(x)\beta_1 + \beta'_1 f(x) \qquad \dots (11a)$$

and, $$\beta_3 = \beta_1 f(x) \qquad \dots (11b)$$

Expression for over all factor of safety F_s

Using Equations (10) and (11), and resolving the forces shown in Figure 1(*b*) normal and parallel to the base of the slice, an expression for

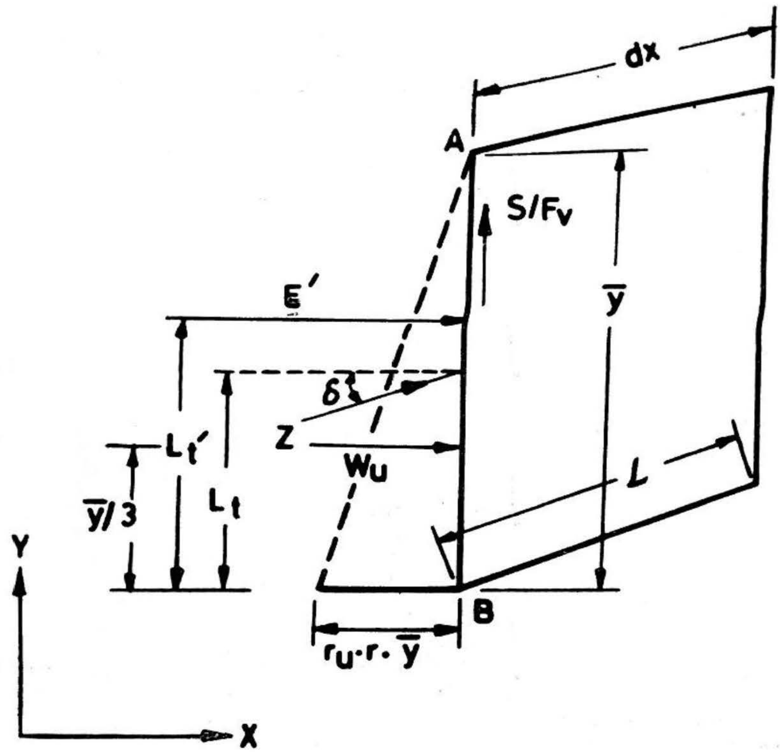

FIGURE 2. Forces acting on an inter-slice boundary AB

the overall factor of safety, F_s, in terms of effective stress is obtained:

$$F_s = \frac{\int_a^b [c' (1+y'^2)+\{\gamma\bar{y}(1-r_u)-y' (\beta_1'\lambda_1+\lambda_2\beta_1) + \tan\theta \ (\beta_2\lambda_1+\beta_3\lambda_2)\} \tan\phi']dx}{\int_a^b [\gamma(1-r_u)\bar{y}y'+\beta_1'\lambda_1+\beta_1\lambda_2+y' \tan\theta \ (\beta_2\lambda_1+\beta_3\lambda_2)]dx} \quad \ldots (12)$$

where, $\lambda_1 = b_1\bar{y}+b_4\bar{y}^2$

$\lambda_2 = b_1\bar{y}+2b_4\bar{y}\bar{y}'$

Transposing Equation (12), an expression for, E', can be obtained in the form :

$$E' = \int_a^b \left(\frac{c' (1+y'^2)+\gamma (1-r_u) \tan\phi'\bar{y} - F_s\gamma (1-r_u)\bar{y}y'}{(y'-f(x) \tan\theta) \tan\phi'+F_s (1+f(x) \tan\theta \ y')} \right) dx \quad \ldots (12a)$$

Force equilibrium condition

Now if the external forces on the potential sliding mass are in equilibrium, the vectorial sum of the inter-slice forces must be zero, in other words, the sum of the horizontal and vertical components of the inter-slice forces must also be zero, i.e.

$$\int_a^b dE' = 0 \qquad \dots (13)$$

and,

$$\int_a^b dX' = 0 \qquad \dots (13a)$$

or,

$$\int_a^b \{\lambda_1\beta_1' + \lambda_1\beta_1\}\, dx = 0 \qquad \dots (14)$$

and,

$$\int_a^b \tan\theta\, (\beta_2\lambda_1 + \beta_3\lambda_2)\, dx = 0 \qquad \dots (14a)$$

Moment equilibrium condition

The moment equilibrium condition remains to be satisfied. The moment equilibrium condition of the potential sliding mass is satisfied by taking moments of the inter-slice forces about the centre of the base of the slice and is equated to zero. After simplifying and proceeding to the limit as $dx \to 0$ it can be shown that :

$$\tan\theta\, f(x)\lambda_1\beta_1 - \left\{\frac{d}{dx}(\lambda_1\beta_1\, y_t') - y\,\frac{d}{dx}(\lambda_1\beta_1)\right\} = 0 \qquad \dots (15)$$

By integrating the Equation (15) we have :

$$M_x = (y_t' - y)\, E' = \int_a^x \lambda_1\beta_1\, [f(x)\tan\theta - y']\, dx \qquad \dots (16)$$

and the equation for no rotation requires that :

$$M_b = 0 = \int_a^b \{\lambda_1\beta_1\, (f(x)\tan\theta - y')\}\, dx \qquad \dots (17)$$

After satisfying Equation (17), values of y_t' can be found from Equation (16) thereby satisfying the moment equilibrium condition at each point along the potential sliding surface.

Average factors of safety and slope of inter-slice forces

The values of overall factor of safety, F_s, average factor of safety, F_v, along the vertical plane, and the slope $\tan\theta$ of the inter-slice forces are determined by satisfying force and moment equilibrium conditions together with the boundary conditions.

The factor of safety, F_s, given by Equation (12) need not necessarily refer to absolute minimum. To determine the minimum factor of safety and the corresponding critical slip surface conditions for minimality have to be satisfied. These conditions are obtained by using variational techniques and the method of obtaining these conditions are explained in the following section.

Conditions for Minimality

Functional $J[y(x)]$

The analysis carried out in the preceding section shows that the problem of stability analysis of slope is essentially a problem of finding the slip surface $y(x)$ which minimizes the functional

$$F_s = J[y(x)] = \frac{[c'(1+y'^2)+\{\gamma\bar{y}(1-r_u)-y'(\beta'_1\lambda_1+\lambda_2\beta_1)}{[\gamma(1-r_u)\bar{y}y'+\beta_1'\lambda_1+\beta_1\lambda_2+y'\tan\theta} \frac{+\tan\theta(\beta_2\lambda_1+\beta_3\lambda_2)\}\tan\phi']}{(\beta_2\lambda_1+\beta_3\lambda_2)]\,dx} \quad \text{... (18)}$$

subject to the boundary conditions $y(a) = Y_a$ and $y(b) = Y_b$. The functional $J[y(x)]$ can be more succintly written as :

$$J[y(x)] = \frac{\int_a^b g_1(x,y,y')dx}{\int_a^b g_2(x,y,y')\,dx} \quad \text{... (19)}$$

where g_1 and g_2 are the known functions of their arguments.

Methods of minimization

Two methods viz., direct and indirect methods have been employed to minimize the function.

Indirect method : The functional (Equation 19) is a specific form of more general functionals of the nature,

$$J[y(x)] = \int_a^b L(x, y, y', k_a(x;y))\,dx \quad \text{... (20)}$$

$$\text{where, } k_a(x;y) = \int_a^b g_a(x, z, y, y')\,dz \quad \text{... (21)}$$

It is observed that minimization problem of the functional defined by (Equation 20) and (Equation, 21) cannot be tackled using the usual methods of classical calculus of variations, since the Lagrangian function contains integrals as well as derivatives as arguments. Functionals where Lagrangian function includes integrals as well as derivatives as arguments are known as the *nonlocal functionals*. Solution of these functionals leads to the nonlocal calculus of variations (Edelen 1967) with Euler equations as integro-differential equations, this is in contrast to the classical calculus of variations, where Euler equations are just differential equations. The necessary and sufficient conditions for extremality of such functionals have been given by Bhatkar (1972). Using these results, the Euler equation for Equation (20) and Equation (21) becomes a second order integrodifferential equation of the form :

$$\frac{\partial L}{\partial y}-\frac{d}{dx}\left(\frac{\partial L}{\partial y}\right)+\int_a^b \sum_{a=1}^{q} \frac{\partial L}{\partial k_a}(z)\left\{\frac{\partial g_a^*}{\partial y}-\frac{d}{dx}\frac{g_a^*}{\partial y'}\right\}dz \quad \dots (22)$$

where,

$$g_a^* = g_a\,[z, x, y\,(x), y'\,(x)] \quad \dots (22a)$$

For the specific case under consideration, namely the minimization problem of the functional defined by Equation (18), the Euler-Lagrange Equation (22) becomes a separable system of integral and differential equations as,

$$F_s = \frac{\int_a^b \{p_1+p_2\,y+p_3\,y^2+p_4y''+p_5yy'+p_6y^2y'+p_7y'^2+p_8yy'^2\}\,dx}{\int_a^b \{p_9+p_{10}y+p_{11}y^2+p_{12}y''+p_{13}yy'+p_{14}y^2y'+p_{15}y'^2+p_{16}yy'^2\}dx} \quad \dots (23)$$

$$(q_1+q_2y+q_3y^2+q_4y'+q_5yy'+q_6y'^2+q_7y''+q_8yy'')$$
$$-F_s(q_9+q_{10}y+q_{11}y^2+q_{12}y'+q_{13}yy'+q_{14}y'^2+q_{15}y''+q_{16}yy'') = 0 \quad \dots (24)$$

where,

$$p_1 = c'+\gamma\,(1-r_u)y_0\,\tan\phi'+(b_1+b_4y_0)\beta_2y_0\,\tan\theta\,\tan\phi'+(b_1+2b_4y_0)\beta_3y_0'\,\tan\theta\,\tan\phi'$$

$$p_2 = -\gamma\,(1-r_u)\,\tan\phi'-(b_1+2b_4y_0)\beta_2\,\tan\theta\,\tan\phi'-2b_4\beta_3y_0'\,\tan\theta\,\tan\phi'$$

$$p_3 = b_4\beta_2\,\tan\theta\,\tan\phi'$$

$$p_4 = -\beta_1'y_0\,(b_1+y_0b_4)\,\tan\phi'-\beta_1y_0'\,(b_1+2b_4y_0)\,\tan\phi'-\beta_3(b_1+2b_4y_0)\,\tan\theta\,\tan\phi'$$

$$p_5 = (b_1+2b_4y_0)\beta_1'\,\tan\phi'+(b_1+2b_4y_0')\beta_1\,\tan\phi'+2b_4\beta_3\,\tan\theta\,\tan\phi'$$

$$p_6 = -b_4\beta_1'\,\tan\phi'$$

$$p_7 = c'+\beta_1\,(b_1+2b_4y_o)\,\tan\phi'$$

$$p_8 = -2b_4\beta_1\,\tan\phi'$$

$$p_9 = \beta_1'y_0\,(b_1+b_4y_0)+\beta_1y_o'\,(b_1+2b_4y_0)$$

$$p_{10} = -(b_1+2b_4y_0)\beta_1'-2b_4\beta_1y_0'$$

$$p_{11} = \beta_1' b_4$$
$$p_{12} = (1-r_u)\gamma y_0 - \beta_1(b_1+2b_4y_0)+(b_1+y_0b_4)y_0\beta_2 \tan\theta+(b_1+2b_4y_0)\beta_3 y_0' \tan\theta$$
$$p_{13} = -(1-r_u)\gamma+2b_4\beta_1-\beta_2(b_1+2b_4y_0)\tan\theta-2b_4y'_0\beta_3\tan\theta$$
$$p_{14} = \beta_2 b_4 \tan\theta+2b_4\beta_3\tan\theta$$
$$p_{15} = -(b_1+2b_4y_0)\beta_3\tan\theta$$
$$p_{16} = 2b_4\beta_3\tan\theta$$
$$q_1 = p_2-p_4'$$
$$q_2 = 2p_3-p_5'$$
$$q_3 = -p_6'$$
$$q_4 = -2p_7'$$
$$q_5 = -2p_8'$$
$$q_6 = -p_8$$
$$q_7 = -2p_7$$
$$q_8 = -2p_8$$
$$q_9 = -p_{10}-p'_{12}$$
$$q_{10} = 2p_{11}-p'_{13}$$
$$q_{11} = -p'_{14}$$
$$q_{12} = -2p'_{15}$$
$$q_{13} = -2p'_{16}$$
$$q_{14} = -p_{16}$$
$$q_{15} = -2p_{15}$$
$$q_{16} = -2p_{16}$$

If now an assumption is made regarding the existence of unique critical slip surface, the Euler equation is both a necessary and sufficient, condition for the absolute minimum of the factor of safety given by Equation (18). Such an assumption is always justified from the physical point of view. Thus the minimization problem is reduced to the problem of finding the solution of the two point boundary-value problem (TPBVP) defined by Equation (23), Equation (24) and the boundary conditions $y_a = Y_A$ and $y_b = Y_B$. The TPBVP can be solved using any one of the numerical methods available in the literature (Sage 1968). Here Warner's method (Fox 1962) has been used for carrying out the numerical computation.

Direct method : Alternatively, the minimization of the functional $J[y(x)]$ defined by Equation (18) can be carried out, using the direct methods such as the Raleigh-Ritz technique (Courant and Hilbert 1966) or the method of local variations (Chernovs'ko 1965). In this paper only Raleigh-Ritz technique has been employed to the analysis. In this technique, a trial solution for $y(x)$ is assumed whose functional dependence on x is chosen, but which includes undetermined constants. The later are found using the standard minimisation technique in the form of partial differentiation. For the problem under consideration, a trial function for the slip surface can be assumed of the form

$$\tilde{y}(x) = \sum_{k=0}^{N} d_k x^k \qquad \ldots (25)$$

which satisfies the specified boundary condition, exactly. The substitution of Equation (25) in Equation (18), after carrying out the integration, yields

$$J[\tilde{y}(x)] = f_a(d_a, \ldots, d_N) \qquad \ldots (26)$$

where f_a is a known function of its parameters. The minimization of f_a can now be carried out using the method of partial differentiation. This yields $(N+1)$ nonliner algebraic equation

$$\frac{\partial f_a}{\partial d_k} = 0, \; k = 0, \ldots, N \qquad \ldots (27)$$

which can be solved using any one of the standard numerical methods of solution. In the present case, a polynomial of the fourth degree was assumed as a trial solution. For the slip surface Warner's method (Fox 1962) was employed for solving Equation (27).

Results and Discussion

General

The stabilily analysis by the variational technique described in this paper has been illustrated by analysing the stability of slope section shown in Figure. 3. This section has also been analysed by Spencer (1967) using

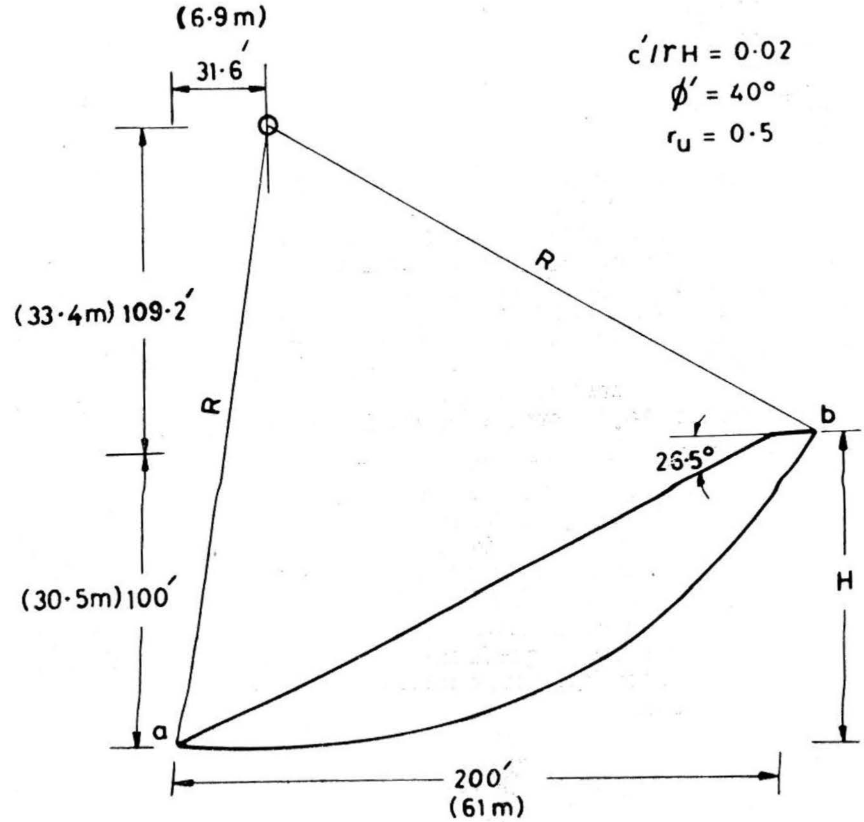

FIGURE 3. Slope section of the illustrated example

the slip-circle analysis, the numerical results obtained by the variational method are compared with those obtained by Spencer.

The soil properties of the slope section considered here are same as those considered by Spencer. The slope section is homogeneous and has a constant pore-pressure coefficient. It is assumed by Spencer as well as in the present analysis that the tension crack does not develop at the top of slope section. The values of the soil properties, viz., the effective angle of shearing resistance, ϕ', stability ratio, $c'/\gamma H$, the pore-pressure coefficient, r_u, and the slop section have been indicated in Figure. 3. The specific function $f(x)$ relating the inter-slice forces E' and X', in terms of effective stresses, is shown in Figure. 4. The analysis has been carried out by direct and indirect methods in calculus of variations in terms of effective stresses.

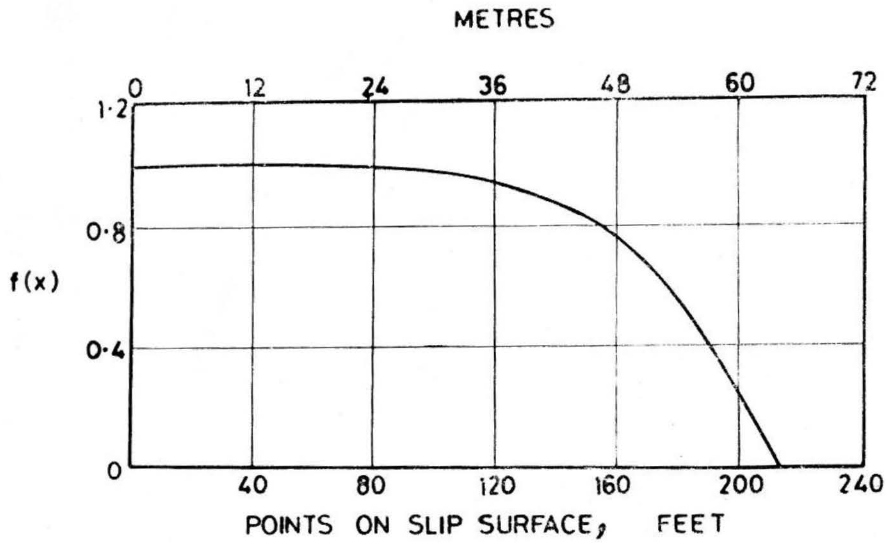

FIGURE 4. Function $f(x)$ showing the relation between E' and X' forces

Slip surface

Figure 5 show the critical slip surfaces obtained by slip-circle analysis, and also by direct and indirect variational methods of analysis for pore-pressure coefficient, r_u=0.5. Curve 1 is the critical slip-circle and curves 2, 3 are the critical slip surfaces obtained by indirect and direct methods respectively. The critical slip surfaces obtained by indirect and direct variational methods lie very close to each other, but, significantly, deviates from the critical slip-circle. The shape of the slip surface obtained by variational method could be very closely compared to the shape of a catenary with the boundaries as the end points, having its apex near the lower end and flat curvature towards the upper end. It is interesting to note that the shape of the critical slip surface obtained by the variational method closely resembles the actual slip surface as observed in the field for a homogeneous embankment (Wolfskill and Lambe 1967).

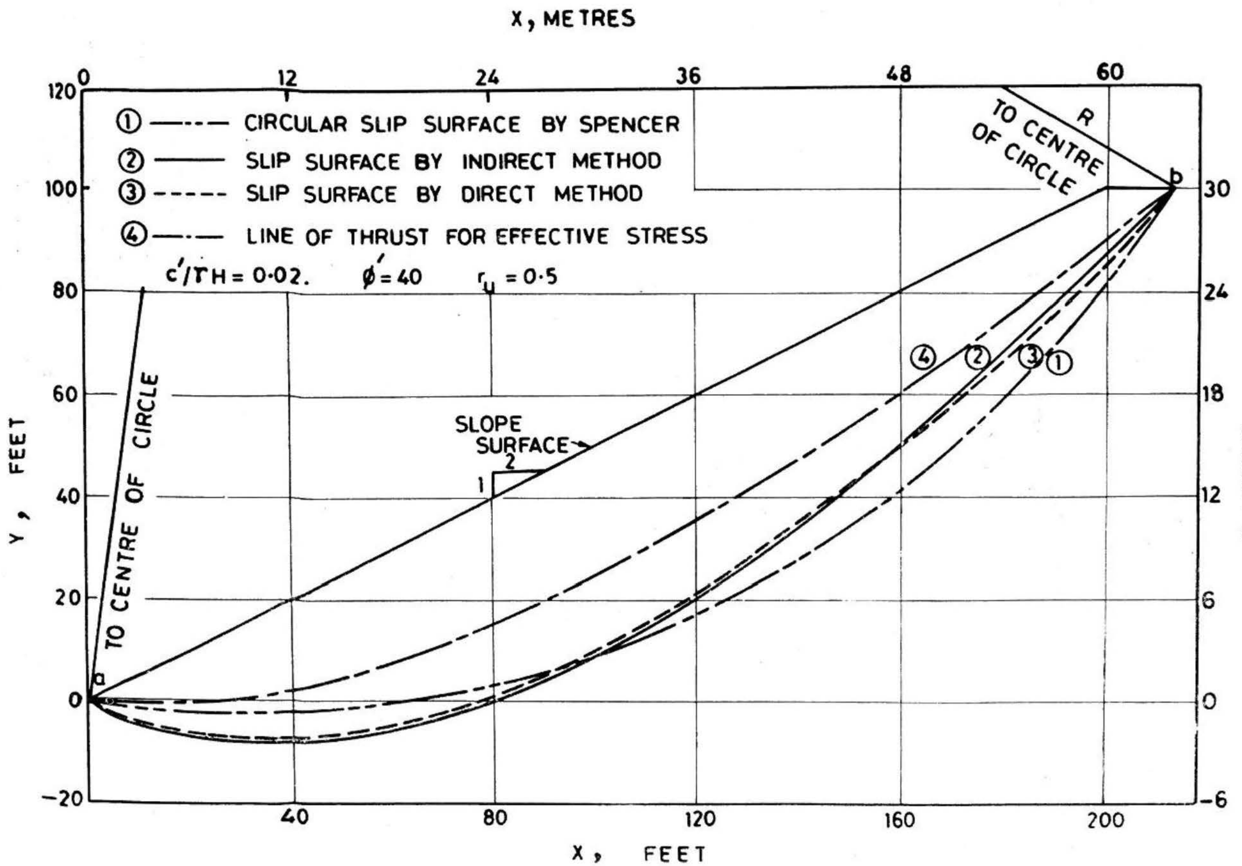

FIGURE 5. Critical slip surface and effective horizontal thrust line

Effective thrust line

Curve 4 of Figure. 5 shows the effective thrust line obtained by the variational method. The curve is fairly smooth throughout the section and lies well within the middle third of the section. The ratio $L_t'/\bar{y}$ obtained along the inter-slice boundaries is given in Table 2. There is a marginal fluctuation in this ratio right upto the end of the section. A maximum ratio of 0.5 is observed at the lower end; the ratio gradually decreases towards the lower third portion of the section, where a minimum ratio of 0.382 is recorded. Thereafter, the ratio increases slightly towards the middle of the section to a value of 0.42 and finally shows a decreasing trend till the upper end of the slip surface. At the top extreme boundary the ratio slightly lies beneath the middle third point of the inter-slice boundary with a minimum ratio of 0.31. The fluctuation in the ratio is due to the variation of the inter-slice forces E' and X' along the inter-slice boundary and also due to the varying intensity of pore-pressure developing at different points along the sliding mass. Low values of $L_t'/\bar{y}$ at the extreme upper end due to the rapid decrease in the magnitude of inter-slice forces and also that at the extreme top end of the slip surface tension cracks are likely to develop as shear progresses.

TABLE 1.

Sl No.	Method		F_s	F_v	θ	$f(x)$	Percentage difference in F_s*
1	Slip-circle analysis by Spencer (1967)		1.0700	—	22.50	1	—
2	Variational method	Direct	1.1261	1.4536	23.41°	Varying as in Fig 4	5.25
		Indirect	1.1235	1.4511	23.72°		5.00

*Percentage difference in factor of safety = Absolute

$$\left[\frac{\text{Factor of safety by Spencer} - \text{Factor of safety variational method}}{\text{Factor of safety Spencer}}\right] \times 100.$$

Influence of pore-pressure co-efficient

In order to illustrate the influence of pore-pressure co-efficient on stability analysis, numerical results are obtained by examining the stability of the specific slope section by assuming two different pore-pressure co-efficients, viz., $r_u=0.3$ and 0.5, and the results are presented in Table 2 and Figure 6. The results presented in Table 2 show that the pore-pressure coefficient plays an important role in the stability analysis of slope. The factors of safety, F_s, F_v, have considerably increased for $r_u=0.3$, and also that the slope of the inter-slice forces, tanθ is different from that obtained taking $r_u=0.5$. It is interesting to note that the ratio, $\frac{L_t'}{\bar{y}}$ for $r_u=0.3$ has also considerably increased and that nowhere the ratio lies beneath the middle third of the inter-slice boundary, which implies that the section

TABLE 2

Sl.No.	r_u	F_s	F_v	θ	$L_t'/\bar{y}$ along inter-slice boundaries										
					Horizonal distance along the slip surface from the lower end in feet (metres)										
					10 (3.05)	30 (10.1)	50 (15.25)	70 (21.35)	90 (27.45)	110 (33.55)	130 (39.65)	150 (45.75)	170 (51.85)	190 (57.95)	210 (64.05)
1	0.5	1.1235	1.4511	23.72	0.5	0.39	0.391	0.382	0.41	0.372	0.382	0.351	0.348	0.342	0.32
2	0.3	1.4102	2.0162	26.45	0.52	0.42	0.41	0.41	0.42	0.42	0.45	0.412	0.382	0.37	0.36

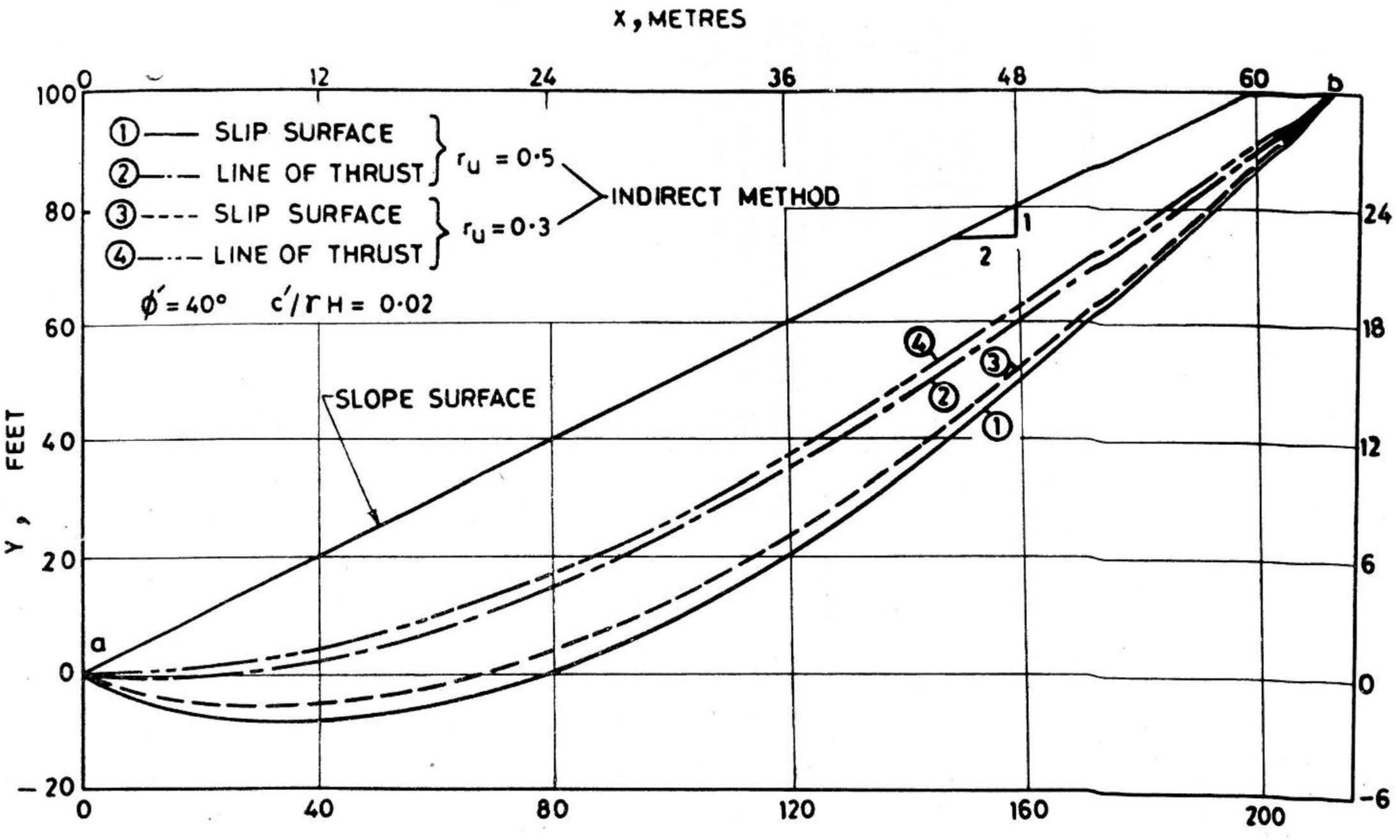

FIGURE 6. Influence of pore-pressure coefficient r_u on critical slip surface and horizontal effective thrust line

with, $r_u=0.3$ presents a more stable state of equilibrium than the section with $r_u=0.5$.

Figure 6 shows the critical slip surfaces obtained by the variational method for $r_u=0.3$ and 0.5, and also shows the positions of the effective horizontal thrust lines. The results obtained for the minimum factor of safety, associated with the critical slip surface for cases with pore-pressure co-efficient, $r_u=0.3$ and 0.5 are significantly different; hence for a rigorous stability analysis it is necessary to consider the actual intensity of pore-pressure developing in the section.

The above results emphasises, that the stability analysis carried out interms of the total stress analysis may lead to misleading results and hence it is important to consider the effective stresses for carrying out rigorous stability analysis of slope.

Factors of safety, F_s and F_v

The results obtained by Spencer and by the variational technique are given in Table 1. Table 2 presents the results obtained by variational method for pore-pressure co-efficients, $r_u=0.3$ and 0.5, and also gives the ratio $\frac{L_t'}{\bar{y}}$ along inter-slice boundaries of the potential sliding surface. By rounding of to two decimal places, the values of overall factor of safety, F_s, and the average factor of safety, F_v, along the vertical boundaries obtained by direct and indirect variational methods (Table 1) are observed to be same. But, the overall factor of safety, F_s, obtained by Spencer using slip-circle analysis shows a significant variation over those obtained by variational methods. The percentage difference in overall factor of safety, F_s, obtained by the variational method over that obtained by slip-circle analysis is of the order of 5% for the specific example considered here.

Figure. 7 (a) and (b) show the variation of factors of safety, F_s and F_v along the critical slip surface obtained by the variational method for pore-pressure co-efficients $r_u=0.3$ and 0.5. The factors of safety remains fairly constant in the middle region of the slip surface, but rapidly increases towards the boundaries and at the boundaries the variation shows an asymptotic nature. The values of factor safety, F_s and F_v obtained for pore-pressure co-efficient, $r_u=0.3$ are significantly higher than for $r_u=0.5$.

The factor of safety, F_s, for pore-pressure co-efficient $r_u=0.5$, has a minimum value which lies in the middle third region of the slip surface. A close observation of the factor of safety along the slip surface shows that there are regions at which the value of factor of safety, F_s, is very close to one and a minimum value of 0.97 is observed at about two-thirds of the distance from the lower-end. This point emphasises, that the slope section is just at the point of progressive failure with local shear failures developing mostly in the middle region. For the section with pore-pressure co-efficient, $r_u=0.3$, the factors of safety, F_s and F_v have significantly increased over those obtained with $r_u=0.5$, and it has also been observed that nowhere, along the slip surface the factor of safety, F_s, is less than one. As a whole, the slope section considered with pore-pressure co-efficient, $r_u=0.3$, leads to a more stable section, ensuring that the state of stable equilibrium is maintained throughout. When the stability of the mass is

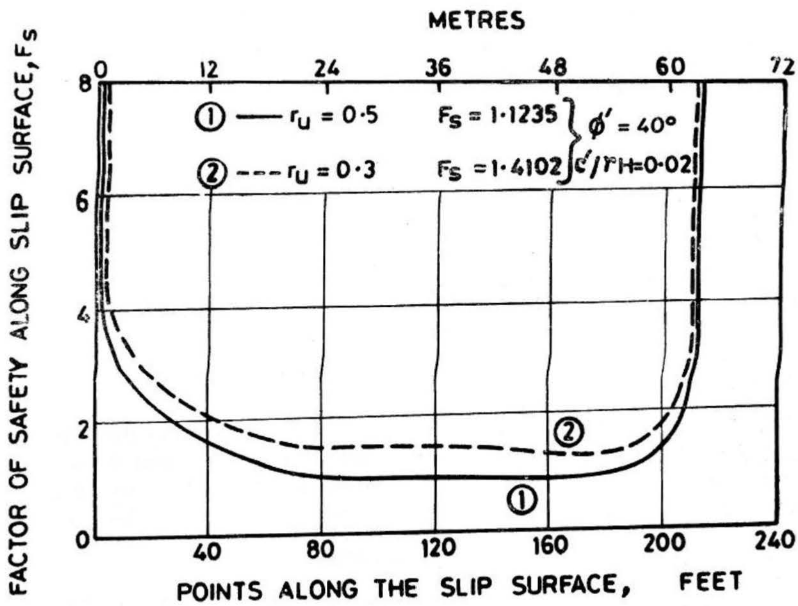

FIGURE 7 (a). Factor of safety, F_s, along the critical slip surface

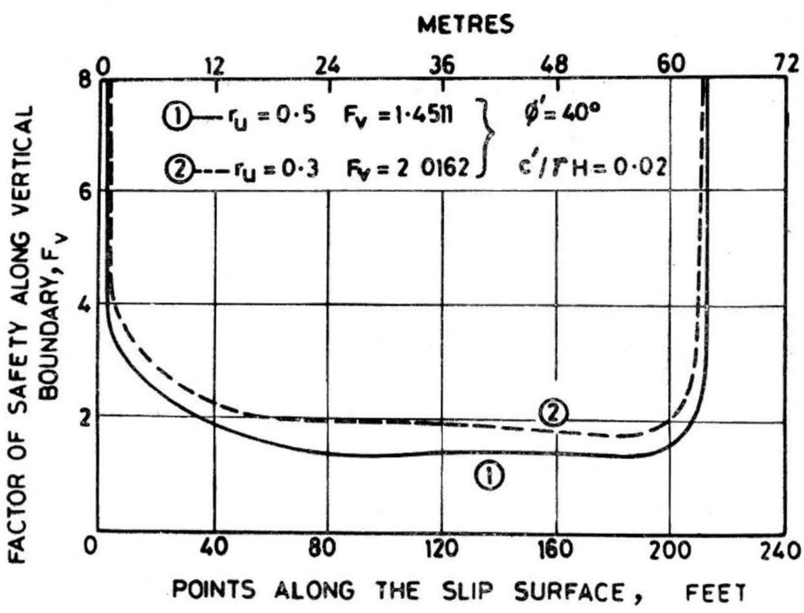

FIGURE 7 (b). Factor of safety, F_v, along the critical slip surface disturbed, progressive shearing develops accompanied by gradual and non-uniform local decrease in soil strength at different regions of the mass and at different rates (Bjerrum 1968, Bishop 1971). As a result, some regions of the soil mass will have local limiting condition (*i. e.* $\tau_s' = \sigma_n' \tan\phi' + c'$) and on other regions a pre-limiting condition (*.i e.* $\tau_s' < \sigma_n' \tan\phi' + c'$), but, as a whole the soil mass will be stable. However, the soil mass fails, if over the entire sliding mass, the failure criterion is satisfied. Hence such situations must be critically examined for a complete safety of the slope. For a complete safety of the section the judgement must not only be on the basis of overall factor of safety but also on the stability at various locations along the potential sliding mass. A critical analysis of this type necessitates understanding of the shear strength parameters, load distribution on the slope, variation in slope section and variation of pore-pressure.

Normal effective stress distribution

The distribution of normal effective stress, σ'_n, along the slip surface obtained by variational method for cases with r_u=0.3 and 0.5 are shown in Figure. 8(a). The distribution of the stress shows a varying intensity along the slip surface. The normal effective stress, is observed to be

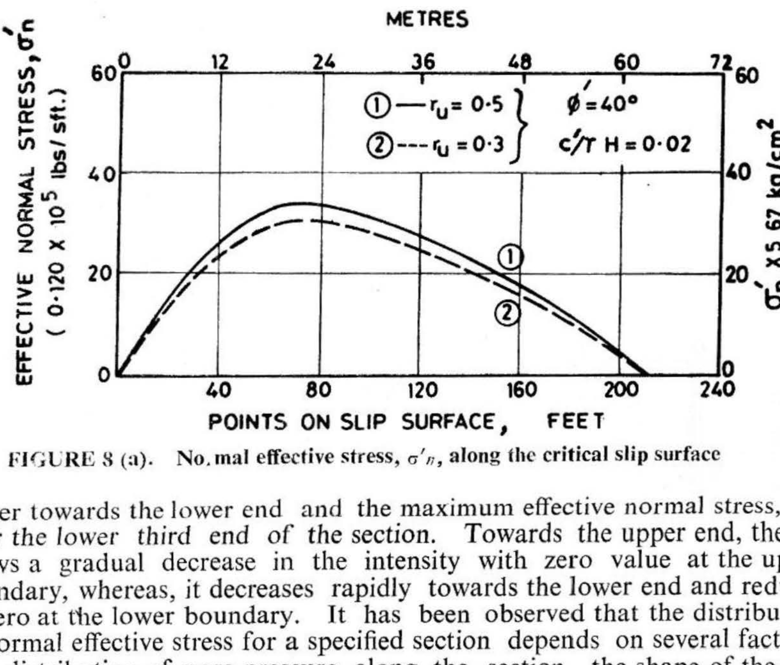

FIGURE 8 (a). No.mal effective stress, σ'_n, along the critical slip surface

larger towards the lower end and the maximum effective normal stress, lies near the lower third end of the section. Towards the upper end, the σ_n' shows a gradual decrease in the intensity with zero value at the upper boundary, whereas, it decreases rapidly towards the lower end and reduces to zero at the lower boundary. It has been observed that the distribution of normal effective stress for a specified section depends on several factors, viz., distribution of pore-pressure along the section, the shape of the slip surface and also on the inter-slice forces, hence it is necessary to treat the normal stress distribution as a dependent function, ınstead of an independent function.

Effective inter-slice force

The effective inter-slice force, E', developing along the vertical inter-slice boundaries for cases with pore-pressure co-efficient, r_u=0.3 and 0.5 are shown in Figure 8(b). At the boundaries the magnitude of E' force

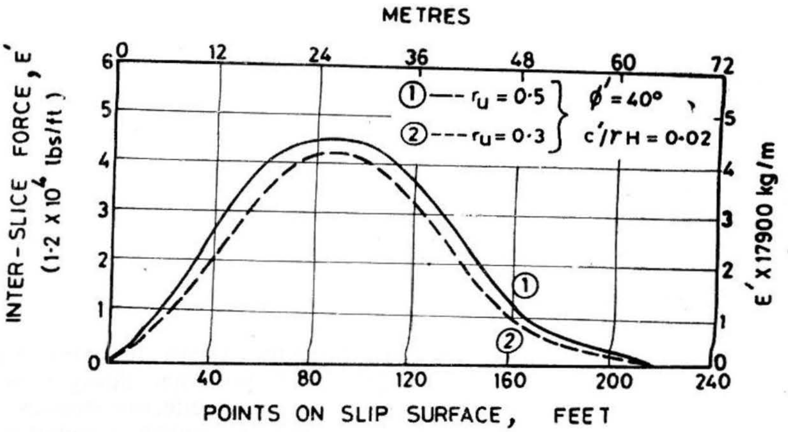

FIGURE 8 (b). Effective inter-slip force E′ along the critical slip surface

reduces to zero, implying, that the condition for force equilibrium is satisfied. Larger values of E' have been observed to be concentrated in the region between middle and one third from the lower end of the section. The values rapidly decrease initially but become gradual near the ends. The maximum values of E' were observed to lie almost at the same point for the cases with r_u =0.3 and 0 5. The E' forces, for cases with r_u=0.3 and 0.5 show a similar pattern of distribution along the slip surface but are different in their magnitude. The magnitude of E' force depends on the actual shear resistance developing along the inter-slice boundary. In this analysis the values of E' forces are obtained by employing Coulomb-Mohr failure criterion along the vertical inter-slice boundary to evaluate the extent of shearing taking place along the inter-slice boundaries. Any local shear failure developing along the vertical inter-slice boundaries could be examined by studying the magnitude of factor of safety, F_v. In the present example, nowhere, the values of F_v, have been found to be less than one, implying that the local shear failure did not occur anywhere along the vertical boundary.

The above discussion of the results obtained by variational methods suggests that variational technique can be successfully employed for a rigorous analysis of slope satisfying all equilibrium and boundary conditions together with conditions for minimum factor of safety and critical slip surface. The analysis also examines for the failure criterion along the sliding surface and hence checks for any local shear failure developing along the slip surface.

Conclusions

A rigorous analytical technique for stability analysis of slope has been developed based solely upon the principles of limiting equilibrium and the analysis satisfies all equilibrium and boundary conditions. The analysis considers the presence of inter-slice forces and assumes that no tension is being developed at the top of the potential slip surface. The technique is developed by framing the stability problem as a minimization problem in the nonlocal calculus of variations. The results obtained by the variational method showed significant variation over those obtained by Spencer (1967) using slip circle analysis. The critical slip surface associated with minimum factor of safety obtained by variation method considerably deviates from the critical slip-circle. Any assumption regarding the internal stress distribution within the potential sliding mass may lead to ill-conditioned functions resulting in misinterpretation of numerical results. The variational method suggests that the assumed function for internal stress distribution must satisfy all equilibrium and boundary conditions and also conditions for minimum factor of safety and critical slip surface. The normal stress distribution along the potential sliding surface is related to the critical slip surface. Though the existing methods of analysis yield results which are meaningful by assuming some normal stress distribution, the results themselves do not necessarily refer to the absolute minimum. The factor of safety, slip surface, normal stress distribution, internal stress distribution and position of horizontal effective thrust line are largely influenced by the pore-pressure developed in the potential sliding mass and hence the slope sections must be analysed in terms of effective stresses. The main advantage of the variation method is that it provides a sound rigorous mathematical technique for analysing slope sections with any configuration and can be extended for slope sections with varying shear strength, pore-pressure distribution and also for sections with non-homogeneous boundary and loading conditions. For such problems the existing methods become cumberome, both analytically as well as computationally. The variational method thus provides a powerful tool for analysing the stability of soil structures.

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Notation

H	Height of slope section
a, b	Slip surface boundary on slope section
dx	Elemental width of slice
α	Slope of base of slice
x, y	Specified co-ordinate system
ϕ'	Angle of shearing resistance } in terms of effective stresses
c'	Cohesion intercept } in terms of effective stresses
R	Radius of the critical slip-circle
γ	Bulk density

r_u	Pore-pressure co-efficient
F_s	Overall factor of safety
F_v	Average factor of safety along vertical inter-slice boundary
δ	Slope of inter-slice force with respect to horizontal
θ	Angle of slope of inter-slice forces with respect to horizontal
E'	Horizontal thrust on the side of slice in terms of effective stress
X'	Vertical shear force on the side of the slice in terms of effective stress
dW	Weight of the slice
dW_T	Tangential component of the weight of slice
u	Pore-pressure
dU	Force due to pore-pressure on the base of slice
dP	Total normal pressure on the elemental base of the slice
dP'	Effective normal pressure on the base of the elemental slice
σ_n	Total normal stress
σ_n'	Effective normal stress
τ'	Shear strength in terms of effective Stress
τ_{xy}	Shear stress
dS	Total shear force available along the base of the slice
dS_m	Total shear force mobilised along the base of the slice
Z	Total inter-slice force
dZ	Resultant of pair of inter-slice forces
W_u	Resultant force due to water pressure acting on the side of slice
L_t	Height of inter-slice force for total stress
L_t'	Height of inter-slice force for effective stress
$f(x)$	Functions of x and prime over
β_1-β_3, p_1-p_{16}, q_1-q_{16}, y', y''	these quantities represent the ordinary derivatives with respect to x, and number of prime represent the order of derivative