# Vertical Stress due to Distributed Loads within Soil Mass

*by*
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## Inroduction

THE vertical stress in soil mass is usually calculated either using Boussinesq's or Westergaard's equation for a point load applied at the surface. These expressions have been integrated over circular and rectangular areas in order to obtain vertical stress beneath the footings and the results are available in the form of charts and tables. Mindlin (1936) gave an expression for the vertical stress increase due to a force at a point in the interior of the semi-infinite body. Earlier in 1932, Melan (1952) gave a solution for two-dimensional case. The solutions of Mindlin is for a three-dimensional case. In this paper, the expressions of Melan and Mindlin have been integrated over a uniformly distributed areas and the results are presented in the form of charts. Because of the complexity of the problem, the integrals have been solved numerically with the help of a high speed digital computer.

## Analysis

*(a) Two-dimensional Case*

If a vertical live load is acting at a depth '*c*' below the surface of a semi-infinite mass [Figure 1(*a*)], then the vertical stress increase due to this live load is given by (Melan)

$$\sigma_z=\frac{P}{\pi}\left[\frac{1}{2(1-\mu)}\left\{\frac{(z-c)^3}{R_1^4}+\frac{(z+c)[(z+c)^2+2cz]}{R_2^4}-\frac{8cz(z+c)x^2}{R_2^6}\right\}+\frac{1-2\mu}{4(1-\mu)}\left(\frac{z-c}{R_1^2}+\frac{3z+c}{R_2^2}-\frac{4zx^2}{R_2^4}\right)\right] \quad \ldots(1)$$

The expression reduces to Boussinesq's case for $c=0$ and $\mu=\frac{1}{2}$. Plot of this expression for $\mu=\frac{1}{2}$ and $\frac{1}{4}$ is available in its literature (Harr, M.E.).

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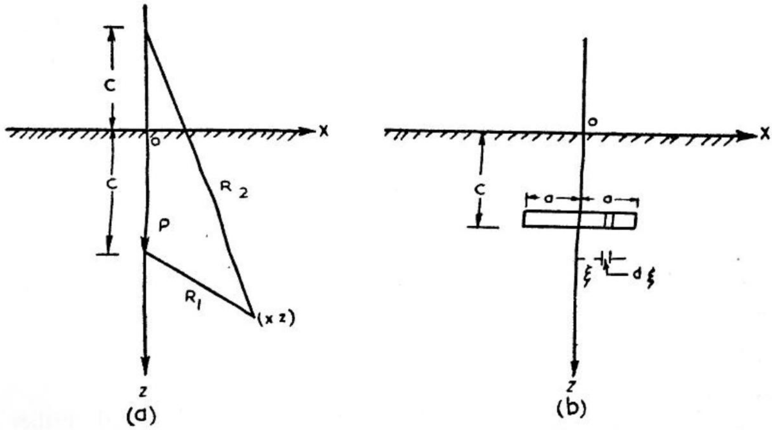


FIGURE 1 (a) : Line load within soil mass, and
(b) : Uniform load over a strip.

*Uniform Normal Load Over Strip* : The stresses at any point due to the distributed load over a strip of width 2a [Figure 1(*b*)] are equivalent to the sum of all loads $q.d§$ acting at $x=§$. Integration of Equation (1) between limits $-a$ to $+$a yields

$$\frac{\sigma_z}{q}=\frac{nk_1}{4\pi(1-\mu)}\left\{\frac{m_2}{n^2m_2^2+k_1^2}-\frac{m_1}{n^2m_1^2+k_1^2}\right\}$$
$$+\frac{4nk_2(k_2-1)}{4\pi(1-\mu)}\left\{\frac{m_2}{(n^2m_2^2+k_2^2)^2}-\frac{m_1}{(n^2m_1^2+k_2^2)^2}\right\}$$
$$+\frac{nk_2+2n(1-2\mu)(k_2-1)}{4\pi(1-\mu)}\left\{\frac{m_2}{n^2m_2^2+k_2^2}-\frac{m_1}{(n^2m_1^2+k_2^2)}\right\}$$
$$+\frac{1}{2\pi}\left\{\tan^{-1}\frac{2nk_2}{n^2m_1m_2+k_2^2}+\tan^{-1}\frac{2nk_1}{n^2m_1m_2+k_1^2}\right\} \quad ...(2)$$

where, $n=a/c$ ; $m_1=\frac{x}{a}-1$ ; $m_2=\frac{x}{a}+1$ ; $k_1=\frac{z}{c}-1$ and

$$k_2=\frac{z}{c}+1$$

Plot of this expression is given in Figures 2(*a*) and 2(*b*) for $\mu=0{\cdot}0$ and $\frac{1}{2}$ which gives vertical stress under the centre of the loaded area.

(*b*) *Three-dimensional Case*

If a point force is applied at a depth '*c*' from the surface (see Figure 3), then the vertical stress increase due to this point force is given by (Mindlin)

$$\sigma_z(r,z)=\frac{P}{8\pi(1-\mu)}\Bigg[-\frac{(1-2\mu)(z-c)}{R_1^3}+\frac{(1-2\mu)(z-c)}{R_2^3}$$
$$-\frac{3(z-c)^3}{R_1^5}-\frac{3(3-4\mu)z(z+c)^2-3c(z+c)(5z-c)}{R_2^5}$$
$$-\frac{30cz(z+c)^3}{R_2^7}\Bigg] \quad ...(3)$$

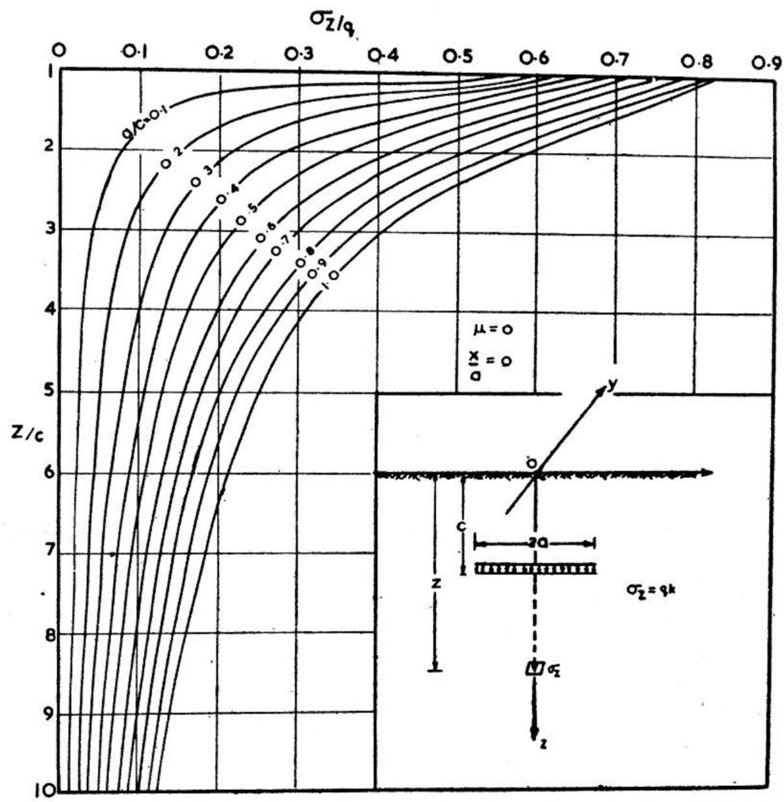


FIGURE 2 (a) : Normal load over a strip. ($\mu$=0)

where,

$$R_1=\left(x^2+y^2+\overline{z-c}^2\right)^{\frac{1}{2}} \text{ and } R_2=\left(x^2+y^2+\overline{+zc}^2\right)^{\frac{1}{2}}.$$

This expression reduces to Boussinesqs case for $c=0$ and $\mu=\frac{1}{2}$. The variation of $\sigma_z$ with the ratio $z/c$ is shown in Figures 3($a$) and 3($b$), wherein $(\sigma_z c^2/P)$ is plotted against $r/c$ for selected values of $z/c$—the Poisson's ratio considered being 0.0 ; 0.3* ; 0 4 and 0.5. This expression like Boussinesq's equation is independent of the unit weight of the mass.

### Stress Beneath Circular Loaded Area

Integrating Equation (3), twice over a circular area of radius $R$, the expression for vertical stress beneath the centre of the circular loaded area is obtained. The expression is given by

$$\frac{\sigma_z}{q}=\frac{1}{8(1-\mu)}\left[2(1-2\mu)\left\{\left[\frac{1}{\dfrac{\left(\dfrac{R}{c}\right)^2}{\left(\dfrac{z}{c}-1\right)^2}+1}\right]^{\frac{1}{2}}-1\right\}\right.$$

* The plot obtained by Mindlin is for $\mu$=0.3 only.

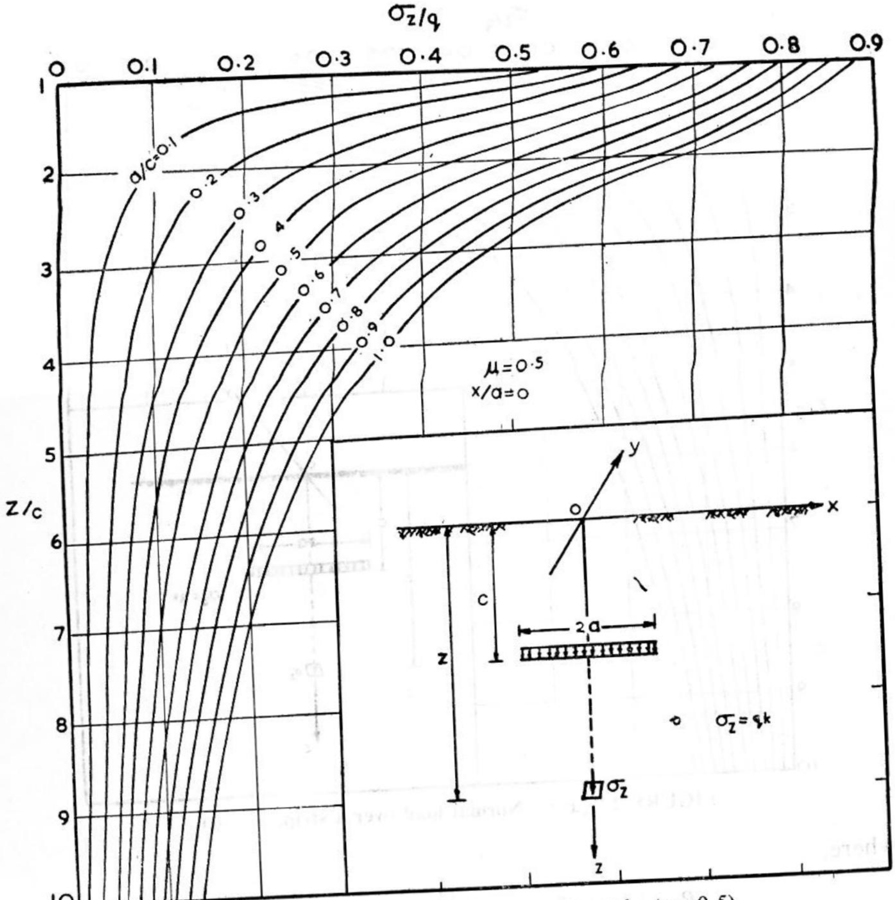


**FIGURE 2 (b) : Normal load over a strip ($\mu$=0.5)**

$$- \frac{2(1-2\;)\left(\frac{z}{c}-1\right)}{\left(\frac{z}{c}+1\right)} \left\{ \frac{1}{\left[\frac{\left(\frac{R}{c}\right)^2}{\left(\frac{z}{c}+1\right)^2}+1\right]^{\frac{1}{2}}} -1 \right\}$$

$$+2\left\{ \frac{1}{\left[\frac{\left(\frac{R}{c}\right)^2}{\left(\frac{z}{c}-1\right)^2}+1\right]^{3/2}} -1 \right\}$$

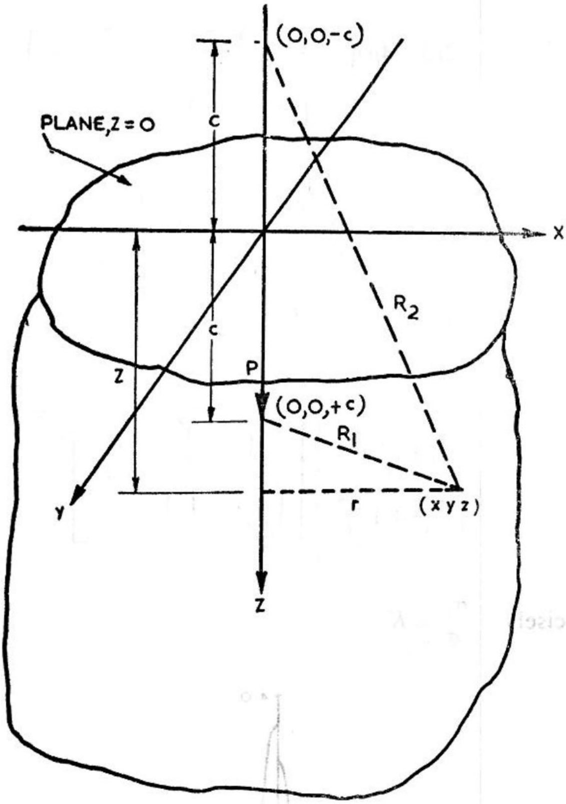


**FIGURE 3 : Concentrated force within mass.**

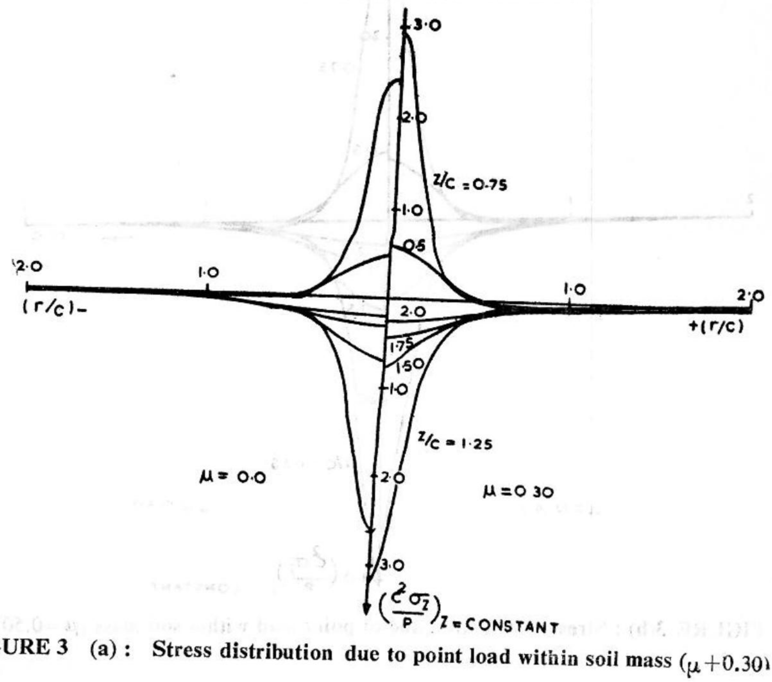


**FIGURE 3 (a) : Stress distribution due to point load within soil mass ($\mu + 0.30$)**

$$+\frac{2(3-4\mu)\left(\frac{z}{c}+1\right)\frac{z}{c}-2\left(\frac{z5}{c}+1\right)}{\left(\frac{z}{c}+1\right)^{2}}$$

$$\left\{\frac{1}{\left[\frac{\left(\frac{R}{c}\right)^{2}}{\left(\frac{z}{c}+1\right)^{2}}+1\right]^{3/2}}-1\right\}$$

$$+\frac{12\frac{z}{c}}{\left(\frac{z}{c}+1\right)^{2}}\left\{\frac{1}{\left[\frac{\left(\frac{R}{c}\right)^{2}}{\left(\frac{z}{c}+1\right)^{2}}+1\right]^{\frac{5}{2}}}-1\right\}\Bigg]\Bigg\} \qquad \ldots(4)$$

or Concisely $\frac{\sigma_z}{q}=K$

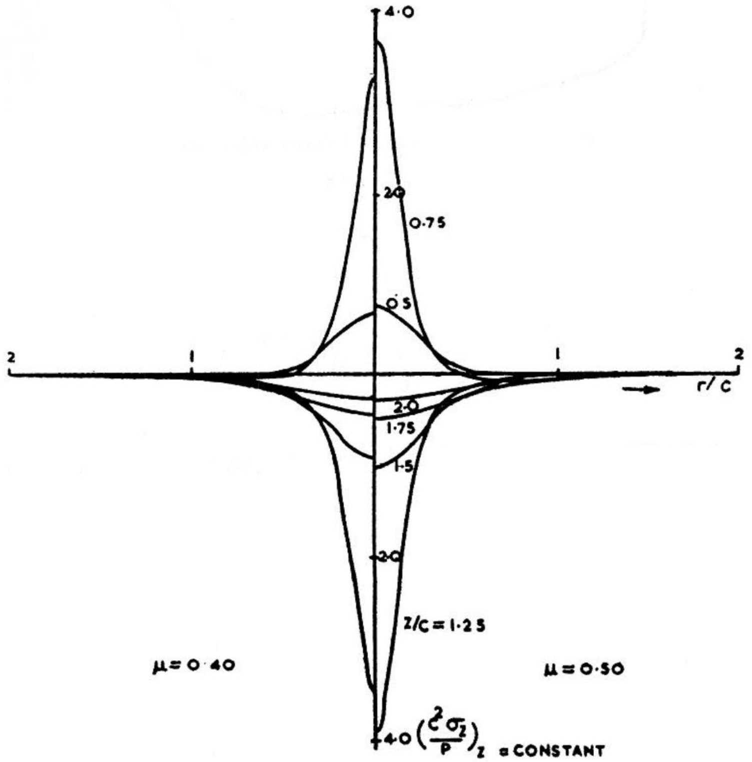


**FIGURE 3(b) : Stress distribution due to point load within soil mass ($\mu$=0.50)**

where, $K$ is equal to the right hand side of Equation (4) and is a dimensionless quantity known as the influence value. The plot of Equation (4) is shown in Figures 4(*a*), 4(*b*) and 4(*c*) for $\mu=0$, $\frac{1}{4}$ and $\frac{1}{2}$ respectively. The influence value is plotted against the ratio $z/c$ to log scale for different ration $R/c$ whichis varied from 0.25 to 20.

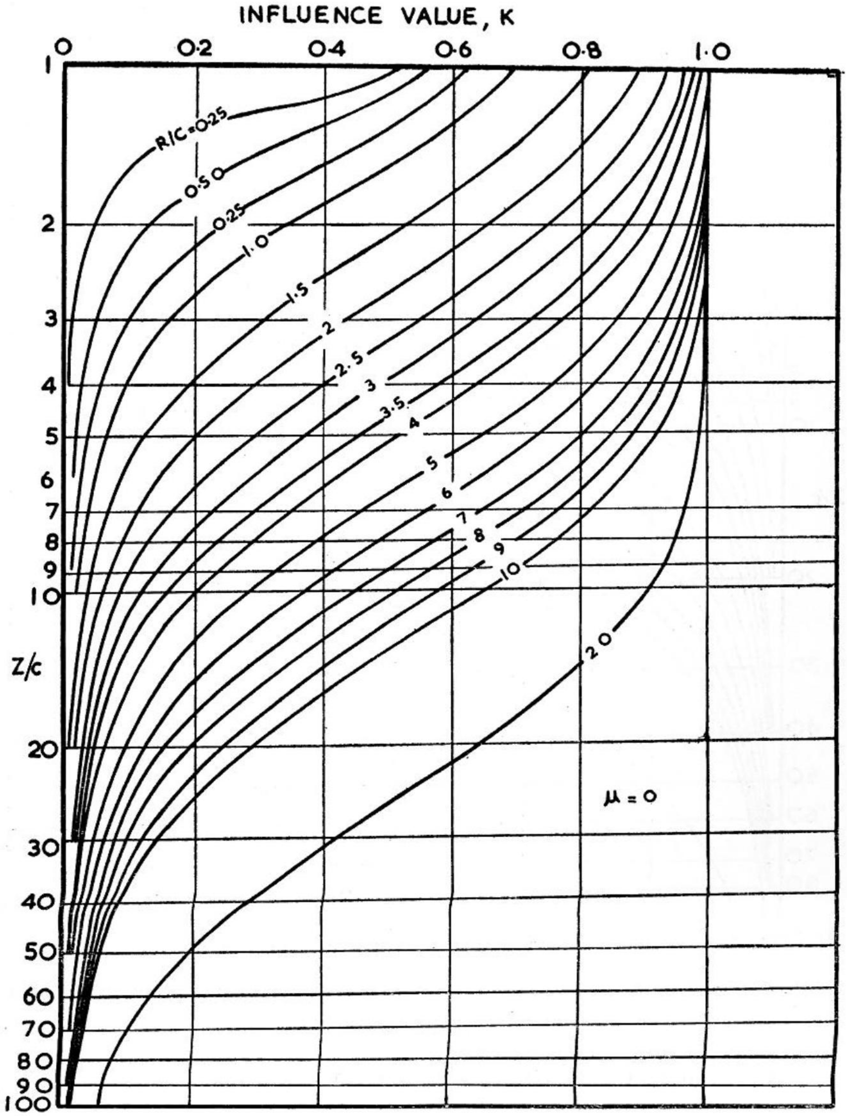


**FIGURE 4 (a): Influence value for distributed load over circular area ($\mu$=0.0).**

If the stresses beneath the circular loaded area other than the centre is required, it is obtained by double integration of Equation (3), where $P$ is replaced by $q.r.dr.d\theta$ and $R_1$ by $(r^2+b^2+\overline{z-c}^2-2br\cos\theta)^{1/2}$ and $R_2$ by $(r^2+b^2+\overline{z+c}^2-2\,br\cos\theta)^{1/2}$

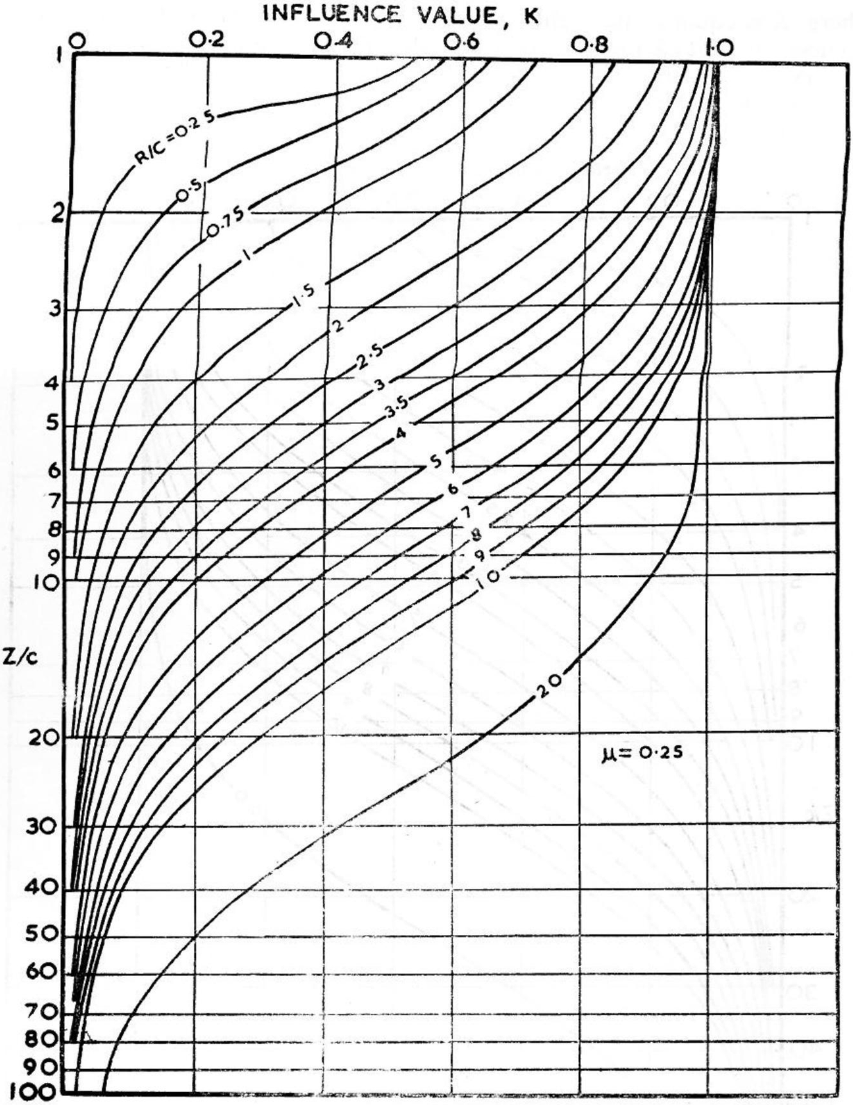


**FIGURE 4 (b) : Influence value for distributed load over circular area ($\mu$=0.25).**

Substituting these values [see Figure 5 (*a*)].

$$\frac{\sigma_z}{q}=\frac{1}{8\pi(1-\mu)}\left[\iint\frac{-(1-2\mu)r.dr.d\theta}{(r^2+b^2+\overline{z-c}^2-2br\cos\theta)^{3/2}}\right.$$

$$+\iint\frac{(1-2\mu)(z-c)r.dr.d\theta}{(r^2+b^2+\overline{z+c}^2-2br\cos\theta)^{\frac{3}{2}}}$$

$$+\iint\frac{-3(z-c)^3\,r.dr.d\theta}{(r^2+b^2+\overline{z-c}^2-2br\cos\theta)^{3/2}}$$

$$+\iint\frac{\{-3(3-4\mu)z(z+c)^2+3c(z+c)(5z-c)\}r.dr.d\theta}{(r^2+b^2+\overline{z+c}^2-2br\cos\theta)^{\frac{5}{2}}}$$

$$+\iint\frac{-30cz(z+c)^3r.dr.d\theta}{(r^2+b^2+\overline{z+c}^2-2br\cos\theta)^{7/2}}\Bigg] \qquad \ldots(5)$$

Now letting $r/c=A$ and $b/c=B$ and remembering that $dr=c.dA$, the above integrals reduce to dimensionless forms :

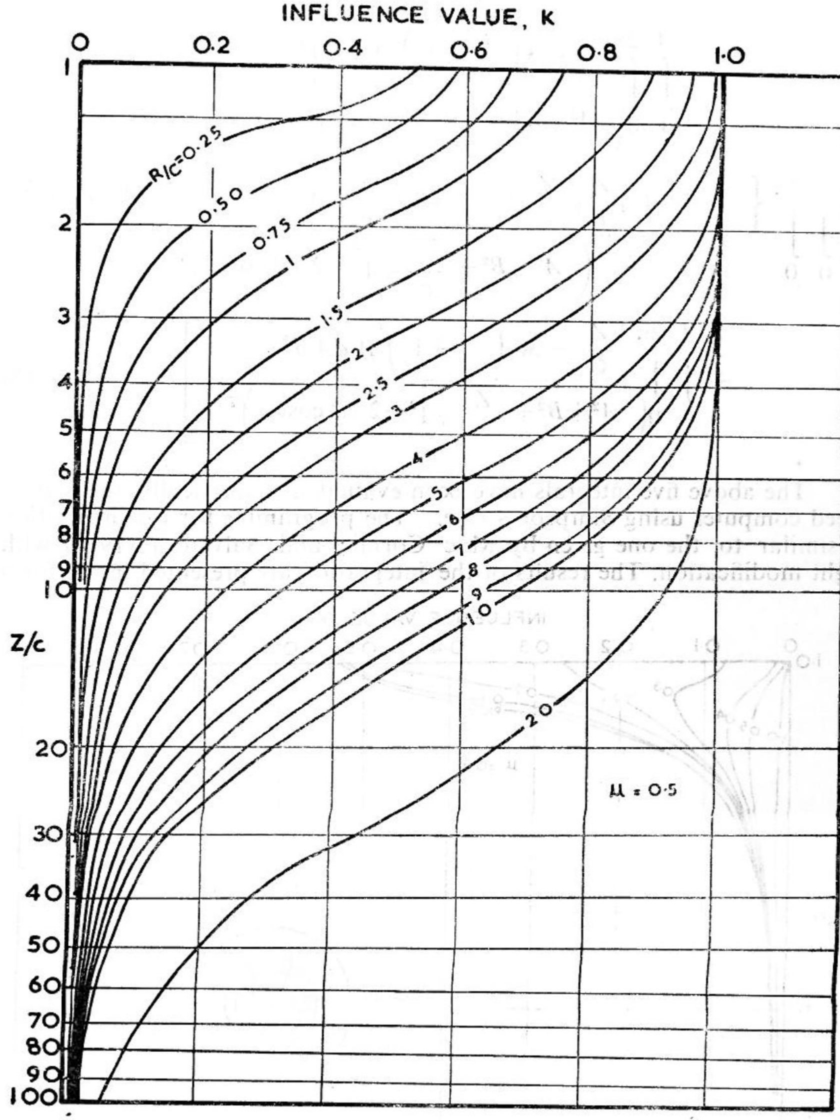


**FIGURE 4 (c) : Influence value for distributed load over circular area ($\mu$=0·5).**

$$\frac{\sigma_z}{q}=\frac{1}{8\pi(1-\mu)}\left[\int_0^A\int_0^{2\pi}\frac{-(1-2\mu)\left(\frac{z}{c}-1\right).A.dA.d\theta}{\left(A^2+B^2+\left(\frac{Z}{c}-1\right)^2-2AB\cos\theta\right)^{\frac{3}{2}}}\right.$$

$$+\int_0^A\int_0^{2\pi}\frac{(1-2\mu)\left(\frac{z}{c}-1\right)A.dA.d\theta}{\left(A^2+B^2+\left(\frac{Z}{c}+1\right)^2-2AB\cos\theta\right)^{\frac{3}{2}}}$$

$$+\int_0^A\int_0^{2\pi}\frac{-3\left(\frac{z}{c}-1\right)^3A.dA.d\theta}{\left(A^2+B^2+\left(\frac{z}{c}-1\right)^2-2AB\cos\theta\right)^{\frac{5}{2}}}$$

$$+\int_0^A\int_0^{2\pi}\frac{\left\{-3(3-4\mu)\frac{Z}{c}\left(\frac{Z}{c}+1\right)^2+3\left(\frac{z}{c}+1\right)\left(5\frac{z}{c}-1\right)\right\}A.dA.d\theta}{\left(A^2+B^2+\left(\frac{z}{c}+1\right)^2-2AB\cos\theta\right)^{\frac{5}{2}}}$$

$$\left.+\int_0^A\int_0^{2\pi}\frac{\frac{Z}{c}-30\left(\frac{z}{c}+1\right)^3A.dA.d\theta}{\left(A^2+B^2+\left(\frac{Z}{c}+1\right)^2-2AB\cos\theta\right)^{\frac{7}{2}}}\right] \quad \ldots(6)$$

The above five integrals have been evaluated numerically in a high speed computer using Simpson's rule. The programme for the integration is similar to the one given by Mac Cormic and Salvadori (1962) with slight modification. The results of the integration are presented in the form

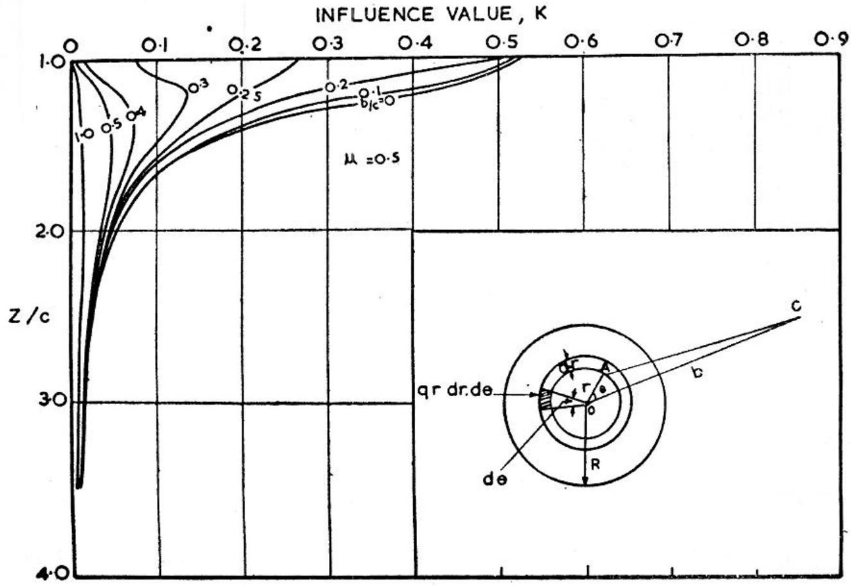


**FIGURE 5 (a) : Influence value for circular area for R/C=0.25.**

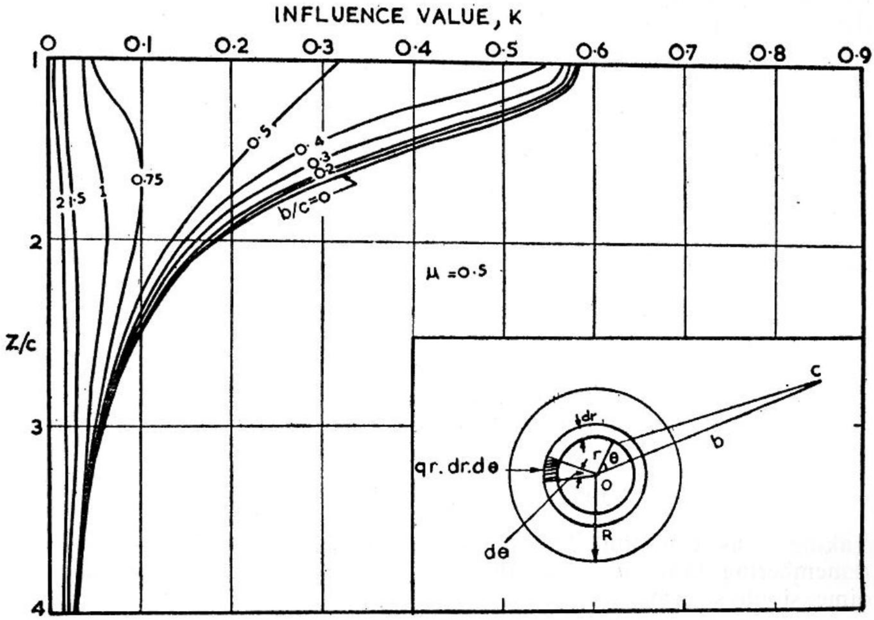


FIGURE 5 (b) : Influence value for circular area for R/C=0.5.

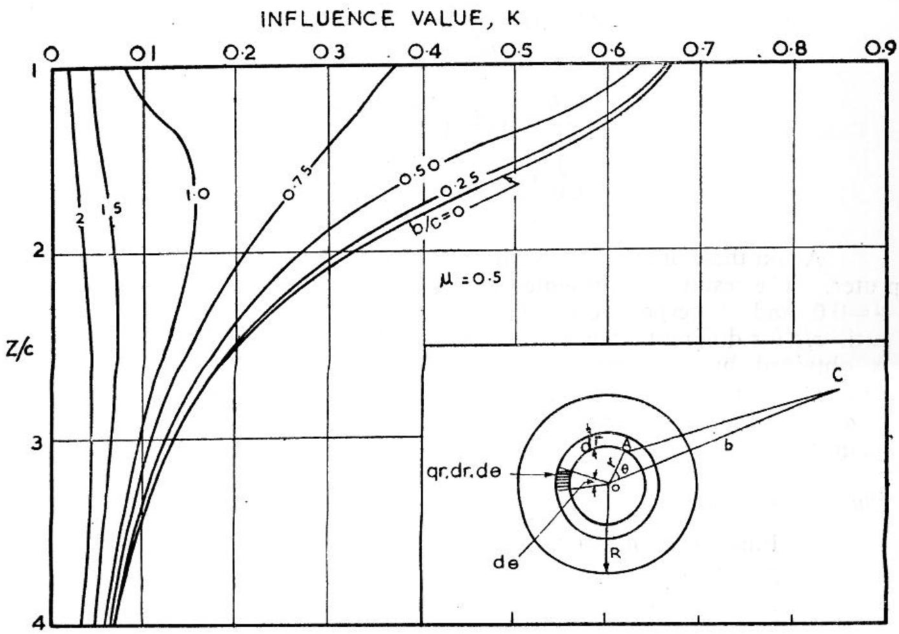


FIGURE 5 (c) : Influence value for circular area for R/C=0.75.

of 5 graphs. The value of $\mu$ is 0·5 in all the cases. The value of *A* is varied from 0.25 to 1.5. The influence value is plotted against the ratio *z/c* for different selected values of the ratio *b/c* (=*B*). The above numerical method fails for $z/c = 1.0$ as it gives a far less influence value than that at $z/c=1.1$. This is absurd as per Equation (4), whose plotting is shown in Figures 4(*a*), 4(*b*) and 4(*c*). So the influence value at $z/c=1.0$ for these cases, is obtained using Equation (4) itself as follows : An influence chart can be constructed for any particular ratio of $z/c$ using Equation (4), very much similar to Newmark's chart. A chart is constructed for the ratio $z/c=1.0$ and the influence value for different ratios of $R/c$ and $r/c$ is obtained by counting the number of squares and multiplying it by the influence value.

## Rectangular Loaded Area

Again integrating Equation (3), twice, one can get the vertical stresses beneath the corner of a rectangular loaded area. Thus,

$$\frac{\sigma_z}{q}=\frac{1}{8\pi(1-\mu)}\left[\iint\frac{-(1-2\mu)\,(z-c)\,dx.\,dy}{\{x^2+y^2+z-c^2\}^{3/2}}+\ldots\right.$$

$$\left.\ldots\ldots+\iint\frac{-30cz\,(z+c)^3\,dx.\,dy}{\{x^2+y^2+z+c^2\}^{7/2}}\right] \qquad \ldots(7)$$

Taking $c$ as common throughout and letting $x/c=a$ and $y/c=b$ and remembering that $dx=cda$ and $dy=cdb$, Equation (6) reduces to dimensionless form :

$$\frac{\sigma_z}{q}=\frac{1}{8\pi(1-\mu)}\left[\int_0^a\int_0^b\frac{-(1-2\mu)\left(\frac{Z}{c}-1\right)da.\,db}{\left\{a^2+b^2+\frac{Z}{c}-1^2\right\}^{3/2}}+\ldots\right.$$

$$\left.\ldots\ldots+\int_0^a\int_0^b\frac{-30\frac{Z}{c}\left(\frac{Z}{c}+1\right)^3 da.\,db}{\left\{a^2+b^2+\frac{Z}{c}+1^2\right\}^{7/2}}\right] \qquad ..(8)$$

Again these integrals have been evaluated numerically in the computer. The results are presented in Figures 6(*a*), 6(*b*), 6(*c*) and 6(*d*), for $\mu=0.0$ and $\frac{1}{2}$ respectively. The influence value is plotted against the ratio $z/c$ for different values of $a$ and $b$. The influence value at $z/c=1.0$ is obtained by constructing an influence chart as explained previously. The values of '*a*' and '*b*' are interchangeable. For any intermediate values of $a$ and $b$ interpolation can be done as explained in the following examples.

*Illustrative Examples*

(*i*) Find the influence value for $a=1$, $b=2.5$ and $z=4$ (say)

Consider $\mu=0{\cdot}5$

From graph, for $a=1$, $b=2$ and $z=4$, we have $K=0.062$

For $a=1$, $b=3$ and $z=4$, we have $K=0.075$.

$\therefore$ For $a=1$, $b=2.5$ and $z=4$, $K=\frac{0.062+0.075}{2}=0.0685$

(*ii*) Find the influence value for $K=1.5$, $b=2$ and $z=4$ (say)

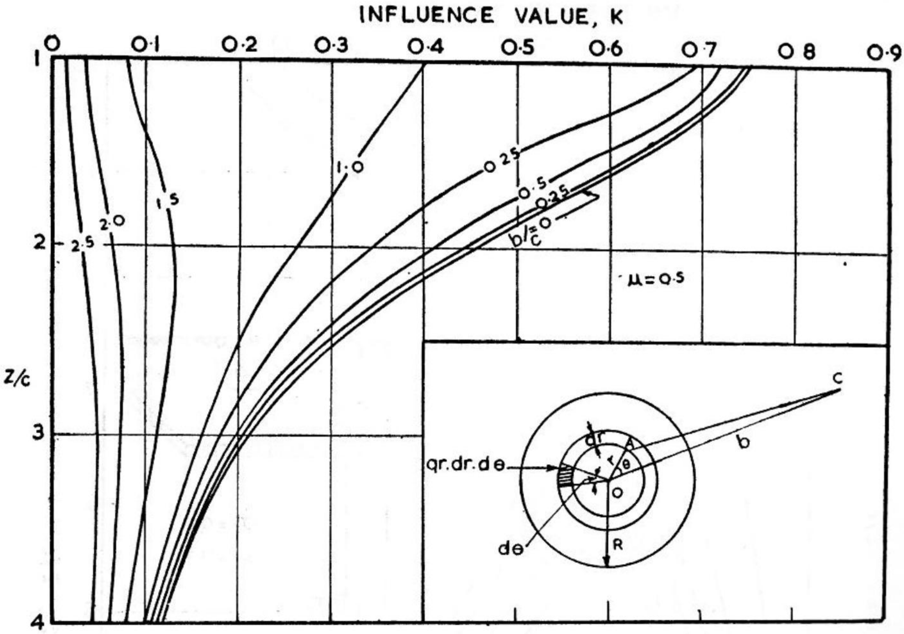


FIGURE 5 (d) : Influence value for circular area for R/C=1.0.

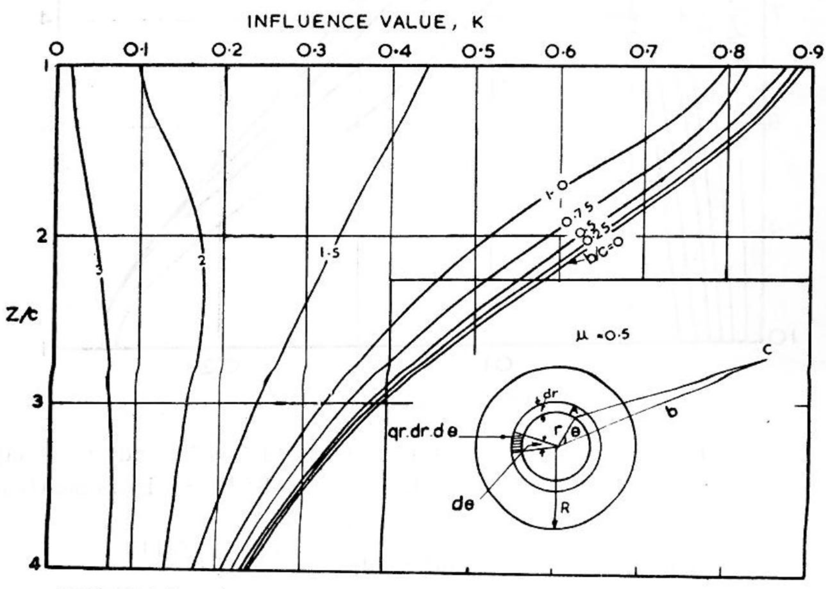


FIGURE 5 (e) : Influence value for circular area for R/C=1.5.

For $a=1$, $b=2$ and $z=4$, we have $K=0.062$

For $a=2$, $b=2$ and $z=4$, we have $K=0.106$

$\therefore$ For $a=1.5$, $b=2$ and $z=4$, $K=\dfrac{0.062+0.106}{2}=0.084$

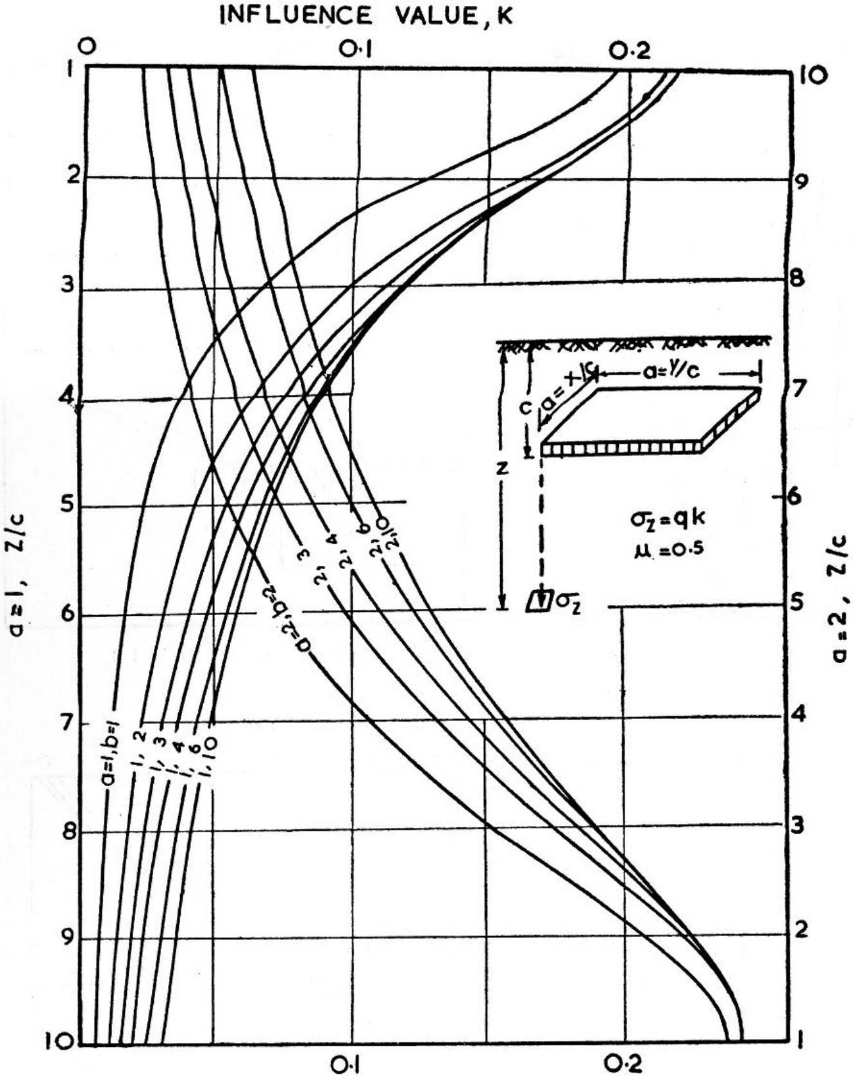


**FIGURE (a) 6 : Influence value for rectangular area ($\mu$=0.5).**

(*iii*) Find the influence value for $a=1.5$ and $b=2.5$ and $z=4$ (say)
For $a=1$, and $b=2.5$, we have $K=0.0685$ [ from Example (*i*) ]
For $a=2$, and $b=2.5$, we have $K=0.118$

$$\therefore \text{ For } a=1.5, \quad b=2.5 \text{ and } z=4, K=\frac{0.0685+0.118}{2}$$

$$=0.0932$$

**Discussion**

(*a*) In Figure 7, is shown the influence value for strip, square and circular loaded areas plotted against the ratio $B/c$ where, $B$ is the width of loaded area. The vertical stress corresponds to a depth equal to $1\frac{1}{2}$ $B$ measured from the plane on which the load is acting. Keeping $B$ as constant, the depth of embedment $c$ is increased and the vertical stress is calculated for the same depth in all the cases. It is seen that as $c$ is increased the vertical stress decreases. In other words, as the depth of embedment $c$ increases, the vertical stress decreases. These influence values shown in different figures, therefore, may serve better than those calculated from the loadings at the surface.

(*b*) *Convergence of Integrals* : The convergence of the integrals given in Equation (7), depends not only on the value of the ratio $Z/c$ but also $A$ and $B$. For small values of $A$ and $B$, the convergence

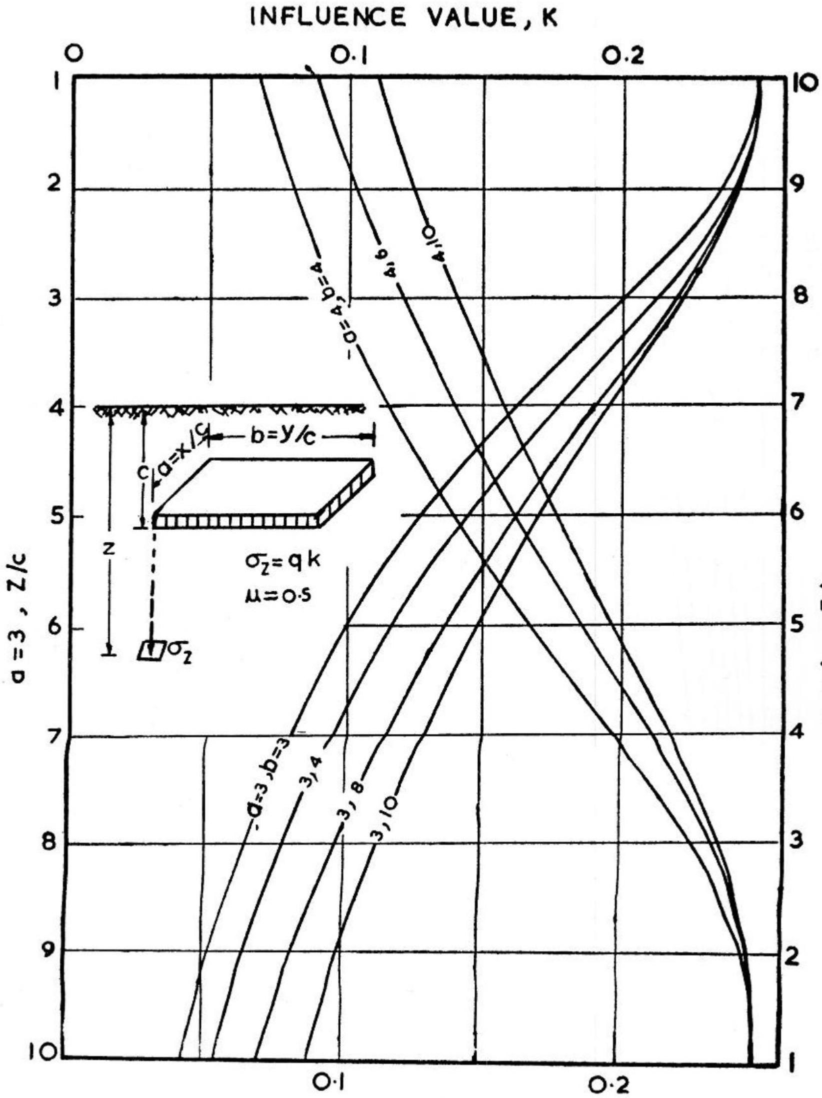


FIGURE 6 (b) : Influence value for rectangular area ($\mu$=0.5).

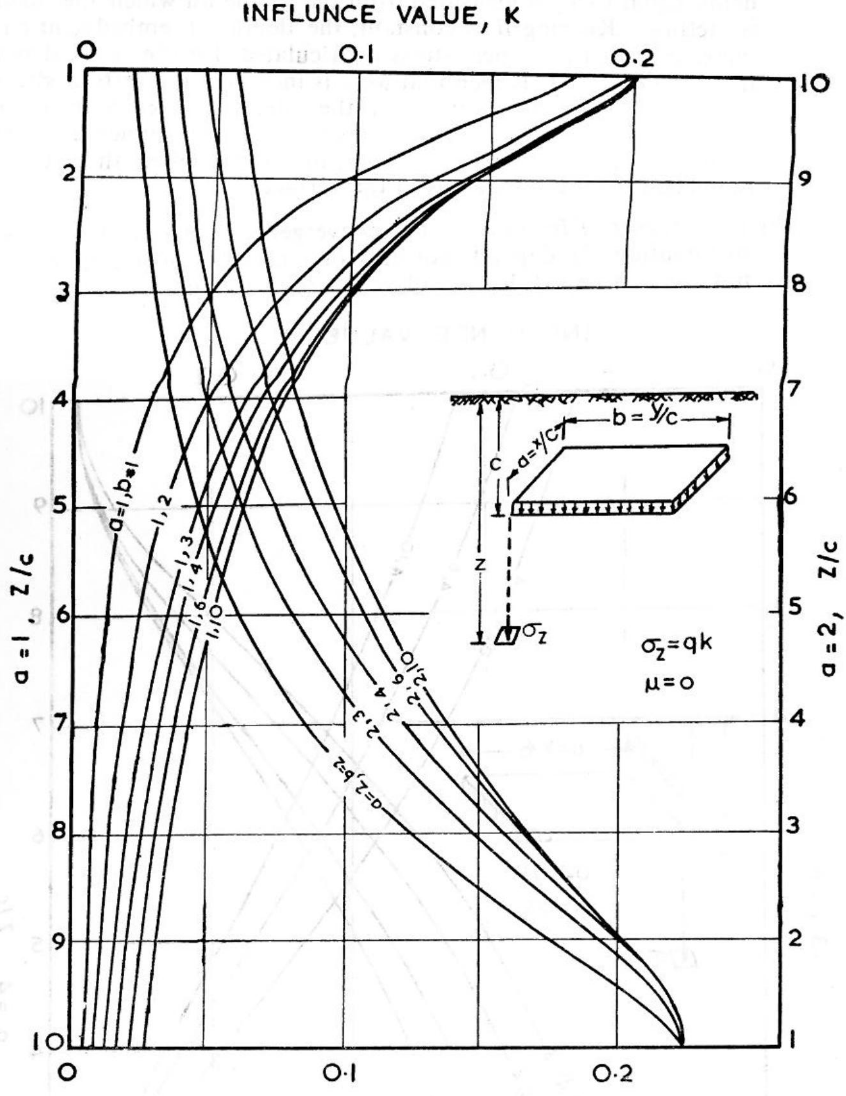


FIGURE 6 (c) : Influence value for rectangular area ($\mu$ =0 0)

is poor for $z/c$ < 1.5 and converges rapidly for $z/c$ > 1.5. But at higher values of $A$ and $B$ the convergence is slow as $z/c$ increases.

The integral [Equation (8)] converges rapidly for all values of $z/c$ > 2 irrespective of $a$ and $b$. But the convergence is fairly good for 1.5 < $z/c$ < 2 and is slow for $z/c$ < 1 5 This only requires a finer mesh of the area to be integrated and requires comparatively long time in the computer to solve the integrals. In any case, convergence can be obtained for all values, but for computers time.

## Conclusions

Melan's and Mindlin's equations have been integrated over circular and rectangular loaded areas and the results are plotted in the form of dimensionless number $K$, the influence value. Vertical stress can be calculated using these charts once the width of the loaded area, depth at which the load is acting and depth from the surface at which vertical stress is desired are known. These values are less than those calculated with surface loadings.

## Notations

$A$ = Dimensionless ratio

$B$ = Dimensionless ratio

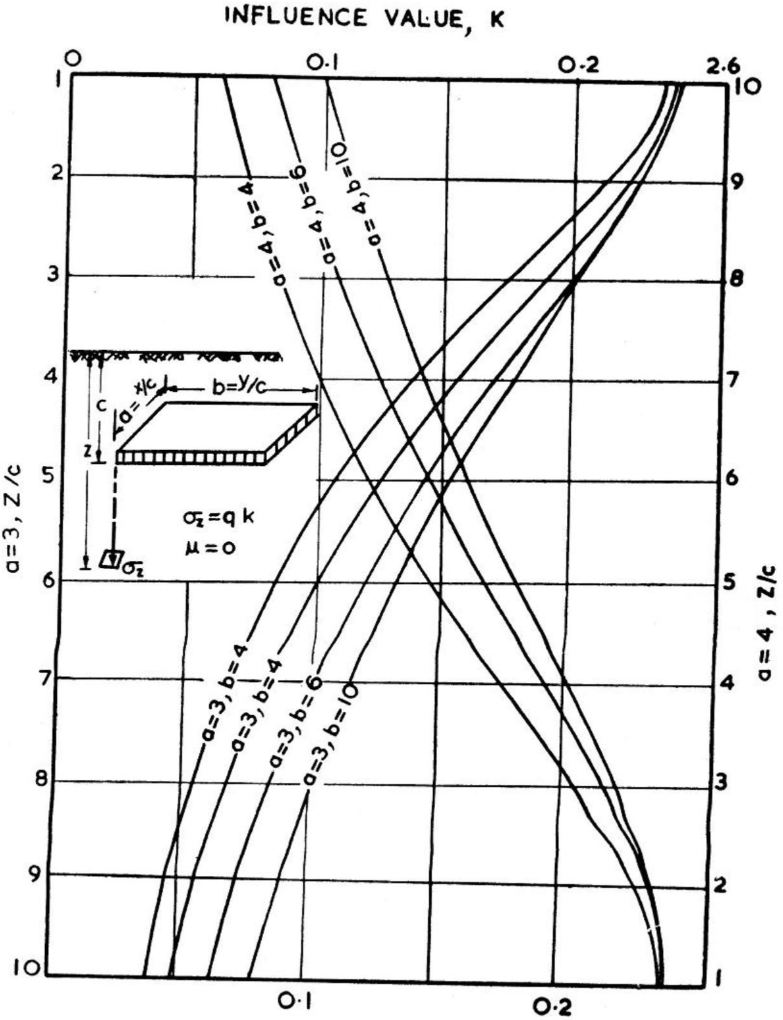


FIGURE 6 (d): Influence value for rectangular area ($\mu$=0.0).

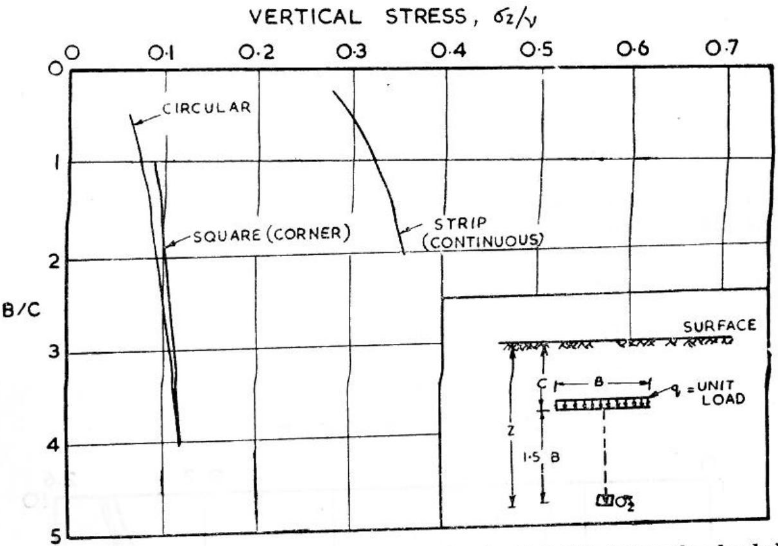


FIGURE 7 : Variation of vertical stress at a depth 1.5 B below the loaded area keeping B constant and varying C.

$K$ = Influence value
$P$ = Linear or point load
$R$ = Radius of loaded area ; Radial distance
$a$ = Dimensionless ratio ; width of loaded area
$b$ = Radial distance, dimensionless ratio
$c$ = Depth at which load is applied from surface
$q$ = Load intensity
$r$ = Radial distance
$x$, $y$, $z$ = Co-ordinates
$\sigma_z$ = Vertical stress
$\mu$ = Poisson's ratio
§ = Running co-ordinates.

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