Axisymmetric Seepage Flow Through Semi-Infinite Porous Media

by

Satish Chandra*

H.C. Misra**

Introduction

For Axisymmetric Flow through porous media, the Laplace equation in cylindrical coordinates is given by :

$$\nabla^2\phi=\frac{\partial^2\phi}{r^2}+\frac{1}{r}\ \frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad ...(1)$$

where ϕ is potential function, r the distance of the point considered in XY plane and z is the distance along z-axis or the point,

The solution of Equation (1) was obtained by Weber (1919) for the potential due to an electrified circular disc. The corresponding hydrodynamic problem for the potential due to a circular disc source has been reported by Sneddon (1951) and Lamb (1945).

This solution has also been obtained by Sunde (1946) for the potential due to an elementary flat circular disc of negligible thickness on the surface of the ground at a point on the ground surface.

The problem of seepage in the half space can be formulated as below (Figure 1).

Assuming z to be positive vertically downwords

$$\nabla^2\phi=0 \quad -\infty<z<0$$

$$-\infty<y<\infty$$

$$-\infty<x<\infty$$

on $z=0$,

$$\phi=-Kh_1 \quad -\infty<x\leqslant -l$$

$$-a\leqslant y\leqslant a$$

$$\phi=-Kh_2 \quad l\leqslant x<\infty$$

$$-a\leqslant y\leqslant a$$

*Professor of Hydrology, University of Roorkee, Roorkee (U.P.)

**Professor and Assistant Director, Birla Institute of Technology and Science, Pilani (Rajasthan).

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and $\frac{\partial \phi}{\partial z}=0 \quad a<y<-a \qquad -a<y<a$

$-\infty<x<\infty$ and $-l<x<l$

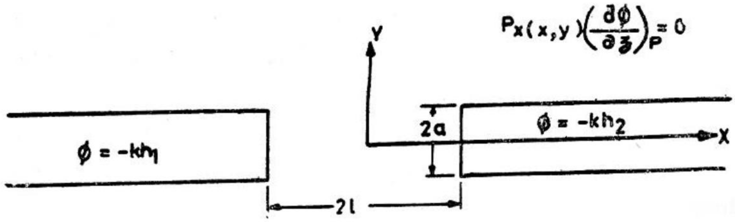


FIGURE 1 : Half space problem.

As is evident it is a mixed boundary value problem involving infinite dimensions. As such, the usual analytical methods, e.g., separation of variables, integral transforms, relaxation, etc., were not found suitable for the solution. A numerical method involving the superposition of the known solutions for the potential due to circular discs on the surface of the half plane has been developed. The method has been extended to include the case in which the seepage region is bounded by an underlying horizontal impervious stratum.

Analysis

The problem described above has been solved by considering the upstream and downstream equipotential regions to consist of a finite number of circular discs placed adjacent to each other, each disc having a constant potential on its face. These potentials have been so adjusted that when the total effect of all the discs is considered, the potential distribution on the upstream and dowstream regions from where and to which the seepage takes place, is constant. Due to the symmetry of the problem, it was found convenient to keep these potentials as +50 percent and −50 percent respectively. The superposition of the solutions due to elementary disc is permissiable because of the linearity of the Laplace's equation. Although the upstream and downstream equipotential surfaces are theoretically of infinite extent, it is clearly not possible in any numerical or experimental method to consider them as such. These surfaces have, there fore, been kept finite. However, the dimensions were determined by-solving the two dimensional problem with finite upstream and downstream equipotential surfaces and determining the size so that the potentials would be reasonably representative of the infinite case. If the canal on either side of the floor is considered to be 1.5 times the length of the floor, the potential distribution will be close to those obtained by considering the equipotential regions extending to infinity (Satish Chandra, 1968).

(*a*) *Infinite Depth of the Porous Media*

To evaluate the potential due to the upstream and downstream constant potential regions from where and to which the seepage takes place, the method of superposition of potentials due to discs has been used assuming these regions to be finite. These two regions B & C of size $2a \times 2b$ are symmetrically located on either side of the Y axis extending from $x=l$

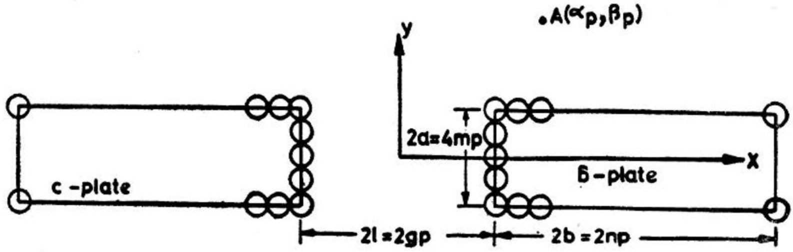


FIGURE 2 : Disc arrangement.

to $x=+(l+2b)$ and $x=-l$ to $x=-(l+2b)$ and about the x-axis from $y=a$ to $y=-a$ (Figure 2). Each of these regions is assumedto contain $(2m+1)$ discs along the Y axis and $(n+1)$ discs along x axis.

In this analysis the two constant potential regions B & C have been assumed to have the length equal $2np$ and width 4 mp. These two regions have been assumed to be separated by a distance $2l=2gp$. For the two dimensional seepage by assuming the constant potential regions to be finite and equal to $3l$ ($2l$ being length of the floor) the potentials obtained at different points along the floor are within 0.2 percent of those obtained when the constant potential regions extend to infinity on either side. Thus, by assuming 2 $np=3gp$. the finite regions of constant potential have been considered to represent the regions extending to infinity.

From the requirements of the problem the resultant potential, after interference, due to all the discs everywhere on the regions B & C has to be constant and of the same positive and negative strenghts respectively.

For this purpose the strength distribution of the discs on the region B & C was determined so that the resulting potentials on B & C regions were constant as required.

Having determined the potential strength of the individual discs, the resulting potential at any point (αP, βp) can be obtained. The potential distribution in the region between the two constant potential regions on $z=0$ plane have been computed.

(b) Finite Depth of Porous Media

For evaluting the potentials due to seepage through the porous media of finite depth the method of images has been employed. For seepage through a media of depth T, an infinite number of similar regions of discs as for the infinite medium problem discussed earlier, placed vertically above and below these regions at distances equal to $2T$, $4T$, $6T$ and so on have been considered which will make the plane $z=T$ an impervious boundary, and the plane $z=0$ also an impervious boundary surface, except for the regions of constant potential. (Figure 3).

Potential Distribution with Confined Seepage Below Floor Level

The method developed, has been used to determine the potential distribution on the area between the constant potential regions for infinite and finite depths of the pervious medium of infinite areal extent. The problems have been solved on digital computer.

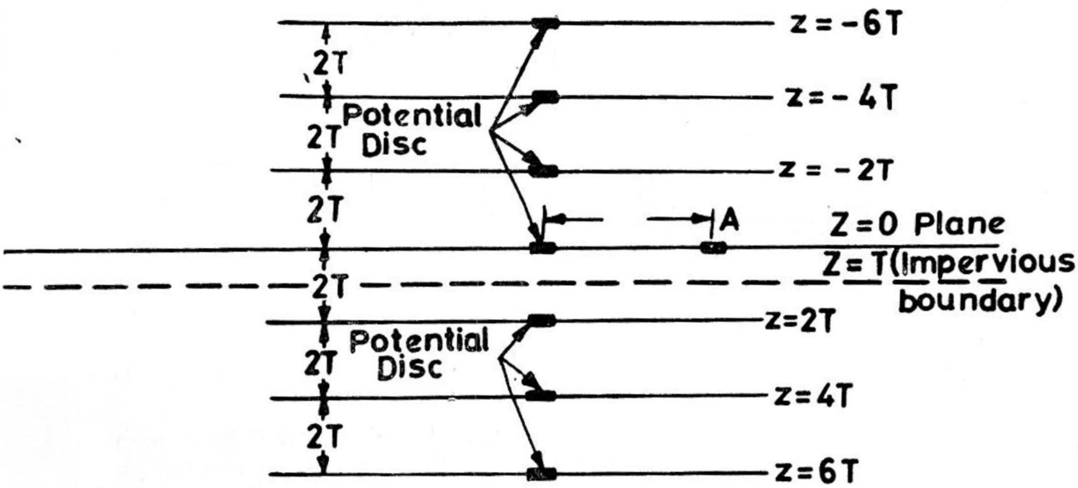


FIGURE 3 : Image system for confined seepage.

In case of infinite depth of the previous medium, the potentials have been determined for the following seven combinations of the g, m and n values :

Sl. No.	g	m	n
1	6	1	9
2	6	2	9
3	6	3	9
4	6	9	9
5	12	2	18
6	12	3	18
7	12	6	18

The first four combinations of g, m and n correspond to a coarse mesh of discs forming the regions from where and to which the seepage takes place representing the length width ratio of the floor of 3, 3/2, 1, and 1/3 respectively. To ascertain the effect of using a finer mesh of discs the order three combinations have been used to give length-width ratios of 3, 2 and 1 respectively. Due to the limitation on the size of the computer it was not possible to try a still finer mesh. The length-width ratio of 1/3 was used to ascertain how close the potentials obtained by this method correspond to the theoretical two dimensional seepage potentials already known. The strength of the discs forming the regions from where and to which the seepage takes place and the resulting potentials under the floor region for all the cases have been determined.

In case of finite depth of the medium, the strengths of the discs, forming the region from where the seepage takes place, have been determined for only one condition, i.e., for $g.m.$ n and T values of 6, 2, 9 and 6 respectively to establish this method and to determine the potential distribution under the floor for finite depth of porous medium.

To compare these results with those obtained by electrical analogy technique, experiments were conducted for length-width ratio of 1, 2 and 3 in an electrical analogy tank, and the potentials obtained.

Infinite Depth of Medium

(*a*) *Strength Distribution*

On examining the strength distribution it is observed that the strength of the discs increase from the middle to the edges of the plate. On moving along the *X* direction (Figure 2) away from the floor, the strengths first reduce and then increase towards the ends. Among the *Y*-direction the strengths are minimum on the *X* axis and increase towards the edges.

Increase in the width of the plates (increase in *m* values) reduces the strength magnitude of all the discs, but the strength distribution pattern, remains the same. The strength distribution of the discs on *X* axis and at $x=g$ along *Y* axis have been presented in Figure 4. These

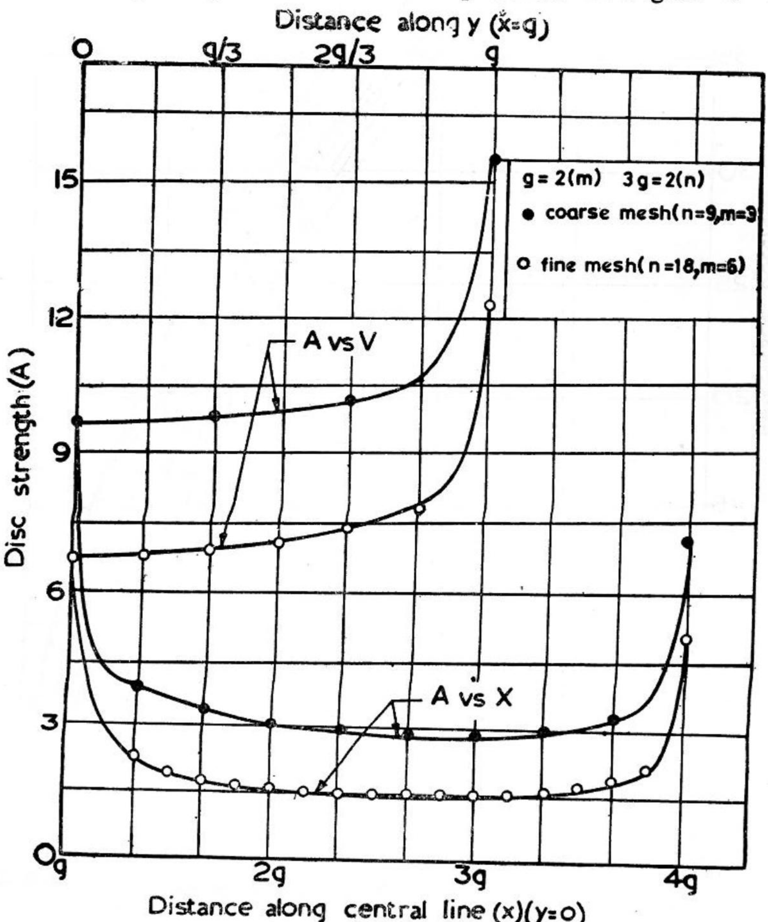
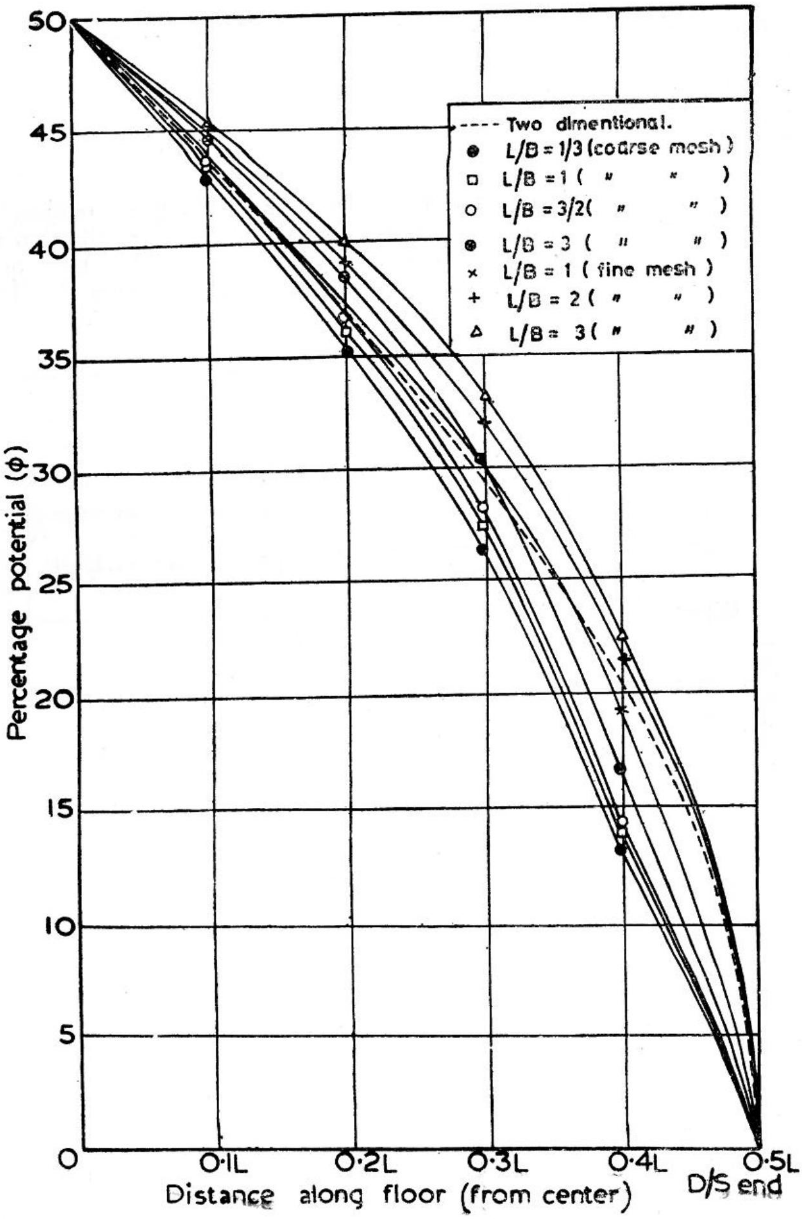


FIGURE 4 : Strength distribution of discs (L/B=1)

strength distributions pertain to coarse and fine set of discs $L/B=1$ of the floor, i.e., g, m, n values of 6, 3, 9 & 12, 6, 18 respectively. It is observed that by making the mesh finer (i.e., by increasing the number of discs on the constant potential regions), the strength distribution of the discs on the region is the same as with coarser mesh but the magnitude of strengths for all the discs is slightly reduced.



FIOURE 5 : Centre line potential distribution under the floor (Computed).

(b) *Potentials along Centre line*

The potentials along the centre line of the floor, for the length-width ratio (L/B) of 1/3, 1, 3/2 and 3 with coarse and 1, 2, 3 with fine mesh of disc forming the constant potential regions respectively have been plotted in Figure 5. The potential distribution under the floor for two dimensional seepage have also been presented for comparison. It is observed that the potentials on the upstream half of the floor decrease by increasing the length width ratio, whereas on the downstream half of the floor the potentials increase by increasing the length-width ratio. The same trend has been observed for the coarse and fine mesh of discs. The results abtained experimentally by electrical analogy method also indicate the same trend. Theoretically, by decreasing the length-width ratio the potential distribution should approach that for two dimensional seepage. Also the potentials on the upstream half for all the length-width ratio should be lower than those for two-dimensional seepage while on the downstream half these should be higher. It has been observed that for the coarse mesh these potentials are always more on the upstream side and less on the downstream side than those two dimensional seepage. In accordance with the theoretical expectation, for all the length-width ratios of the floor considered, with fine mesh, the potentials on the downstream side are more than those for two dimensional seepage except for $L/B=1$ close to the downstream end. It may be concluded that if the computer size would permit a still finer mesh than that considered here, this would give the desired potential distribution on the floor with greater accuracy.

The effect of change in the L/B ratio on the magnitude of potentials has been observed to be maximum at the quarter points, the floor being 2 percent for change of L/B ratio from 1 to 2 and 1.5 percent for change of L/B ratio from 2 to 3. The magnitude of the change reduces to zero on moving towards the centre and ends of the floor. On comparing these results with the experimental results, it is found that on the downstream half of the floor the potentials obtained experimentlly are higher than those obtained by the method of discs (Figures 6, 7, and 8). This difference is expected to be eliminated if a finer mesh of discs is used to represent the constant potential regions. The results obtained by electrical analogy experiments at U.S. Waterways Experimental Station (1963) have also been presented in these figures. These results comare well with the results obtained in the present investigation. The minor differences are due to the use of different tank and model dimensions in the two investigations.

(c) *Potential Distribution Across the Floor*

Referring to Figures 6, 7 and 8, it is observed that on the downstream half of the floor the potentials along the edge are higher than those on the central line for all length-width ratios. The same trend has been observed with the computed and experimental results. The maximum difference in the end line and central line potentials is about 3 percent in computed and 4 percent in experimental resuls for all the length-width ratios. This maximum difference is at about one tenth of the length from the ends of the floor reducing towards the centre and ends where there is no difference in potentials at the sides and at the central line.

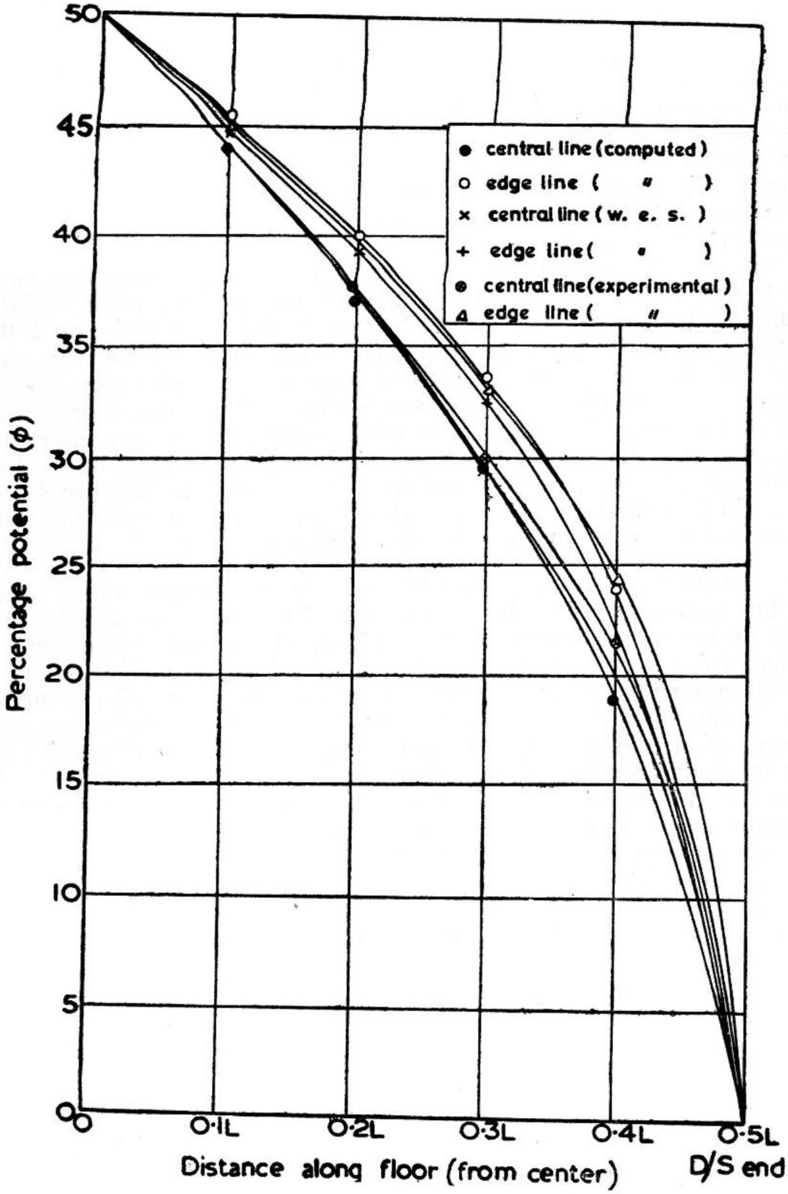


FIGURE 6 : Potential distribution under the floor (L/B=1)

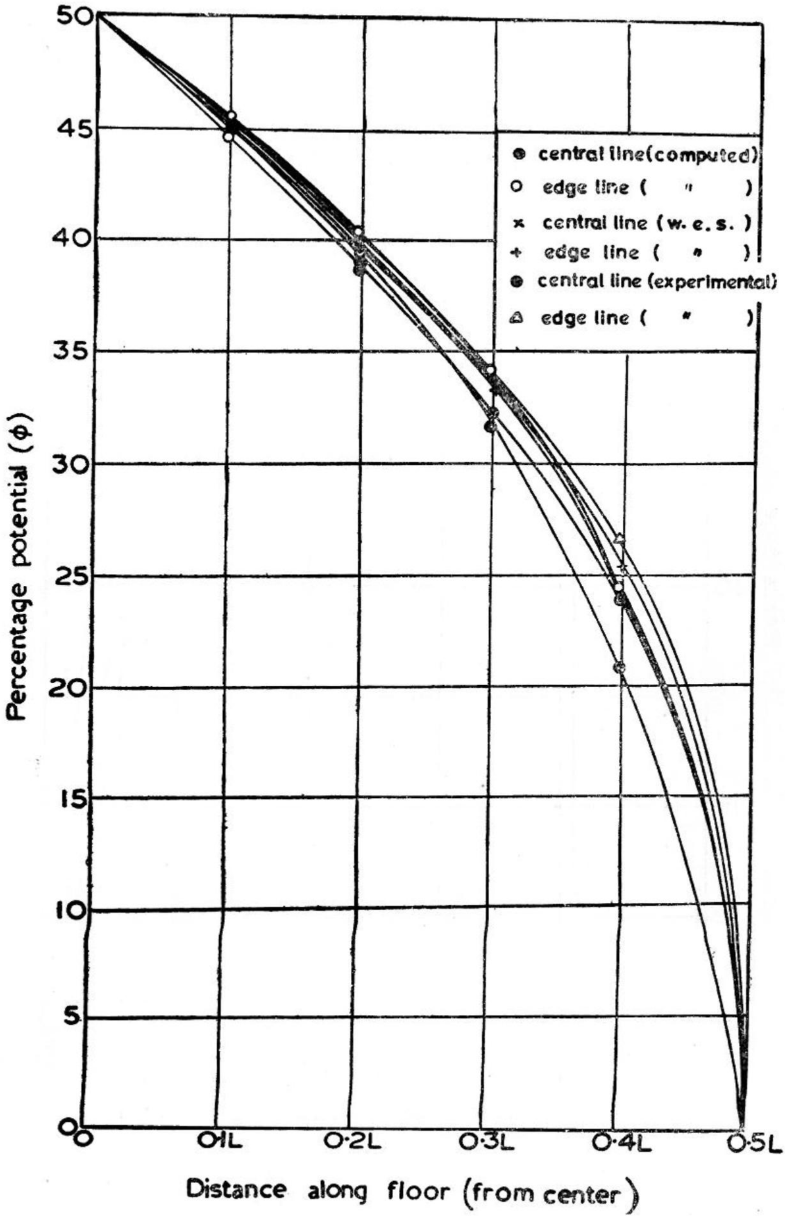


FIGUre 7 : Potential distribution under the floor (L/B=2).

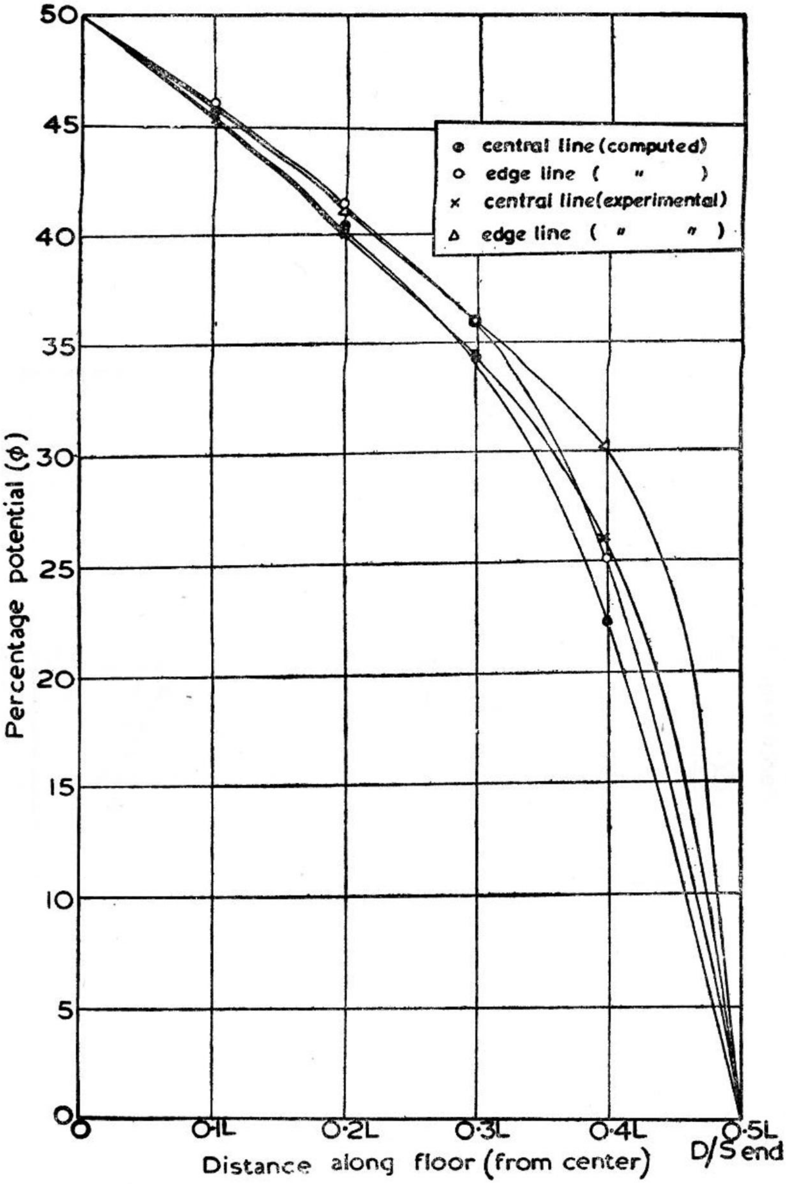


FIGURE 8 : Potential distribution under floor (L/B=3).

The potential distribution at the level of the floor computed in the region close to the constant potential plates has been presented in Figure 9.

Finite Depth of the Medium

The potentials below the floor for seepage through a pervious medium of finite depth equal to half the length of the floor with length-

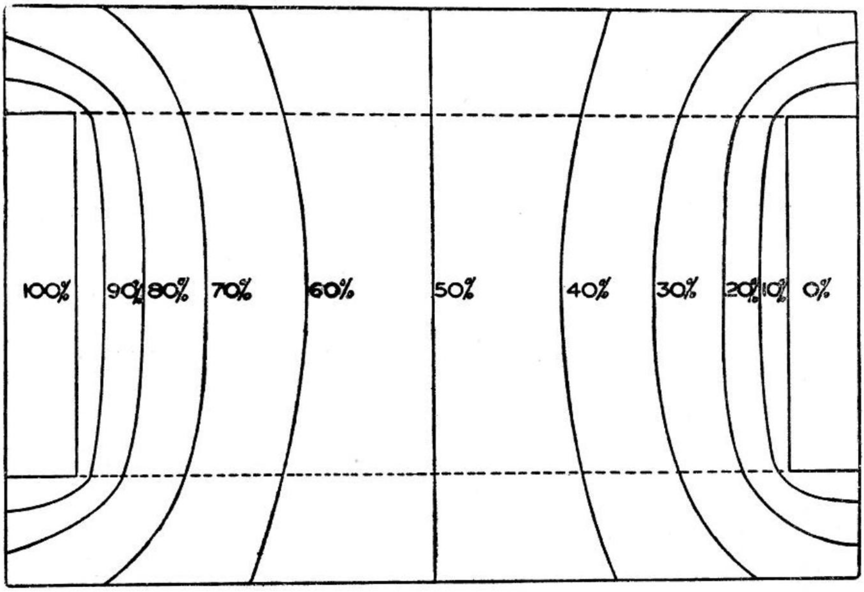


FIGURE 9: Potential distribution at floor level (L/B=2, fine mesh of discs)

width ratio of 2 have been plotted in Figure 10. For comparison the potentials obtained with the same length-width ratio and infinite depth of pervious medium and same number of discs representing the constant potential regions, have also been plotted, for the downstream half of the floor. The central and edge line potentials for finite depth of the medium are found to be lower than those for infinite depth of the medium. The maximum difference in potentials has been observed at distances of 0.2 L from the ends reducing to zero towards the centre and ends of the floor.

In two dimensional secpage for the finite depth of the medium equal to $(L/2)$, the difference in potentials is within 1 percent of those for the infinite medium (Leliavsky, 1965). The larger difference in three dimensional seepage is due to the larger concentration of stream lines towards the sides.

Conclusions.

Using this mcthod it has been found that the potential due to three dimensional seepage in half space with infinite depth of the medium are higher than those for two dimensional seepage under the downstream half of the floor, and under the upstream half of the floor lower than those for two dimensional case

By increasing length-width ratio, these potentials increase for the downstream half of the floor and decrease for the upstream half. This has been confirmed experimentally also.

The reduction in depth of the pervious stratum has been found to reduce the potentials on the downstream half of the floor and increase on the upstream half of the floor. This is similar to that for the two-dimensional seepage.

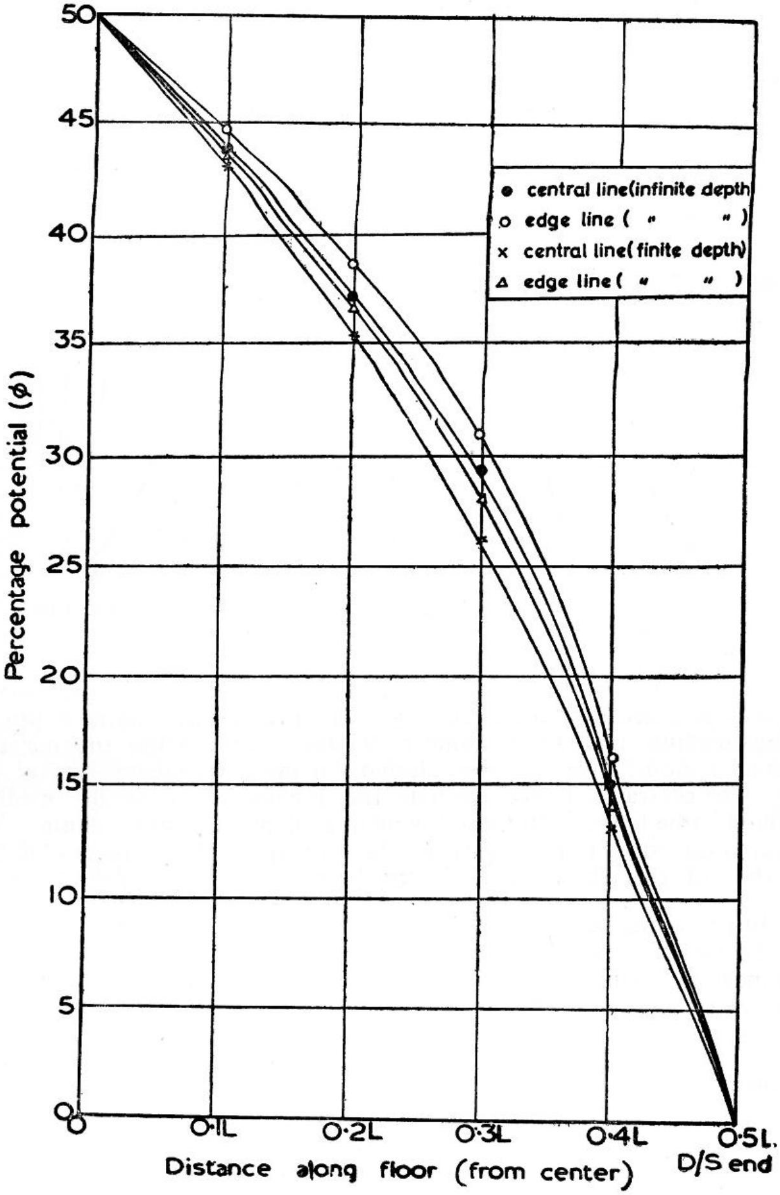


FIGURE 10 : Potential distribution under the floor (L/B=2, finite depth medium)

Notations

$\left.\begin{array}{l} 2_a \\ B \\ 4_{np} \end{array}\right\}$ = Width of the impervious floor

p = Coordinates of a point along x-axis

B_p = Coordinates of a joint along y-axis

$\left.\begin{array}{l}2_p\\2_{np}\end{array}\right\}$ = Finite length of the seepage region of constant potential

K = permeability of the medium.

$\left.\begin{array}{l}2_l\\L\\2_{gp}\end{array}\right\}$ = length of the impervious floor

ϕ = potential function

p = radius of the elementary disc

r = radial distance

T = Finite depth of the pervious region

X = distance along x-axis

Y = distance along y-axis

Z = distance along z-axis

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