# A New Pseudo-Approximate Solution of the Consolidation Equation

*by*

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## 1. Introduction

THE one-dimensional primary consolidation of soils is governed by the parabolic equation of Terzaghi, *viz.*,

$$\frac{\partial U}{\partial t} = C_v \frac{\partial^2 U}{\partial Z^2} \qquad ...(1)$$

where $U(Z, t)$ is the pore-water pressure in excess of hydrostatic, and $C_v$ is the coefficient of consolidation. Exact solutions to the linear partial differential Equation (1) can be obtained by the Fourier methods or by Laplace Transforms for simple initial and boundary conditions. However, in practice, problems may arise with conditions to which solutions have not been attained analytically or cannot be found in the available references. Also with complicated conditions rigorous mathematical solutions become difficult. Because of the assumptions made in deriving Equation (1), even the most elaborate mathematical analysis is only an approximation to the settlement process. Hence, if compromises are to be made, it appears reasonable to do so with respect to mathematical rigour. It is, therefore, advisable to introduce approximate methods for handling solutions so that practical results can be achieved with the degree of accuracy justified in the light of the amount of information available on the soil properties at the site and the importance of the structure. Approximate methods based on finite-difference, iteration and relaxation have already been proposed. This paper proposes a new technique which is shown to yield results very close to the exact rigorous solution.

## 2. The Pseudo-Approximate Method

The quintessence of the new method is that discretisation is carried out only in space and not in both space and time. Hence, difference equations are substituted for the spatial partial derivative $\frac{\partial^2 U}{\partial Z^2}$ only and Equation (1) is left continuous in time. The resulting pseudo-approximate recurrence equation is solved using the Laplace Transform.

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Using Taylor's series, it can be shown that,

$$\frac{\partial^2 u}{\partial z^2} = \underset{\triangle z \to 0}{Lt} \left[ \frac{u\,(z+\triangle Z,\, t) + u\,(z-\triangle z,\, t) - 2u\,(Z,\, t)}{\triangle Z^2} \right]. \quad ...(2)$$

Substitution of Equation (2) into Equation (1) and upon taking the limit $\triangle Z$ to a small finite distance, we obtain

$$\frac{\partial U}{\partial t} = \frac{C_v}{\triangle Z^2} \left[ u\,(z+\triangle z,\, t) + u\,(z-\triangle z,\, t) - 2u\,(z,\, t) \right] \quad ...(3)$$

Suppose that the depth $H$ of the clay layer (Figure 1) is discretised into four finite layers. In this case there are five nodal points and the $\triangle Z$'s are equal as shown in Figure 1. Then, we can write

$$\triangle Z = n \triangle z \quad ...(4)$$

where $n$ is a positive integer which can take values 0, 1, 2, 3, 4, in this discretisation. Furthermore,

$$u\,(z,\, t) = u\,(n \triangle z,\, t) = u_n\,(t) \quad ...(5)$$

$$u\,(z+\triangle z,\, t) = u\,[(n+1) \triangle z,\, t] = u_{n+1}\,(t) \quad ...(6)$$

$$u\,(z-\triangle z,\, t) = u\,[(n-1) \triangle z,\, t] = u_{n-1}\,(t) \quad ...(7)$$

$$\frac{\partial u}{\partial t}\,(z,\, t) = \frac{\partial u}{\partial t}\,(n \triangle z,\, t) = \frac{\partial u_n\,(t)}{\partial t} \quad ...(8)$$

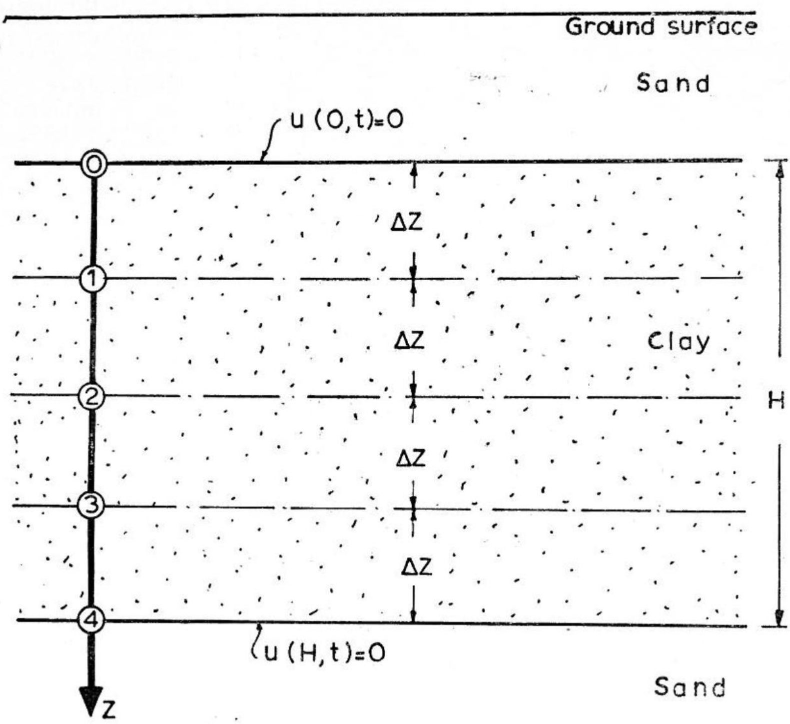


**FIGURE 1: Clay layer discretised in space.**

Substitution of Equations (5) through (8) in Equation (3) gives,

$$\frac{\partial U_n(t)}{\partial t} = \frac{C_v}{\triangle Z^2}\Big[U_{n+1}(t) + U_{n-1}(t) - 2\,U_n(t)\Big] \qquad ...(9)$$

Equation (9) is a first order differential equation that controls the excess pore-pressure at the $n$th node. It is, in fact, a recurrence relationship ; the differential equations for the pore-pressure at the internal nodes can be obtained on substituting $n = 1$, 2, and 3.

$$\frac{\partial U_1(t)}{\partial t} = M\Big[u_2(t)+u_o(t)-2u_1(t)\Big] \qquad ...(10)$$

$$\frac{\partial U_2(t)}{\partial t} = M\Big[u_3(t)+u_1(t)-2u_2(t)\Big] \qquad ...(11)$$

$$\frac{\partial U_3(t)}{\partial t} = M\Big[u_4(t)+u_2(t)-2\,u_3(t)\Big] \qquad ...(12)$$

where $M$ is a modulus defined by

$$M = \frac{C_v}{\triangle Z^2} \qquad ...(13)$$

From Figure 1, the boundary conditions are

$$u(0, t) = 0 \text{ (free-drainage)} \qquad ...(14)$$

$$u(H, t) = 0 \text{ (free-drainage)} \qquad ...(15)$$

Conditions (14) and (15) imply,

$$u_o(t) = 0 \qquad ...(16)$$

$$u_4(t) = 0 \qquad ...(17)$$

Substituting Equations (16) and (17) in Equations (10) through (12), we obtain,

$$\frac{\partial U_1(t)}{\partial t} = M\Big[u_2(t)-2u_1(t)\Big] \qquad ...(18)$$

$$\frac{\partial U_2(t)}{\partial t} = M\Big[u_3(t)+u_1(t)-2\,u_2(t)\Big] \qquad ...(19)$$

$$\frac{\partial U_3(t)}{\partial t} = M\Big[u_2(t)-2\,u_3(t)\Big] \qquad ...(20)$$

The initial condition for the consolidation problem may be arbitrarily assumed as,

$$u = (Z, 0) = G(Z) \qquad ...(21)$$

condition (21), for our purposes imply,

$$u_n(0) = G(Z) \qquad ...(22)$$

Taking Laplace transforms of Equations (18), (19) and (20) and using (22), we obtain,

$$s\bar{u}_1(s) - u_1(0) = M[\bar{u}_2(s) - 2\bar{u}_1(s)] \quad ...(23)$$

$$s\bar{u}_2(s) - u_2(0) = M[\bar{u}_3(s) + \bar{u}_1(s) - 2\bar{u}_2(s)] \quad ...(24)$$

$$s\bar{u}_3(s) - u_3(0) = M[\bar{u}_2(s) - 2\bar{u}_3(s)] \quad ...(25)$$

where $L[u_1(t)] = \bar{u}_1(s)$ and so on. The problem has now been transformed into the Laplace Plane and we have to solve Equation (23) through (25) simultaneously to obtain $\bar{u}_1(s)$ through $\bar{u}_3(s)$. Rearranging Equation (23) through (25) we get the following forms :

$$(s+2M)\bar{u}_1(s) - M\bar{u}_2(s) = u_1(0) \quad ...(26)$$

$$-M\bar{u}_1(s) + (s+2M)\bar{u}_2(s) - M\bar{u}_3(s) = u_2(0) \quad ...(27)$$

$$-M\bar{u}_2(s) + (s+2M)\bar{u}_3(s) = u_3(0) \quad ...(28)$$

Equations (26) through (28) may be written as,

$$\left.\begin{aligned} a_{11}\bar{u}_1(s) + a_{12}\bar{u}_2(s) + a_{13}\bar{u}_2(s) &= k_1 \\ a_{21}\bar{u}_1(s) + a_{22}\bar{u}_2(s) + a_{23}\bar{u}_3(s) &= k_2 \\ a_{31}\bar{u}_1(s) + a_{32}\bar{u}_2(s) + a_{33}\bar{u}_3(s) &= k_3 \end{aligned}\right\} \quad ...(29)$$

The condition for the set of Equations (29) to have a unique solution is

$$-A\,| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \quad ...(30)$$

If condition (30) is satisfied, then the set of Equations (29) have a unique solution given by,

$$\bar{U}_1(s) = \frac{D_1}{|A|}$$

$$\bar{U}_2(s) = \frac{D_2}{|A|} \quad ...(31)$$

and $$\bar{U}_3(s) = \frac{D_3}{|A|}$$

where

$$D_1 = \begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix} \qquad \ldots(32)$$

$$\text{and } D_3 = \begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}$$

The following values of $a$'s and $k$'s hold from Equation (29) through (32)

$$\begin{array}{lll} a_{11} = s + 2M & a_{12} = \ -M & a_{13} = \ 0 \\ a_{21} = -M & a_{22} = s + 2M & a_{23} = \ -M \\ a_{31} = 0 & a_{32} = -M & a_{33} = s + 2M \end{array}$$

and

$$k_1 = u_1(0)$$
$$k_2 = u_2(0)$$
$$k_3 = u_3(0)$$

The determinant $|A|$ can be expanded after inserting the elements,

$$|A| = \begin{vmatrix} s + 2M & -M & 0 \\ -M & s + 2M & -M \\ 0 & -M & s + 2M \end{vmatrix}$$

$$= s^3 + 6s^2M + 10sM^2 + 4M^3 \qquad \ldots(33)$$

To facilitate the inversion process, the third degree polynomial (33) can be broken as,

$$s^3 + 6s^2M + 10sM^2 + 4M^3 = (s+\alpha)(s+\beta)(s+\gamma) \qquad \ldots(34)$$

Equating the coefficients of equal powers of $s$ in Equation (34), and solving for $\alpha$, $\beta$ and $\gamma$, we obtain,

$$\left.\begin{array}{l} \alpha = 3{\cdot}414\ M \\ \beta = 0{\cdot}586\ M \\ \gamma = 2{\cdot}000\ M \end{array}\right\} \qquad \ldots(35)$$

Thus $|A|$ is completely known. By defining the following constants,

$$d_o = M^2\ [3u_1(0) + 2u_2(0) + u_3(0)] \qquad \ldots(36)$$

$$d_1 = M\ [4u_1(0) + u_2(0)] \qquad \ldots(37)$$

$$h_o = 2M^2 [\ u_1 (0) + 2\, u_2 (0) + u_3 (0)] \qquad ...(38)$$

$$h_1 = M\ [\ u_1 (0) + 4\, u_2 (0) + u_3 (0)] \qquad ...(39)$$

$$l_o = M^2\ [\ u_1 (0) + 2\, u_2 (0) + 3u_3 (0)] \qquad ...(40)$$

$$l_1 = M\ [\ u_2 (0) + 4\, u_3 (0)] \qquad ...(41)$$

Using (36) through (41) in Equations (31), the final solution in the Laplace-Plane is obtained in the form,

$$\bar{u}_1 (s) = \frac{s^2 u_1 (0) + s\, d_1 + d_o}{(s+\alpha)(s+\beta)(s+\gamma)} \qquad ...(42)$$

$$\bar{u}_2 (s) = \frac{s^2 u_2 (0) + s\, h_1 + h_o}{(s+\alpha)(s+\beta)(s+\gamma)} \qquad ...(43)$$

$$\bar{u}_3 (s) = \frac{s^2 u_3 (0) + s\, l_1 + l_o}{(s+\alpha)(s+\beta)(s+\gamma)} \qquad ...(44)$$

The form of Equation (42) through (44) suggests that the inverse Laplace Transform can be best obtained using the complex inversion formula and Cauchy's residue theorem. The inversion process is illustrated in the typical example that follows.

## 3. A Typical Example

Consider the initial condition shown in Figure 2, encountered in the investigation of consolidation settlement resulting from a uniform increase in excess pore-water pressure with depth. This case, despite its apparent simplicity, is representative of the state of initial excess pore-water pressures commonly assumed to exist within a soil specimen in a standard consolidation test upon the application of an increment of load. The initial condition is given by,

$$U (z, 0) = P_o = \text{constant} \qquad ...(45)$$

and the boundary conditions are given by Equations (14) and (15).

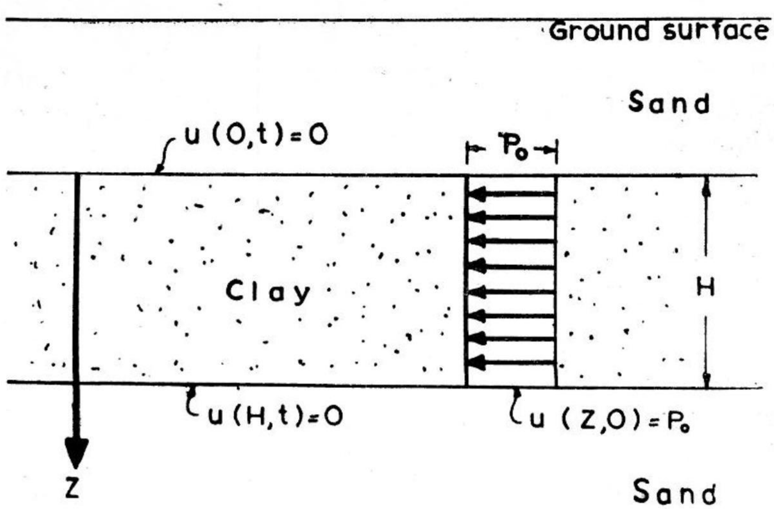


**FIGURE 2 : Initial condition for the example.**

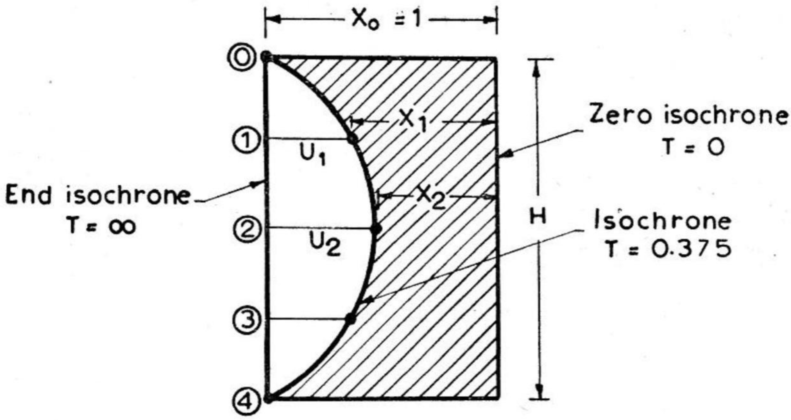


**FIGURE 3 : Isochrone for typical model computation for line (4), Table I.**

The clay-layer is discretised into four equal layers and the five nodes are numbered as shown. It is seen, from symmetry, that,

$$u_o(t) = u_4(t) = 0 \text{ (boundary conditions)}$$
$$u_1(t) = u_3(t) \qquad \ldots(46)$$

The constants defined by Equations (36) through (41) may be calculated using the following fact,

$$u_1(0) = u_2(0) = u_3(0) = P_o \qquad \ldots(47)$$

The following values can be obtained on substitution,

$$d_o = 6\,M^2\,P_o\,;\; h_1 = 6\,M\,P_o$$
$$d_1 = 5\,M\,P_o\,;\; l_o = 6\,M^2\,P_o \qquad \ldots(48)$$
$$h_o = 8\,M^2\,P_o\,;\; l_1 = 5\,M\,P_o$$

The value of the modulus is given by Equation (13),

where $Z = H/4$.

$$M = \frac{C_v}{\left[\frac{H}{4}\right]^2} = \frac{16\,C_v}{H^2} \qquad \ldots(49)$$

Using Equations (47) and (48) in (42) through (44) we obtain,

$$\bar{u}_1(s) = \frac{P_o\,[s^2 + 5\,M\,s + 6\,M^2]}{(s+\alpha)\;(s+\beta)\;(s+\gamma)}$$
$$\bar{u}_2(s) = \frac{P_o\,[s^2 + 6\,M\,s + 8\,M^2]}{(s+\alpha)\;(s+\beta)\;(s+\gamma)} \qquad \ldots(50)$$

and

$$\bar{u}_3(s) = \frac{P_o\,[s^2 + 5\,M\,s + 6\,M^2]}{(s+\alpha)\;(s+\beta)\;(s+\gamma)}$$

Thus $\bar{u}_1(s) = \bar{u}_3(s)$ as has already been expected (46) from the symmetry of the initial condition (45).

To invert $\bar{u}_1(s)$, consider the function,

$$F(s) = e^{st}\,\bar{u}_1(s) \qquad ...(51)$$

Then $F(s)$ has three simple poles defined by $s=-\alpha$, $s=-\beta$ and $s=-\gamma$. The residues, at each of the poles, of the function $F(s)$, are calculated as follows :

Residue at the first pole,

$$\psi_1 = \underset{s\to(-\alpha)}{Lt}\ [s-(-\alpha)]\,F(s)$$

$$= \underset{s\to(-\alpha)}{Lt}\ (s+\alpha)\,e^{st}\,\bar{u}_1(s)$$

$$= \underset{s\to(-\alpha)}{Lt}\ (s+\alpha)\,e^{st}\,\frac{P_o\,(s^2+5\,M s+6\,M^2)}{(s+\alpha)\,(s+\beta)\,(s+\gamma)}$$

$$= \underset{s\to(-\alpha)}{Lt}\ e^{st}\,\frac{P_o\,(s^2+5M\,s+6M^2)}{(s+\beta)\,(s+\gamma)}$$

$$= \frac{P_0\,(\alpha^2-5M\,\alpha+6M^2)\,e^{-\alpha t}}{(\beta-\alpha)\,(\gamma-\alpha)} \qquad ...(52)$$

Similarly,

Residue of $F(s)$, at the second simple pole,

$$\psi_2 = \frac{P_o\,[\beta^2-5M\,\beta+6M^2]\,e^{-\beta t}}{(\alpha-\beta)\,(\gamma-\beta)} \qquad ...(53)$$

Residue of $F(s)$, at the third simple pole,

$$\psi_3 = \frac{P_o\,[\gamma^2-5M\,\gamma+6M^2]\,e^{-\gamma t}}{(\alpha-\gamma)\,(\beta-\gamma)} \qquad ...(54)$$

By the complex inversion theorem, we have,

$$L^{-1}\,[\bar{u}_1(s)] = \Sigma \text{ Residues of } e^{st}\,\bar{u}_1(s)$$

*i.e.*, $$u_1(t) = \psi_1+\psi_2+\psi_3 \qquad ...(55)$$

Using Equations (35) and (49) in (55) and substituting,

$$\frac{4\,C_v\,t}{H^2} = T \text{ (time factor)} \qquad ...(56)$$

we obtain,

$$u_1(t) = \frac{P_o}{6{\cdot}775}\,e^{-13{\cdot}656T} + \frac{P_o}{1{\cdot}171}\,e^{-2{\cdot}344\,T} \qquad ...(57)$$

Using the complex inversion process outlined above, $\bar{u}_2(s)$ can also be inverted. The following equation is obtained for $u_2(t)$,

$$u_2(t) = P_o\left[\,1{\cdot}207\,e^{-2{\cdot}344\,T} - \frac{1}{4{\cdot}875}\,e^{-13{\cdot}656\,T}\right] \qquad ...(58)$$

Using Equations (57) and (58), the values $u_1(t)$ and $u_2(t)$ are calculated for various values of $T$. The results are shown in Table I. The average consolidation $U_{av}$ is calculated for each $T$ using Simpson's rule. As an example, the complete set of calculations for line (4) in Table I, are given below :

Line (4) :

$$T = 0{\cdot}375$$

Table I :

$$\frac{u_1(T)}{P_o} = \frac{1}{6{\cdot}775}\, e^{-5{\cdot}121} + \frac{1}{1{\cdot}171}\, e^{-0{\cdot}879}$$

$$= 0{\cdot}001 + 0{\cdot}354 = 0{\cdot}355$$

$$\frac{u_2(T)}{P_o} = 1{\cdot}207\, e^{-0{\cdot}879} - \frac{1}{4{\cdot}875}\, e^{-5{\cdot}121}$$

$$= 0{\cdot}501 - 0{\cdot}001 = 0{\cdot}500$$

The isochrone for $T = 0{\cdot}375$ is shown in Figure 3. The shaded-area, using Simpson's rule,

$$= \frac{1}{3}\left[\frac{H}{4}\right]\left[x_0 + 4\,x_1 + 2\,x_2 + 4\,x_3 + x_4\right]$$

$$= \frac{H}{6}\left[x_o + 4\,x_1 + x_2\right] \text{ (owing to symmetry)}$$

$$= \frac{H}{6}\left[1 + 4\,(0{\cdot}645) + 0{\cdot}5\right]$$

$$= \frac{H}{6}\left[4{\cdot}08\right] = 0{\cdot}68\,H$$

Since the total area between the zero and end isochrones is $H$, the average percentage consolidation at $T = 0{\cdot}375$ is given by,

$$U_{av} = \frac{0{\cdot}68\,H}{H} \times 100 = 68 \text{ per cent.}$$

This completes the set of computations for line (4) of Table I.

**TABLE I**

*Computation for the Example using Author's Pseudo-Approximate Method.*

| Line | $T$ | $U_1(T)$ | $U_2(T)$ | $U_{av}$ per cent (author) | $U_{av}$ per cent (Exact) |
|---|---|---|---|---|---|
| 1 | 0·000 | 1·000 | 1·000 | 0 | 0 |
| 2 | 0·125 | 0·664 | 0·863 | 41 | 40 |
| 3 | 0·250 | 0·480 | 0·665 | 56·5 | 56 |
| 4 | 0·375 | 0·355 | 0 500 | 68 | 67 |
| 5 | 0·500 | 0·265 | 0·371 | 76 | 76 |
| 6 | 0·625 | 0·196 | 0·277 | 82 | 82 |

TABLE II

*Comparison of Author's Method with the Finite-Difference Method.*

| $T$ | Average consolidation (per cent) | | |
|---|---|---|---|
| | Finite-difference method | Author's method | Exact method |
| 0 | 0 | 0 | 0 |
| 0·125 | 33 | 41 | 40 |
| 0·250 | 54 | 56·5 | 56 |
| 0·375 | 66·5 | 68 | 67 |
| 0·500 | 77 | 76 | 76 |
| 0·625 | 83 | 82 | 82 |

The last column in Table I, gives the average consolidation values obtained by an exact solution using Fourier method for this initial condition. It is seen that the author's method gives values remarkably close to the exact solution.

To compare the pseudo-approximate solution with the traditional finite difference method, the same example is worked out using the finite-difference method and the values are entered in Table II. On comparison, it is seen that in the finite-difference method, the error is large and the error exists always. But in the pseudo-approximate method the error is negligibly small and there is no residual error as the time passes on.

## 4. Conclusions

The pseudo-approximate method presented herein appears to be the most powerful and elegant for solving the consolidation Equation (1) subjected to any type of initial condition. The operation is rather easy and it does not involve the tedium encountered in an exact solution. However, the results are surprisingly close to the exact values.

## 5. Acknowledgement

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