Buckling Resistance of Piles in Layered Clays

by
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Introduction

SOFT clays or silts overlying a deep-seated stratum of rock, sand or gravel, often require the use of long bearing piles to support engineering structures. The primitive methods of design were either to ignore the effect of surrounding ground and consider the pile as a free standing column, or to assume that the surrounding ground offers an infinitely rigid support implying that almost any load can be placed on the pile consistent with the strength of the material of the pile or the strength of the supporting soil strata or rock. The truth lies between these two extremes. Spolford (1936), Cummings (1938) and Casagrande (1947) have reported that piles in soft clay bear loads well in excess of their Euler loads. Bjerrum (1957) has reported the failure of piles under axial loads inducing stresses much below the yield stress. The piles when extracted were seen to have failed in a very soft stratum. This shows that in the design of such piles, besides considering the supporting nature of the surrounding ground, the problem of buckling also has to be studied.

An early approach to the problem of buckling of piles surrounded by homogeneous clay was due to Granholm (1929). Timoshenko's method of analysing the buckling of a bar on an elastic foundation can be used to obtain the crippling loads on bearing piles surrounded by homogeneous clay. Both assume an elastic ground implying a linear pressure-deflection relationship. The former obtains the crippling load by solving the differential equation governing the problem, whereas the latter adopts the Raleigh-Ritz energy method to derive the same. There is no trace of literature reporting the behaviour of piles under axial loads in layered clays. An effort is made to evolve a method of calculating the buckling resistance of piles in a two-layered system of clay, each layer having a different foundation modulus as shown in Figure 1 (*a*).

Energy Method for Determining Crippling Load on Pile Surrounded by Layered Clay

The method is based on the following assumptions :

(1) The pressure-deflection relationship for clay is linear.
(2) The foundation modulus of clay is constant with depth.
(3) The adhesion between the pile surface and the surrounding soil medium is negligible.
(4) The ends of the pile are hinged.

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This paper was received on 5 June 1971. It is open for discussion up to December 1972.

Figure 1 (*b*) shows a slender vertical bearing pile hinged at both ends, surrounded by a two-layered system of clay, and subjected to axial load *P*. The deflected shape of the elastica can be represented by Fourier sine series,

$$y=\sum_{n=1}^{n=\infty} a_n \sin\frac{n\pi x}{l} \qquad ...(1)$$

For small deflections the loss of potential energy of the compressive force *P* to a very close approximation is given by (Timoshenko),

$$\triangle T=\frac{P}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx=\frac{P\pi^2}{4l}\sum_{n=1}^{n=\infty} n^2\, a_n^{\,2} \qquad ...(2)$$

The energy stored during bending of the pile to the same degree of approximation is given by (Timoshenko),

$$\triangle U_1=\frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx=\frac{\pi^4 EI}{4l^3}\sum_{n=1}^{n=\infty} n^4\, a_n^{\,2} \qquad ...(3)$$

Strain energy stored during deformation of the surrounding soil is given by,

$$\triangle U_2=\frac{k\beta}{2}\int_0^{\alpha l} y^2 dx+\frac{\beta}{2}\int_{\alpha l}^{l} y^2\, dx \qquad ...(4)$$

Conservation of energy requires,

$$\triangle T=\triangle U_1+\triangle U_2 \qquad ...(5)$$

Substituting the value of y from Equation (1) in Equation (4) and simplifying, we get,

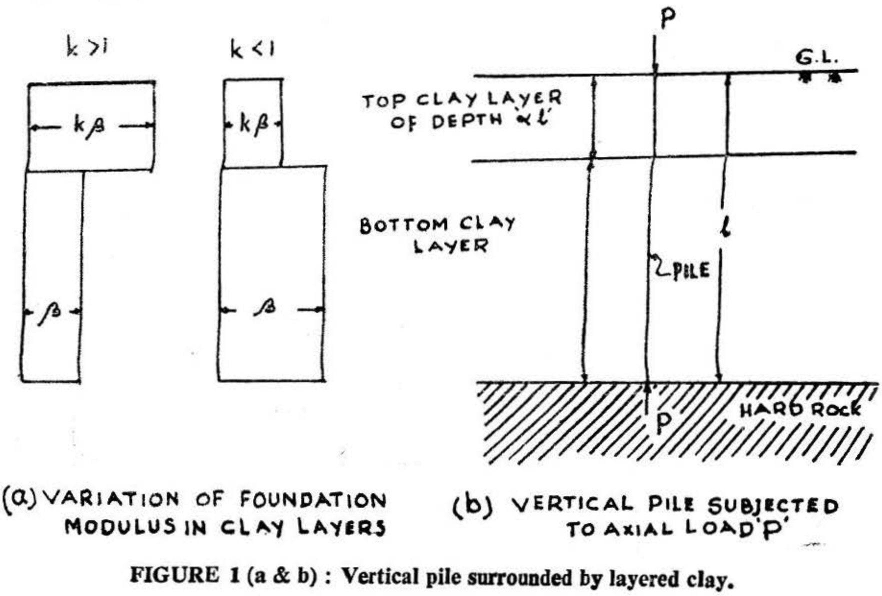


FIGURE 1 (a & b) : Vertical pile surrounded by layered clay.

$$\Delta U_2=\frac{k\beta l}{2}\left[\sum_{n=1}^{n=\infty}\left(\frac{a_n^2\alpha}{2}-\frac{a_n^2}{4n\pi}\sin 2n\pi\alpha\right)+\sum_n\sum_m\frac{a_n\,a_m}{\pi}\left\{\frac{\sin(m-n)\,\pi\alpha}{m-n}-\frac{\sin(m+n)\pi\alpha}{m+n}\right\}\right]$$

$$+\frac{\beta l}{2}\left[\sum_{n=1}^{n=\infty}\left(\frac{a_n^2}{2}\overline{1-\alpha}+\frac{a_n^2}{4n\pi}\sin 2n\pi\alpha\right)+\sum_n\sum_m\frac{a_n a_m}{\pi}\left\{\frac{\sin(m+n)\pi\alpha}{m+n}-\frac{\sin(m-n)\pi\alpha}{m-n}\right\}\right] \quad ...(6)$$

Incorporating the value of ΔU_2 from Equation (6) in Equation (5) and denoting,

$$p=\frac{P}{\frac{\pi^2 E\,I}{l^2}} \quad \text{and} \quad \gamma=\frac{\beta l^4}{EI\pi^4}$$

We obtain;

$$p=\frac{1}{\sum_{n=1}^{n=\infty} n^2\,a_n^2}\left[\sum_{n=1}^{n=\infty} n^4 a^2{}_n+4\gamma\left\{\frac{k}{4}\sum_{n=1}^{n=\infty}\left(a_n^2\alpha-\frac{a_n^2\sin 2n\pi\alpha}{2n\pi}\right)\right.\right.$$

$$+\frac{k}{2\pi}\sum_n\sum_m a_n a_m\left(\frac{\sin\overline{m-n}\,\pi\alpha}{m-n}-\frac{\sin\overline{m+n}\,\pi\alpha}{m+n}\right)+\frac{1}{4}\sum_{n=1}^{n=\infty}\left(a_n^2\,\overline{1-\alpha}\right.$$

$$\left.\left.\left.+\frac{a_n^2\sin 2n\pi\alpha}{2n\pi}\right)+\frac{1}{2\pi}\sum_n\sum_m a_n a_m\left(\frac{\sin\overline{m+n}\,\pi\alpha}{m+n}-\frac{\sin\overline{m-n}\,\pi\alpha}{m-n}\right)\right\}\right] \quad ...(7)$$

For given values of α, γ and k, Equation (7) determines p in terms of a_1, a_2,...... . To find the lowest p, we have to select coefficients a_1, a_2, ... so as to make Equation (7) a minimum. This requires that the partial derivatives of p with respect to each of the coefficients must vanish, *i.e.*,

$$\left.\begin{aligned}&\frac{\partial p}{\partial a_1}=0\\&\frac{\partial p}{\partial a_2}=0\\&\cdots\cdots\cdots\\&\frac{\partial p}{\partial a_n}=0\\&\cdots\cdots\cdots\end{aligned}\right\} \qquad \ldots(8)$$

Thus we get as many number of Equations as there are coefficients. The partial derivative of p with respect to coefficient a_n is obtained as,

$$\frac{\partial p}{\partial a_n}=a_n\left\{2(k-1)\,\gamma\,\frac{\sin 2n\pi\alpha}{2n\pi}-2\gamma\alpha\,(k-1)-2\gamma+2pn^2-2n^4\right\}$$
$$+\frac{2\gamma\,(k-1)}{\pi}\sum_m a_m\left\{\frac{\sin\,(m+n)\,\pi\alpha}{m+n}-\frac{\sin\,(m-n)\,\pi\alpha}{m-n}\right\}=0 \qquad \ldots(9)$$

In Equation (9) the summation of the second term is extended over all values of m different from n. Thus substituting $n=1, 2, 3, \ldots\ldots$ in Equation (9), we obtain,

$$a_1\left\{2\,(k-1)\frac{\gamma\sin 2\pi\alpha}{2\pi}-2\gamma\alpha\,(k-1)-(2\gamma+2p-2\right\}+\frac{2\gamma(k-1)}{\pi}$$
$$\left\{a_2\left(\frac{\sin 3\pi\alpha}{3}-\frac{\sin\pi\alpha}{1}\right)+a_3\left(\frac{\sin 4\pi\alpha}{4}-\frac{\sin 2\pi\alpha}{2}\right)+\ldots\ldots\right\}=0$$
$$a_2\left\{2\,(k-1)\,\frac{\gamma\sin 4\pi\alpha}{4\pi}-2\gamma\alpha\,(k-1)-2\gamma+8p-32\right\}+\frac{2\gamma(k-1)}{\pi}$$
$$\left\{a_1\left(\frac{\sin 3\pi\alpha}{3}-\frac{\sin\pi\alpha}{1}\right)+a_3\left(\frac{\sin 5\pi\alpha}{5}-\frac{\sin\pi\alpha}{1}\right)+\ldots\ldots\right\}=0$$
$$a_3\left\{2\,(k-1)\,\frac{\gamma\sin 6\pi\alpha}{6\pi}-2\gamma\alpha(k-1)-2\gamma+18p-162\right\}+\frac{2\gamma(k-1)}{\pi}$$
$$\left\{a_1\left(\frac{\sin 4\pi\alpha}{4}-\frac{\sin 2\pi\alpha}{2}\right)+a_2\left(\frac{\sin 5\pi\alpha}{5}-\frac{\sin\pi\alpha}{1}\right)+\ldots\ldots\right\}=0$$
$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \qquad \ldots(10)$$

The trivial solution $a_1=a_2=\ldots\ldots=0$ is not the one that is sought, as in that case there will be no buckling. For a non-trivial solution, from Cramer's rule, the determinant formed by the coefficients of a_1, a_2,...... must vanish, *i.e.*,

$$\begin{vmatrix} 2\,(k-1)\,\dfrac{\gamma \sin 2\pi\alpha}{2\pi}-2\gamma\alpha(k-1)-2\gamma+2p-2 & \dfrac{2\gamma(k-1)}{\pi}\left(\dfrac{\sin 3\pi\alpha}{3}-\dfrac{\sin \pi\alpha}{1}\right) & \dfrac{2\gamma\,(k-1)}{\pi}\left(\dfrac{\sin 4\pi\alpha}{4}-\dfrac{\sin 2\pi\alpha}{2}\right) & \cdots\cdots\cdots \\ \dfrac{2\gamma\,(k-1)}{\pi}\left(\dfrac{\sin 3\pi\alpha}{3}-\dfrac{\sin \pi\alpha}{1}\right) & 2\,(k-1)\dfrac{\gamma \sin 4\pi\alpha}{4\pi}-2\gamma\alpha\,(k-1)-2\gamma+8p-32 & \dfrac{2\gamma\,(k-1)}{\pi}\left(\dfrac{\sin 5\pi\alpha}{5}-\dfrac{\sin \pi\alpha}{1}\right) & \cdots\cdots\cdots \\ \dfrac{2\gamma\,(k-1)}{\pi}\left(\dfrac{\sin 4\pi\alpha}{4}-\dfrac{\sin 2\pi\alpha}{2}\right) & \dfrac{2\gamma\,(k-1)}{\pi}\left(\dfrac{\sin 5\pi\alpha}{5}-\dfrac{\sin \pi\alpha}{1}\right) & 2\,(k-1)\dfrac{\gamma \sin 6\pi\alpha}{6\pi}-2\gamma\alpha\,(k-1)-2\gamma+18p-162 & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0 \qquad \ldots(11)$$

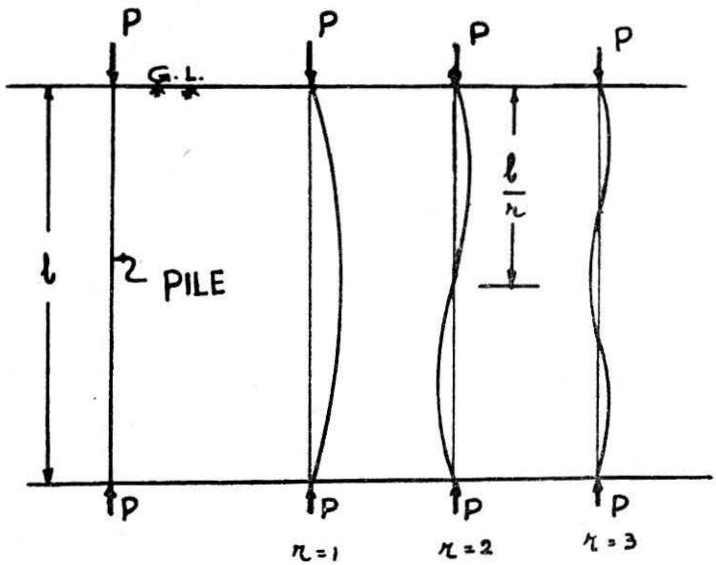
For given values of α, γ and k, the solution of Equation (11) gives a set of values for p; the number of values of p in the set being equal to the number of terms considered in the series [Equation (1)]. The lowest value of p in the set gives p_c. In problems of this type, the greater the number of terms considered in the series [Equation (1)], the more exactly the elasitca is defined and the more accurately the crippling load is evaluated, for, it then converges to a definite minimum value.

Timoshenko's and Granholm's (1929) expressions for crippling load P_c in case of homogeneous clays, are obtained as follows. If $k=1$ is substituted in Equation (11), which implies that the pile is surrounded by a homogeneous clay of foundation modulus β, we get,

$$\left.\begin{array}{l} a_1\,(-2\gamma+2p-2)=0 \\ a_2\,(-2\gamma+8p-32)=0 \\ a_3\,(-2\gamma+18p-162)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n(-2\gamma+2pn^2-2n^4)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \qquad \ldots(12)$$

If the series [Equation (1)] contain n terms, the solution of the set of n Equations (12) yields n values of p of which one value of p will be minimum say, rth. This requires all other coefficients except a_r in the set of Equations (12) to vanish. Thus, in the case of homogeneous clays, the elastica of the pile is represented by,

$$y=a_r \sin \frac{r\pi x}{l} \qquad \ldots(31)$$



$$P_c = \frac{\pi^2 EI}{l^2}\left(r^2 + \frac{\gamma}{r^2}\right)$$

FIGURE 2 : Vertical pile surrounded by homogeneous clay.

From the rth equation of the set of Equations (12) we get,

$$(-2\gamma+2pr^2-2r^4)=0 \qquad \ldots(14)$$

From which,

$$p=p_c=\frac{r^4+\gamma}{r^2} \qquad \ldots(15)$$

Hence,

$$P_c=\frac{\pi^2 EI}{l^2}\left(\frac{r^4+\gamma}{r^2}\right) \qquad \ldots(16)$$

To find the value of r, the number of half sine waves into which the pile will buckle as shown in Figure 2, we take the derivative of P_c with respect to r and equate to zero, *i.e.*,

$$\frac{dP_c}{dr}=\frac{\pi^2 EI}{l^2}(2r-2\gamma r^{-3})=0 \qquad \ldots(17)$$

From which,

$$r=\sqrt[4]{\gamma} \qquad \ldots(18)$$

If $\alpha=1$ is substituted in Equation (11), implying that the pile is surrounded by a homogeneous clay of foundation modulus $k\beta$, we shall obtain,

$$P_c=\frac{\pi^2 EI}{l^2}\left(\frac{r^4+k\gamma}{r^2}\right) \qquad \ldots(19)$$

and,

$$r=\sqrt[4]{k\gamma} \qquad \ldots(20)$$

Numerical Example

It is required to find the crippling load for a steel pile ISHB 450 hinged at both ends and driven through a two layered system of clay to bear on hard rock as shown in Figure 1. The following data is available.

$l=14.81$ mt
$I_{min.}=3045$ cm^4
$E=2\times10^6$ kg/cm^2
$\alpha=0.5$
$\beta=0.0616$ kg/cm^2
$k=148$

The calculations given below are carried out by means of Wang *Electronic Calculator 320 KT Model supplied* by Wang Laboratories, Inc., Teuksbury, U.S.A. :

$$\gamma=\frac{\beta l^4}{EI\,\pi^4}=\frac{0.0616\times14.81^4\times100^4}{2\times10^6\times3045\times3.14^4}=0.5$$

Euler's crippling load$=\dfrac{\pi^2 EI}{l^2}=27.40$ tonnes.

As a first trial, consider only the first term in the series [Equation (1)]. Upon substituting the relevant values for α, γ and k, Equation (11) reduces to,

$$2p-76.5=0$$

From which, $p_c=38.250$

Now consider the first two terms in the series [Equation (1)]. Then Equation (11) will reduce to,

$$\begin{vmatrix} 2p-76.5 & -62.1 \\ -62.1 & 8p-106.5 \end{vmatrix}=0$$

From which the lowest value of $p=p_c=5.900$

Proceeding similarly with the first three terms in the series [Equation (1)], we find Equation (11) reducing to,

$$\begin{vmatrix} 2p-76.5 & -62.1 & 0 \\ -62.1 & 8p-106.5 & -37.4 \\ 0 & -37.4 & 18p-236.5 \end{vmatrix}=0$$

From which the lowest value of $p=p_c=4.905$

Next consider the first four terms in the series [Equation (1)]. Equation (11) reduces to,

$$\begin{vmatrix} 2p-76.5 & -62.1 & 0 & +24.8 \\ -62.1 & 8p-106.5 & -37.4 & 0 \\ 0 & -37.4 & 18p-236.5 & -40 \\ +24.8 & 0 & -40 & 32p-586.5 \end{vmatrix}=0$$

From which the lowest value of $p=p_c=4.870$

Therefore, the crippling load on the pile$=4.87\times\frac{\pi^2 EI}{l^2}=133.44$ tonnes. A suitable factor of safety may be adopted to arrive at the working load.

To have an idea as to the the effect of layering, the crippling loads for the pile surrounded by homogeneous clays of foundation moduli β and $k\beta$ respectively are evaluated.

Firstly, if the pile is surrounded by homogeneous clay of foundation modulus $\beta=0.0616$ kg/cm²

$$r=4\sqrt{\gamma} \qquad =4\sqrt{0.5} \qquad =0.84$$

As the pile bends into an integral number of half sine waves the value of r is rounded off to the next higher integer 1.0. Hence,

$$P_c=\left(\frac{1+0.5}{1}\right)\frac{\pi^2 EI}{l^2}=1.5\frac{\pi^2 EI}{l^2}=41.1 \text{ tonnes.}$$

Secondly, if the pile is surrounded by homogeneous clay of foundation modulus $k\beta=9.0168$ kg/cm²

$$r=\sqrt[4]{148\times0.5} \qquad =4\sqrt{74} \qquad =2.94$$

Rounding off r to 3.0, we get,

$$P_c=\left(\frac{81+74}{9}\right)\frac{\pi^2 EI}{} =17.22\frac{\pi^2 EI}{l^2}=471.80 \text{ tonnes.}$$

Conclusions

(1) The existing theories to estimate the crippling loads on the piles surrounded by homogeneous clay do not correctly predict the crippling loads on piles surrounded by layered clay.

(2) Where the pile is surrounded by a two-layered system of clay, the method developed by the authors for calculating the crippling loads may be used to achieve greater degree of accuracy in estimating the pile capacity.

(3) In practice where the end conditions are between the ideal fixed and hinged end conditions, the pile will bear greater loads than those estimated by the theory.

(4) Since a portion of the axial load on the pile is resisted by adhesion, the pile capacity based on the above analysis is conservative.

Acknowledgement

The authors express their gratitude to Professor Abid Ali, Principal, College of Engineering, Osmania University, Hyderabad-7, Andhra Pradesh, for his encouragement and to Dr. S. Badruddin, Professor and Head of the Department of Civil Engineering, College of Engineering, Osmania University, Hyderabad-7, Andhra Pradesh, for his valuable suggestions during the preparation of the paper. We are thankful to Dr. N.V.R.L.N. Rao, Professor of Civil Engineering, College of Engineering, Osmania University, Hyderabad-7, Andhra Pradesh, for his interest in the work.

Notations

The following symbols are used in this paper:

EI=flexural rigidity of the pile,

k=ratio of the foundation moduli of top and bottom layers of clay,

l=length of the pile,

P=vertical load on the pile,

p=ratio between the vertical load on the pile and Eulers' crippling load,

P_c=crippling load on the pile,

p_c=ratio between the crippling load on the pile and Eulers' crippling load,

y=deflection of the pile at depth x,

α=ratio of the depth of top clay layer to the total depth of pile embedment,

β=foundation modulus of bottom clay layer,

$\gamma=\frac{\beta\, l^4}{EI\, \pi^4}$=dimensionless factor.

References

BJERRUM, L. : Personal Communication to Golder, H Q. and Skipp, B.O., "The Buckling of Piles in Soft Clay". *Proc. IVth Int. Conf. on Soil Mech. and Found. Engg.*, Vol. II, 35, (1957).

CASAGRANDE, A. : "The Pile Foundation for the New John Hancock Building, Boston". *J. Boston Soc. Civ. Engrs.*, 34, No. 4, 297, (1947).

CUMMINGS, A.E. : "The Stability of Foundation Piles against Buckling under Axial Load." *Proc. 18th Ann. Meet. Highway Res. Board*, Pt. 2, 112-19, (1938).

GRANHOLM, H. : "On the Elastic Stability of Piles Surrounded by a Supporting Medium". *Ingen Vetensk Akad. Handl.* Nr. 89 Svenska-Bokhandelcentralen, A.B. Stockholm, (1929).

SPOLFORD, C.M. : "Reconstruction of a Pier in Bostan, Massachusetts, Harbour". *Proc. Am. Sec. Civ. Engrs.*, Pt. 1, 64, No. 8, 1608, (1936).

TIMOSHENKO. S.P. and GERE, J.M. : "Theory of Elastic Stability". *Mc Graw-Hill Book Company., Inc.*