Seepage through an Imperfect Cut-off Wall

by

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Introduction

WHEN a pervious foundation is of limited depth and it is desirable to reduce the seepage and to remove the danger of uplift forces due to seepage, a cut-off such as that shown in Figure 1 (*a*) is used, especially in water supply dams and water power dams. The survey of literature reveals that all types of cut-off, with the exception of clay cut-off established in an open cut with sloping sides, may turn out to be defective in spite of conscientious supervision (1967). The effectiveness of the cut-offs drops sharply due to imperfections and may become practically ineffective (1948, 1953, 1961 & 1967). In the case of a complete cut-off, the cut-off is subjected to pressures of considerable magnitude and for long duration. Most chemical grouts being extremely compressible may get punctured during the service period (1967). Moreover, presence of any layer in the flow region which can make the water destructive will initiate the weakening of the cut-off and may ultimately puncture the cut-off. Therefore, in this study a thin cut-off with an opening at various locations and of many sizes has been studied. The influence of thickness of the cut-off on the seepage pattern is not considered here.

The problem of seepage in presence of a cut-off belongs to a general class of confined flow problems for which the method of solution by conformal mapping was pointed out by Pavlovskey in early twenties and his works have been described in many monographs (1959 & 1962). The closed form results of this paper were derived by making use of the conformal mapping technique.

Theoretical Analysis

The geometry of the problem and boundary values of the stream function, ψ, and the potential function, ϕ, on the boundaries are shown in Figure 1(*a*). Figures 1(*a*) and 1(*b*) denote the points in the $z=x+iy$ and $w=\phi+i\psi$ planes respectively. The problem is to find the values of w for the corresponding values of z or in other words, to obtain a functional relationship $w=f(z)$.

In order to achieve the functional relationship $w=f(z)$ the flow region in each of these (z & w) is mapped conformally onto the same half of an auxiliary t-plane [Figure 1 (*c*)] by means of the Schwarz-Christoffel transformation. This process yields the functions $z=f_1(t)$ and $w=f_2(t)$ and then, by eliminating the variable, t, the function $w=f(z)$ is established.

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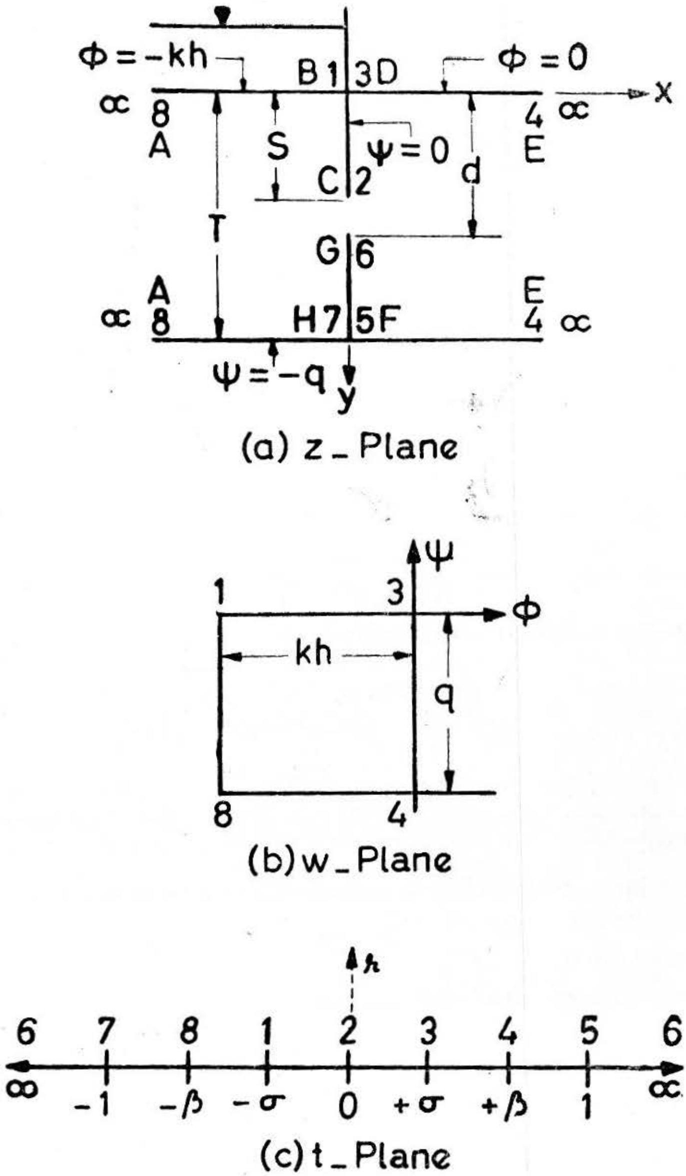

FIGURE 1 (a, b and c) : Illustrations of the problem.

DETERMINATION OF $z=f_1(t)$

For the transformation of the polygon *ABCDEFGHA* in the z-plane, Schwarz-Christoffel Equation takes the form :

$$\frac{dz}{dt}=\frac{M_1 t}{(t^2-\beta^2)\sqrt{(t^2-1)(t^2-\sigma^2)}} \qquad ...(1)$$

and
$$z=M_1\left[\frac{1}{\sqrt{(\beta^2-\sigma^2)(1-\beta^2)}}\left\{\tan^{-1}\sqrt{(\sigma^2-t^2)}\sqrt{\frac{(1-\beta^2)}{(\beta^2-\sigma^2)(1-t^2)}}-\tan^{-1}\sigma\sqrt{\frac{(1-\beta^2)}{(\beta^2-\sigma^2)}}\right\}\right]+N_1 \quad \text{(1963)} \quad ...(2)$$

At point 3, $z=0$ and $t=\sigma$ and therefore from Equation (2)

$$N_1=\frac{M_1}{\sqrt{(\beta^2-\sigma^2)(1-\beta^2)}}\tan^{-1}\sigma\sqrt{\frac{(1-\beta^2)}{(\beta^2-\sigma^2)}} \quad ...(3)$$

Substitution of Equation (3) in Equation (2) yields:

$$z=\frac{M_1}{\sqrt{(\beta^2-\sigma^2)(1-\beta^2)}}\tan^{-1}\sqrt{1+\frac{(\sigma^2-1)}{(1-t^2)}}\sqrt{\frac{(1-\beta^2)}{(\beta^2-\sigma^2)}} \quad ...(4)$$

At point 5, $z=iT$ and $t=1$; therefore, from Equation (4)

$$M_1=\frac{2iT}{\pi}\sqrt{(\beta^2-\sigma^2)(1-\beta^2)} \quad ...(5)$$

Substitution of Equation (5) in Equation (4) yields:

$$z=\frac{2iT}{\pi}\tan^{-1}\sqrt{\frac{(\sigma^2-t^2)}{(1-t^2)}}\sqrt{\frac{(1-\beta^2)}{(\beta^2-\sigma^2)}} \quad ...(6)$$

and after simplification Equation (6) reduces to

$$t=\pm\sqrt{1+\frac{(\sigma^2-1)}{1+\dfrac{(\beta^2-\sigma^2)\tanh^2\left(\dfrac{\pi z}{2T}\right)}{(1-\beta^2)}}} \quad ...(7)$$

The plus sign applies to the right half of the t-plane, the minus sign to the left half.

At point 2, $z=iS$ and $t=0$; therefore, from Equation (6) :

$$\frac{\sigma^2}{\tan^2\left(\dfrac{\pi S}{2T}\right)}=\frac{(\beta^2-\sigma^2)}{(1-\beta^2)} \quad ...(8)$$

Substitution of Equation (8) in Equation (7) yields:

$$t=\pm\sqrt{1+\frac{(\sigma^2-1)}{1+\dfrac{\sigma^2\tanh^2\left(\dfrac{\pi z}{2T}\right)}{\tan^2\left(\dfrac{\pi S}{2T}\right)}}} \quad ...(9)$$

At point 6, $z=id$ and $t=\infty$; therefore, from Equation (6)

$$\sigma^2=\frac{\tan^2\left(\dfrac{\pi S}{2T}\right)}{\tan^2\left(\dfrac{\pi d}{2T}\right)} \quad ...(10)$$

Substitution of Equation (10) in Equation (9) yields:

$$t = \pm \sqrt{\frac{\tan^2\left(\frac{\pi S}{2T}\right) + \tanh^2\left(\frac{\pi z}{2T}\right)}{\tan^2\left(\frac{\pi d}{2T}\right) + \tanh^2\left(\frac{\pi z}{2T}\right)}} \qquad \text{...(11)}$$

DETERMINATION OF $w = f_2(t)$

For the transformation of the polygon *ABDEA* in the w-plane, the Schwarz-Christoffel Equation becomes

$$\frac{dw}{dt} = \frac{M_2}{\sqrt{(\sigma^2 - t^2)(\beta^2 - t^2)}} \qquad \text{...(12)}$$

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and $$w = \frac{M_2}{\beta}\, Sn^{-1}\left(\frac{t}{\sigma}, \frac{\sigma}{\beta}\right) + N_2 \qquad \text{...(13)}$$

At point 3, $w = 0$ and $t = \sigma$; therefore, from Equation (13)

$$N_2 = -\frac{M_2}{\beta} K \qquad \text{...(14)}$$

Substituting Equation (14) in Equation (13) yields :

$$w = \frac{M_2}{\beta}\, Sn^{-1}\left(\frac{t}{\sigma}, \frac{\sigma}{\beta}\right) - \frac{M_2}{\beta} K \qquad \text{...(15)}$$

At point 8, $w = -(kh + iq)$ and $t = -\beta$; therefore, from Equation (15)

$$-(kh + iq) = -\frac{2M_2}{\beta} K - i\,\frac{M_2}{\beta} K' \qquad \text{...(16)}$$

Equating real points in left hand side and right hand side of Equation (16)

$$M_2 = \frac{kh}{2} \cdot \frac{\beta}{K} \qquad \text{...(17)}$$

$$q = \frac{kh}{2} \cdot \frac{K'}{K} \qquad \text{...(18)}$$

Substitution of Equations (17) and (18) into Equation (15) yields :

$$t = \sigma\, Sn\left(\frac{2wK}{kh} + K\right) \qquad \text{...(19)}$$

DETERMINATION OF β AND σ PARAMETERS

Substitution of Equation (10) into Equation (8) yields:

$$\beta = \frac{\cos\left(\frac{\pi d}{2T}\right)}{\cos\left(\frac{\pi S}{2T}\right)} \qquad \text{...(20)}$$

$$\sigma = \frac{\tan\left(\frac{\pi S}{2T}\right)}{\tan\left(\frac{\pi d}{2T}\right)} \qquad \text{...(21)}$$

The functional relationships between $z=f_1(t)$ and $w=f_2(t)$ have been established in Equations (11) and (19) respectively and from these two equations follows the following point to point correspondence between z-plane and w-plane

$$\pm\sqrt{\frac{\tan^2\left(\frac{\pi S}{2T}\right)+\tanh^2\left(\frac{\pi z}{2T}\right)}{\tan^2\left(\frac{\pi d}{2T}\right)+\tanh^2\left(\frac{\pi z}{2T}\right)}}=\frac{\tan\left(\frac{\pi S}{2T}\right)}{\tan\left(\frac{\pi d}{2T}\right)}Sn\left(\frac{2wK}{kh}\right)+K \quad ...(22)$$

Thus the problem in principle has been solved.

DETERMINATION OF EXIT GRADIENT

The gradient at any point in an isotropic flow region is

$$I=-\frac{dh}{ds}$$

where s is the direction of the streamline and h is the total head at that point. Again along a streamline $\frac{dw}{dt}=\frac{d\phi}{dt}$ and when the streamline intersects the tail-water equipotential boundary at right angles $\frac{dz}{ds}=i$. Therefore, the exit gradient along DE

$$I_E=-\frac{dh}{ds}=\frac{1}{k}\cdot\frac{d\phi}{ds}=\frac{1}{k}\cdot\frac{d\phi}{dt}\cdot\frac{dt}{dz}\cdot\frac{dz}{ds}$$

$$=\frac{i}{k}\left(\frac{dw}{dt}\cdot\frac{dt}{dz}\right)_{\psi=\text{Constant}} \quad ...(23)$$

Substitution of Equations (1) and (12) in Equation (23) yields :

$$I_E=\frac{\pi h}{4TK}\cdot\frac{\beta}{\sigma}\left[\frac{(1-t^2)(\beta^2-t^2)}{(\beta^2-\sigma^2)(1-\beta^2)}\right]^{\frac{1}{2}} \quad ...(24)$$

To obtain the exit gradient at any point along DE the value of t at that point needs to be substituted in Equation (24) and substituting $t=\sigma$ for point 3, which is the critical point in this case, in Equation (24) one finds,

$$I_E=\frac{\pi h}{4TKm}\cdot\frac{1}{\sin\left(\frac{\pi d}{2T}\right)} \quad ...(25a)$$

where the modulus is :

$$m=\frac{\sigma}{\beta}=\frac{\sin\left(\frac{\pi S}{2T}\right)}{\sin\left(\frac{\pi d}{2T}\right)} \quad ...(25b)$$

The values of I_E at point 3 for various combinations of S/T and d/T have been computed from Equation (25) with the help of a digital computer and are shown in Figure 2.

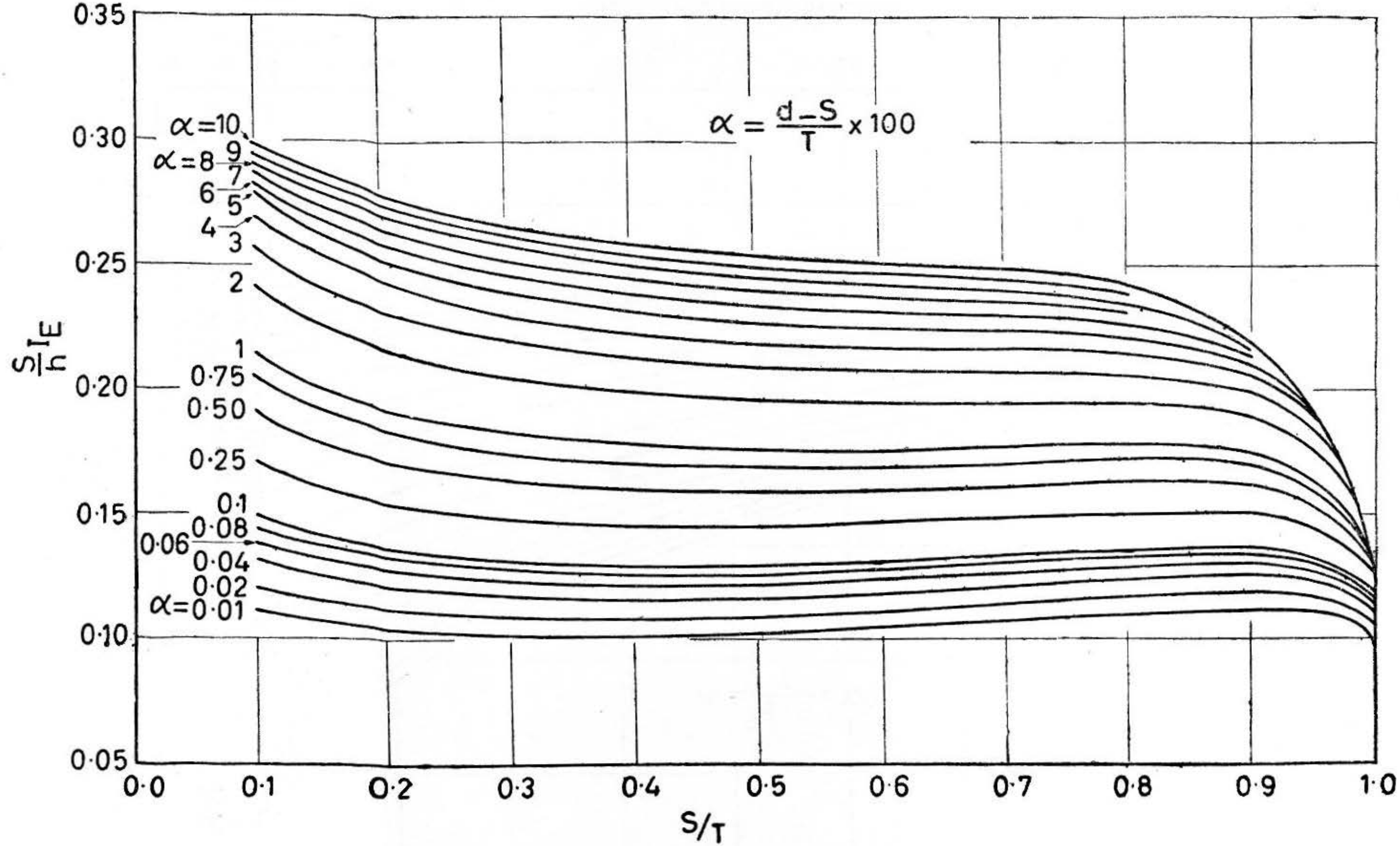

FIGURE 2 : Influence of S/T and d/T on exit gradient.

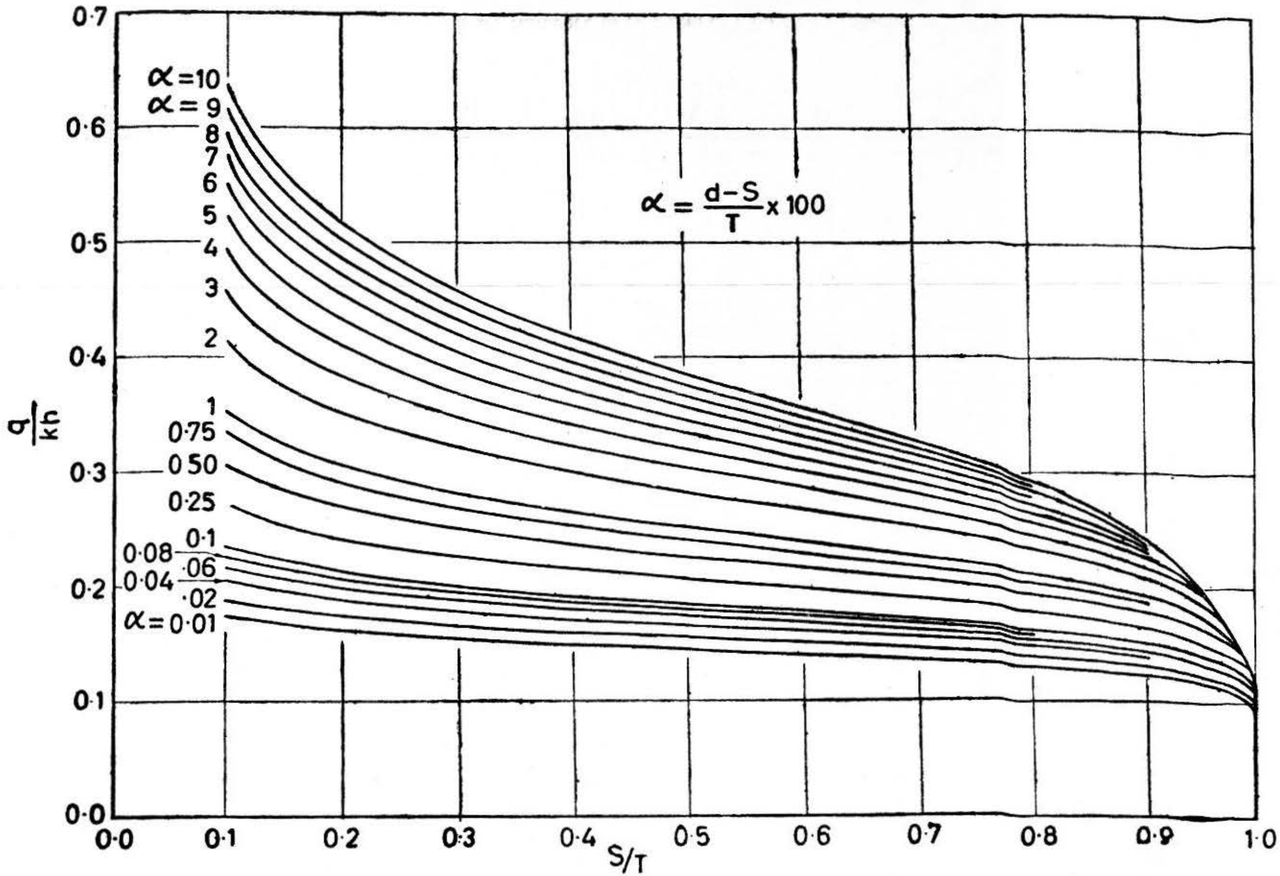

FIGURE 3 : Effects of S/T and d/T on discharge.

DETERMINATION OF DISCHARGE

Using Equation (18), the ratio q/kh has been computed for various values of S/T and d/T and is shown in Figure 3.

Discussion of Results

Figure 3 shows that a small opening of theorder of 0·0001 T with S/T of 0·99 increases the discharge from theoretical value of zero discharge to $q/kh=0{\cdot}105$ and for the same opening with S/T of 0·1, q/kh is 0·175. In other words the quantity of seepage through a particular size opening depends on the location of the opening. Again for an opening of $0{\cdot}1T$ with $S/T=0{\cdot}1$ and 0·99, q/kh are 0·635 and 0·142 respectively. It may, therefore, be noted that the influence of the location of an opening on the rate of increase in discharge is more pronounced when the size of the opening is greater. As regards the exist hydraulic gradient, Figure 2 shows that I_E decreases rapidly with a little increase in S/T when S/T is close to zero and 1, as it is expected. But it is interesting to note that the rate of decrease in I_E is not very suspectible to the location when it is between $S/T=0{\cdot}2$ and 0·7. The I_E is also found to be dependent on the size of the opening.

It needs to be mentioned that the solutions presented here can be used for both partially penetrated and fully penetrated cut-off walls without any opening. In the case of a fully penetrated cut-off without any opening S equals d and consequently $m=1$. Therefore, Equation (18) yields $q=0$ and Equation (25) yields $I_E=0$. Again for the case of a partially penetrated cut-off wall with an opening at the lowest end of the wall, d becomes equal to T and consequently $m=\sin\left(\frac{\pi S}{2T}\right)$. With this value of m, Equations (18) and (25) yield $q=\frac{kh}{2}\cdot\frac{K'}{K}$ and $I_E=\frac{\pi h}{4IKm}$ respectively when $m=\sin\left(\frac{\pi S}{2T}\right)$. These results are identical to the solutions obtained by Polubarinova-Kochina for q and Harr for I_E (1962).

Conclusions

*The general solution to the problem of seepage through a leaky cut-*off is presented using the conformal transformation technique. Established solutions for a partial cut-off wall and for a complete cut-off wall are verified from the solution. Although the solutions presented are valid for all values of S/T and d/T, the observations made earlier are based on the calculated data presented in graphical form, to be used for analysis and design purposes. In the case of an anisotropic porous media, the charts apply for the equivalent isotropic section.

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Notations

d = depth to the lower limit of opening in the cut-off wall;
h = hydraulic head;
I = hydraulic gradient;
I_E = exit gradient;
$K = k(m) = F(m, \pi/2)$ } complete elliptic integral of the first
$K' = k(m') = F(m', \pi/2)$ } kind;
k = coefficient of permeability;
M_1, M_2 = complex constants;
m = modulus of elliptic integrals;
m' = comodulus $= \sqrt{1-m^2}$;
N_1, N_2 = constants of integration;
q = discharge or quantity of seepage;
S = depth to the upper limit of opening in the cut-off wall;
T = depth to the base of the porous stratum;
t = a complex variable, a point in the t-plane;
w = a complex variable, a point in the w-plane;
z = a complex variable, a point in the z-plane;
σ, β = parameters;
ϕ = potential function; and
ψ = stream function.

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